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A new model of (I+S)-type preconditioner for system of linear equations

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Abstract. In this paper, we design a new model of preconditioner for systems of linear equations. The convergence properties of the proposed methods have been analyzed and compared with the classical methods. Numerical experiments of convection-diffusion equations show a good improvement on the convergence, and show that the convergence rates of proposed methods are superior to the other modified iterative methods.

Keywords: Preconditioning, H-matrices, Convection-Diffusion equation, Comparison theorems.
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1 Introduction

Consider the following linear system equations

$$Ax = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is nonsingular and $x, b \in \mathbb{R}^n$. Suppose $\text{diag}(A) = I$ and $A = I - L - U$, where I, is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular matrices of A, respectively. For any

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splitting, $A = M - N$ with $\det(M) \neq 0$, the basic iterative methods for solving Eq. (1) is

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b,\ i = 0, 1, \ldots. \tag{2}$$

This iterative process converges to the unique solution $x = A^{-1}b