A numerical algorithm for solving a class of matrix equations

Huamin Zhang†‡§, Hongcai Yin† and Rui Ding†∗

†Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi 214122, P.R. China
Emails: zhangeasymail@126.com, rding12@126.com
‡School of Management Science and Engineering, Anhui University of Finance & Economics, Bengbu 233000, P.R. China
Email: hongcaiyin@sina.com
§Department of Mathematics & Physics, Bengbu College, Bengbu 233030, P.R. China

Abstract. In this paper, we present a numerical algorithm for solving matrix equations \((A \otimes B)X = F\) by extending the well-known Gaussian elimination for \(Ax = b\). The proposed algorithm has a high computational efficiency. Two numerical examples are provided to show the effectiveness of the proposed algorithm.

Keywords: Gaussian elimination, Kronecker product, matrix equation.

AMS Subject Classification: 15AXX, 65FXX.

1 Introduction

Numerical solutions or iterative algorithms for different matrix equations have received much attention \([34, 22, 23, 11]\). For example, Charnsethikul presented a numerical algorithm for solving \(n \times n\) linear equations \(AX = b\) with parameters covariances \([2]\). The iterative algorithms can solve linear matrix equations \([10, 9, 25, 29, 17]\) but the Gaussian elimination method is direct and important for solving linear equations \([20, 15]\). In order to avoid

*Corresponding author.
Received: 26 January 2014 / Revised: 19 February 2014 / Accepted: 19 February 2014.

© 2014 University of Guilan http://research.guilan.ac.ir/jmm
the error accumulations and to improve the numerical stability, several pivoting strategies have been adopted [15, 14], e.g., the partial pivoting strategy, the complete pivoting strategy and the rook pivoting strategy. Studies on Gaussian elimination include the pivoting strategies [28], stabilities [27] and coefficient matrices [15].

The matrix equations play an important role in system theory [32, 12, 3, 5], control theory [26, 31, 30, 18], stability analysis [21, 24, 13, 4]. A conventional method for solving equations $AXB = F$ is to use the Kronecker product [15]. However, high dimensions of the associated matrices result in heavy computational burden [15]. There exist many methods which transform the matrix into forms for which solutions may be readily computed, such as the Jordan canonical form [19], the companion form [1] and the Hessenberg-Schur form [16]. However, these methods require computing additional matrix transformations or decompositions. Besides these methods, the iterative algorithms [32, 33] and the hierarchical identification principle [6, 7, 8] have also been used to solve the linear equations. Recently, the solution of matrix equation $AXB = F$ has been discussed under different conditions [6]. In this paper, we consider the matrix equation $(A \otimes B)X = F$ and present a new and efficient algorithm based on the Gaussian elimination.

This paper is organized as follows. Section 2 introduces the Gaussian elimination for equations $AX = F$. Section 3 discusses numerical algorithms for matrix equations $(A \otimes B)X = F$. Section 4 gives two numerical examples to illustrate the effectiveness of the proposed algorithm. Finally, we provide some concluding remarks in Section 5.

2 Gaussian elimination for $AX=F$

Consider the following matrix equation

$$AX = F, \quad (1)$$

where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ and $F \in \mathbb{R}^{n \times m}$ are given constant matrices, $X \in \mathbb{R}^{n \times m}$ is the unknown matrix to be solved. Let

$$F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \in \mathbb{R}^{n \times m}, \quad f_i \in \mathbb{R}^{1 \times m}, \quad i = 1, 2, \ldots, n.$$
Assume that $A$ is invertible and let $[A|F]^{(1)} := [A|F]$ be the augmented matrix of system (1), and denoted as

$$[A|F]^{(1)} = \begin{bmatrix}
|a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & f_1^{(1)} \\
|a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & f_2^{(1)} \\
| \vdots & \vdots & \ddots & \vdots & \vdots \\
|a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & f_n^{(1)} \\
\end{bmatrix},$$

where

$$a_{ij}^{(1)} = a_{ij}, \ i, j = 1, 2, \ldots, n,$$

$$f_i^{(1)} = f_i \in \mathbb{R}^{1 \times m}, \ i = 1, 2, \ldots, n.$$

With these symbols, we give the Gaussian elimination for solving matrix equations $AX = F$.

**Algorithm 1.**

1. For $i = 1$, let

$$|a_{j1}^{(1)}| := \max\{|a_{11}^{(1)}|, |a_{21}^{(1)}|, \ldots, |a_{n1}^{(1)}|\},$$

interchange the 1st row and $j$th row. If $A$ is invertible, then $a_{11}^{(1)} \neq 0$ can be used to eliminate $a_{21}^{(1)}, a_{31}^{(1)}, \ldots, a_{n1}^{(1)}$. Let $m_{k1} := a_{k1}^{(1)}/a_{11}^{(1)}$, $k = 2, 3, \ldots, n$, we have

$$[A|F]^{(2)} := \begin{bmatrix}
|a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & f_1^{(1)} \\
|0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & f_2^{(1)} \\
| \vdots & \vdots & \ddots & \vdots & \vdots \\
|0 & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & f_n^{(1)} \\
\end{bmatrix},$$

where

$$a_{kj}^{(2)} = a_{kj}^{(1)} - m_{k1}a_{1j}^{(1)}, \ k = 2, 3, \ldots, n, \ j = 2, 3, \ldots, n,$$

$$f_k^{(2)} = f_k^{(1)} - m_{k1}f_1^{(1)}, \ k = 2, 3, \ldots, n.$$

2. For $i = 2$, let

$$|a_{j2}^{(2)}| := \max\{|a_{22}^{(2)}|, |a_{32}^{(2)}|, \ldots, |a_{n2}^{(2)}|\},$$
interchange the 2nd row and jth row. If A is invertible, then \( a_{22}^{(2)} \neq 0 \) can be used to eliminate \( a_{32}^{(2)}, a_{42}^{(2)}, \ldots, a_{n2}^{(2)} \). Set

\[
m_{k2} := \frac{a_{k2}^{(2)}}{a_{22}^{(2)}}, \quad k = 3, 4, \ldots, n,
\]

and subtract \( m_{k2} \) times the second row of \([A|F]^{(2)}\) from the kth row gives

\[
[A|F]^{(3)} := \left[ \begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & f_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & f_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & f_3^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} & f_n^{(3)} \end{array} \right],
\]

where

\[
a_{kj}^{(3)} = a_{kj}^{(2)} - m_{k2}a_{2j}^{(2)}, \quad k = 3, 4, \ldots, n, \quad j = 3, 4, \ldots, n,
\]

\[
f_k^{(3)} = f_k^{(2)} - m_{k2}f_2^{(2)}, \quad k = 3, 4, \ldots, n.
\]

3. For \( i = 3, 4, \ldots, n \), continuing in this way, let

\[
|a_{ji}^{(i)}| = \max\{|a_{ii}^{(i)}|, |a_{i+1,i}^{(i)}|, \ldots, |a_{ni}^{(i)}|\},
\]

interchange the ith row and jth row. If A is invertible then \( a_{ii}^{(i)} \neq 0 \), \( i = 3, 4, \ldots, n \). Set

\[
m_{ki} := \frac{a_{ki}^{(i)}}{a_{ii}^{(i)}}, \quad i = 3, 4, \ldots, n, \quad k = i + 1, i + 2, \ldots, n
\]

and subtract \( m_{ki} \) times the ith row of \([A|F]^{(i)}\) from the kth row. After \( n - 3 \) steps we end up with

\[
[A|F]^{(n)} := \left[ \begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & f_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & f_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & f_3^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn}^{(n)} & f_n^{(n)} \end{array} \right]. \quad (2)
\]
Here $a_{kj}^{(i+1)}$ and $f_k^{(i+1)}$ satisfy

$$a_{kj}^{(i+1)} = a_{kj}^{(i)} - m_{ki}a_{ij}^{(i)}, \quad i = 3, 4, \ldots, n,$$
$$k = i + 1, i + 2, \ldots, n, \quad j = i + 1, i + 2, \ldots, n,$$
$$f_k^{(i+1)} = f_k^{(i)} - m_{ki}f_i^{(i)}, \quad i = 3, 4, \ldots, n, \quad k = i + 1, i + 2, \ldots, n.$$

4. Referring to (2), we can get the linear system,

$$
\begin{bmatrix}
  a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\
  0 & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\
  \vdots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & a_{nn}^{(1)}
\end{bmatrix}
\begin{bmatrix}
  X_1 \\
  X_2 \\
  \vdots \\
  X_n
\end{bmatrix}
= 
\begin{bmatrix}
  f_1^{(1)} \\
  f_2^{(1)} \\
  \vdots \\
  f_n^{(1)}
\end{bmatrix},
$$

(3)

where

$$X := 
\begin{bmatrix}
  X_1 \\
  X_2 \\
  \vdots \\
  X_n
\end{bmatrix} \in \mathbb{R}^{n \times n}, \quad X_i \in \mathbb{R}^{1 \times m}, \quad i = 1, 2, \ldots, n.$$

From (3), we have

$$X_n = \frac{f_n^{(n)}}{a_{nn}^{(n)}} := P_n. \quad (4)$$

The current augmented matrix corresponding to (3) is denoted as

$$[A|F]^{(n)}_{(1)} = 
\begin{bmatrix}
  a_{11}^{(1)} & \cdots & a_{1,n-2}^{(1)} & a_{1,n-1}^{(1)} & a_{1n}^{(1)} & f_1^{(1)} \\
  0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
  \vdots & \ddots & a_{n-2,n-2}^{(n-2)} & a_{n-2,n-1}^{(n-2)} & a_{n-2,n}^{(n-2)} & f_{n-2}^{(n-2)} \\
  0 & \ddots & 0 & a_{n-1,n-1}^{(n-1)} & a_{n-1,n}^{(n-1)} & f_{n-1}^{(n-1)} \\
  0 & \cdots & 0 & 0 & 1 & P_n
\end{bmatrix}.$$

5. According to (3) and (4), we get

$$X_{n-1} = \frac{1}{a_{n-1,n}^{(n-1)}} \left[ f_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)} X_n \right] := P_{n-1}. \quad (5)$$
The current augmented matrix corresponding to (3) is denoted as

$$[A|F]^{(n)}_{(2)} = \begin{bmatrix}
a^{(1)}_{11} & \cdots & a^{(1)}_{1,n-2} & a^{(1)}_{1,n-1} & a^{(1)}_{1,n} & f^{(1)}_1 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & a^{(n-2)}_{n-2,n-2} & a^{(n-2)}_{n-2,n-1} & a^{(n-2)}_{n-2,n} & f^{(n-2)}_{n-2} \\
0 & \cdots & 0 & 1 & 0 & P_{n-2} \\
0 & \cdots & 0 & 0 & 1 & P_n
\end{bmatrix}.$$  

6. According to (3), (4) and (5), we have

$$X_i = \frac{1}{a^{(i)}_{ii}} \left[ f^{(i)} - \sum_{j=i+1}^{n} a^{(i)}_{ij} X_j \right] := P_i, \quad i = n-2, n-3, \ldots, 1. \quad (6)$$

It follows from (4), (5) and (6) that

$$\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix} \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix} = \begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_n
\end{bmatrix},$$

and its augmented matrix is denoted as

$$[A|F]^{(n)}_{(n)} = \begin{bmatrix}
1 & 0 & \cdots & 0 & P_1 \\
0 & 1 & \cdots & 0 & P_2 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & P_n
\end{bmatrix}.$$  

From the above discussion, we get a solution to the equation $AX = F$ by Algorithm 1. In the following section we will tackle matrix equation $(A \otimes B)X = F$ by using the result in Section 2.

3 The matrix equation $(A \otimes B)X = F$

Consider the matrix equation

$$(A \otimes B)X = F,$$  

where $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$ and $F \in \mathbb{R}^{(nm) \times l}$ are given constant matrices, $X \in \mathbb{R}^{(nm) \times l}$ is the unknown matrix to be solved.
Let \( I_n \) denote an \( n \times n \) identity matrix. For an \( m \times l \) matrix

\[
Y = [y_1, y_2, \ldots, y_l] \in \mathbb{R}^{m \times l}, \quad y_i \in \mathbb{R}^m,
\]

Let \( \text{col}[Y] \) represent an \( ml \)-dimensional vector formed by the columns of \( Y \), i.e.,

\[
\text{col}[Y] := \begin{bmatrix} y_1 \\
y_2 \\
\vdots \\
y_l \end{bmatrix} \in \mathbb{R}^{ml}.
\]

Using the relationship \( A \otimes B = (A \otimes I_m)(I_n \otimes B) \) in [35] and from Eq. (7), we have

\[
(A \otimes I_m)(I_n \otimes B)X = F.
\]

It follows that

\[
\begin{bmatrix}
a_{11}I_m & a_{12}I_m & \cdots & a_{1n}I_m \\
a_{21}I_m & a_{22}I_m & \cdots & a_{2n}I_m \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}I_m & a_{n2}I_m & \cdots & a_{nn}I_m
\end{bmatrix}
\begin{bmatrix}
B & 0 & \cdots & 0 \\
0 & B & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_n
\end{bmatrix},
\]

where

\[
X := \begin{bmatrix}
X_1 \\
X_2 \\
\vdots \\
X_n
\end{bmatrix}, \quad F := \begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_n
\end{bmatrix}, \quad X_i \in \mathbb{R}^{m \times l}, \quad F_i \in \mathbb{R}^{m \times l}.
\]

Eq. (8) can be written as

\[
\begin{bmatrix}
a_{11}I_m & a_{12}I_m & \cdots & a_{1n}I_m \\
a_{21}I_m & a_{22}I_m & \cdots & a_{2n}I_m \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}I_m & a_{n2}I_m & \cdots & a_{nn}I_m
\end{bmatrix}
\begin{bmatrix}
BX_1 \\
BX_2 \\
\vdots \\
BX_n
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
\vdots \\
F_n
\end{bmatrix},
\]

or in a compact form

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
\{\text{col}[(BX_1)^T]\}^T \\
\{\text{col}[(BX_2)^T]\}^T \\
\vdots \\
\{\text{col}[(BX_n)^T]\}^T
\end{bmatrix}
= \begin{bmatrix}
\{\text{col}[F_1^T]\}^T \\
\{\text{col}[F_2^T]\}^T \\
\vdots \\
\{\text{col}[F_n^T]\}^T
\end{bmatrix}. \quad (9)
\]
Let
\[
G := \begin{bmatrix}
\{\text{col}[F_1]\}^T \\
\{\text{col}[F_2]\}^T \\
\vdots \\
\{\text{col}[F_n]\}^T
\end{bmatrix} \in \mathbb{R}^{n \times (ml)},
\tag{10}
\]
and \([A|G]\) be the augmented matrix of Eq. (9). According to Algorithm 1, simplifying \([A|G]\) gives
\[
[A|G]_{(n)}^{(n)} = \begin{bmatrix}
1 & 0 & \cdots & 0 & P_1 \\
0 & 1 & \ddots & \vdots & P_2 \\
\vdots & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & 1 & P_n
\end{bmatrix}.
\tag{11}
\]
Thus, we obtain an important intermediate result
\[
\{\text{col}[(BX_i)^T]\}^T = P_i \in \mathbb{R}^{1 \times (ml)}, \ i = 1, 2, \ldots, n.
\]

Let
\[
\begin{align*}
P_i &= [P_{i1}, P_{i2}, \ldots, P_{im}], \ P_{ij} \in \mathbb{R}^{1 \times l}, \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, m, \\
H_i &= \begin{bmatrix} P_{i1} \\ P_{i2} \\ \vdots \\ P_{im} \end{bmatrix} \in \mathbb{R}^{m \times l}, \ i = 1, 2, \ldots, n.
\end{align*}
\tag{12}
\]
According to the definition of \(\text{col}[X]\), we have \(BX_i = H_i, \ i = 1, 2, \ldots, n\). This means that
\[
B[X_1, X_2, \ldots, X_n] = [H_1, H_2, \ldots, H_n].
\tag{13}
\]
Then the solution of Eq. (7) can be obtained by Algorithm 1. The above procedures can be summarized as Algorithm 2.

**Algorithm 2.**

1. Form \(G\) by (10).
2. According to Algorithm 1, simplify the augmented matrix \([A|G]\) by (11).
3. Form \(H_i\) by (12).
4. Obtain the solution of Eq. (7) by solving (13).
4 Numerical examples

Example 1. Suppose that \((A \otimes B)X = F\), where

\[
A = \begin{bmatrix}
1 & 1 \\
2 & -1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 \\
-1 & 1
\end{bmatrix}, \quad F = \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \begin{bmatrix}
7 & 15 \\
13 & 7 \\
5 & 6 \\
-7 & 2
\end{bmatrix}.
\]

According to Algorithm 2, we construct matrix \(G\). Letting \([A|G]^{(1)} := [A|G]\) gives

\[
[A|G]^{(1)} = \begin{bmatrix}
1 & 1 & 7 & 15 & 13 & 7 \\
2 & -1 & 5 & 6 & -7 & 2
\end{bmatrix}.
\]

Consider the entries of the first column, due to \(2 > 1\), interchange these two rows, we have

\[
\begin{bmatrix}
2 & -1 & 5 & 6 & -7 & 2 \\
1 & 1 & 7 & 15 & 13 & 7
\end{bmatrix}.
\]

Adding \(-1/2\) times the first row to the second row gives

\[
[A|G]^{(2)} = \begin{bmatrix}
2 & -1 & 5 & 6 & -7 & 2 \\
0 & 1.5 & 4.5 & 12 & 16.5 & 6
\end{bmatrix}.
\]

Dividing the second row of \([A|G]^{(2)}\) by \(a^{(2)}_{22} = 1.5\) gives

\[
[A|G]^{(2)}_{(1)} = \begin{bmatrix}
2 & -1 & 5 & 6 & -7 & 2 \\
0 & 1 & 3 & 8 & 11 & 4
\end{bmatrix}.
\]

Adding the second row to the first row of the matrix \([A|G]^{(2)}_{(1)}\), we have

\[
\begin{bmatrix}
2 & 0 & 8 & 14 & 4 & 6 \\
0 & 1 & 3 & 8 & 11 & 4
\end{bmatrix}.
\]

Dividing the first row by \(a^{(1)}_{11} = 2\) gives

\[
[A|G]^{(2)}_{(2)} = \begin{bmatrix}
1 & 0 & 4 & 7 & 2 & 3 \\
0 & 1 & 3 & 8 & 11 & 4
\end{bmatrix}.
\]

Then we have

\[
P = \begin{bmatrix}
P_1 \\
P_2
\end{bmatrix} = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} = \begin{bmatrix}
4 & 7 & 2 & 3 \\
3 & 8 & 11 & 4
\end{bmatrix},
\]

\[
H_1 = \begin{bmatrix}
P_{11} \\
P_{12}
\end{bmatrix} = \begin{bmatrix}
4 & 7 \\
2 & 3
\end{bmatrix}, \quad H_2 = \begin{bmatrix}
P_{21} \\
P_{22}
\end{bmatrix} = \begin{bmatrix}
3 & 8 \\
11 & 4
\end{bmatrix},
\]
\[
[B|H_1, H_2] = \begin{bmatrix}
1 & 1 & 4 & 7 & 3 & 8 \\
-1 & 1 & 2 & 3 & 11 & 4
\end{bmatrix}.
\]

According to Algorithm 1, we have
\[
[B|H_1, H_2]^{(2)} = \begin{bmatrix}
1 & 0 & 1 & 2 & -4 & 2 \\
0 & 1 & 3 & 5 & 7 & 6
\end{bmatrix}.
\]

Finally, we obtain the solution for the equation \((A \otimes B)X = F\) with
\[
X = \begin{bmatrix}
1 & 2 \\
3 & 5 \\
-4 & 2 \\
7 & 6
\end{bmatrix}.
\]

**Example 2.** Consider matrix equation \((A \otimes B)X = F\), where
\[
A = \begin{bmatrix}
2 & -3 \\
-1 & -2
\end{bmatrix}, \quad B = \begin{bmatrix}
3 & -2 & 1 \\
4 & 0 & 2 \\
-1 & -3 & -4
\end{bmatrix},
\]
\[
F = \begin{bmatrix}
F_1 \\
F_2
\end{bmatrix} = \begin{bmatrix}
-60 & -77 \\
-58 & -84 \\
31 & 44 \\
-19 & -28 \\
-6 & -42 \\
-12 & 13
\end{bmatrix}.
\]

According to Algorithm 2, \(G\) can be obtained by
\[
G = \begin{bmatrix}
\{\text{col}[F_1^T]\}^T \\
\{\text{col}[F_2^T]\}^T
\end{bmatrix} = \begin{bmatrix}
-60 & -77 & -58 & -84 & 31 & 44 \\
-19 & -28 & -6 & -42 & -12 & 13
\end{bmatrix},
\]
and the augmented matrix \([A|G]\) can be written as
\[
[A|G]^{(1)} = \begin{bmatrix}
2 & -3 & -60 & -77 & -58 & -84 & 31 & 44 \\
-1 & -2 & -19 & -28 & -6 & -42 & -12 & 13
\end{bmatrix}.
\]

Simplifying the augmented matrix \([A|G]^{(1)}\) gives
\[
[A|G]^{(2)} = \begin{bmatrix}
1 & 0 & -9 & -10 & -14 & -6 & 14 & 7 \\
0 & 1 & 14 & 19 & 10 & 24 & -1 & -10
\end{bmatrix},
\]
\[
P = \begin{bmatrix}
P_{11} & P_{12} & P_{13} \\
P_{21} & P_{22} & P_{23}
\end{bmatrix} = \begin{bmatrix}
-9 & -10 & -14 & -6 & 14 & 7 \\
14 & 19 & 10 & 24 & -1 & -10
\end{bmatrix}.
\]
Numerical algorithm for solving a class of matrix equations

Constructing the matrix

\[
[H_1, H_2] = \begin{bmatrix}
P_{11} & P_{21} \\
P_{12} & P_{22} \\
P_{13} & P_{23}
\end{bmatrix} = \begin{bmatrix}
-9 & -10 & 14 & 19 \\
-14 & -6 & 10 & 24 \\
14 & 7 & -1 & -10
\end{bmatrix},
\]

we write the augmented \([B|H_1, H_2]\),

\[
[B|H_1, H_2] = \begin{bmatrix}
3 & -2 & 1 & -9 & -10 & 14 & 19 \\
4 & 0 & 2 & -14 & -6 & 10 & 24 \\
-1 & -3 & -4 & 14 & 7 & -1 & -10
\end{bmatrix},
\]

which can be transformed into

\[
[B|H_1, H_2]^{(3)} = \begin{bmatrix}
1 & 0 & 0 & -2 & 1 & 1 & 5 \\
0 & 1 & 0 & 0 & 4 & -4 & -1 \\
0 & 0 & 1 & -3 & -5 & 3 & 2
\end{bmatrix}.
\]

Finally, we obtain the solution for equation \((A \otimes B)X = F\),

\[
X = \begin{bmatrix}
-2 & 1 \\
0 & 4 \\
-3 & -5 \\
1 & 5 \\
-4 & -1 \\
3 & 2
\end{bmatrix}.
\]

5 Conclusions

A new and efficient algorithm for solving linear matrix equation \((A \otimes B)X = F\) has been presented by using the Gaussian elimination. Two examples have illustrated the effectiveness of the proposed algorithm.

References


[34] H.M. Zhang and F. Ding, A property of the eigenvalues of the symmetric positive definite matrix and the iterative algorithm for coupled Sylvester matrix equations, J. Franklin Inst. 351 (1) (2014) 340-357.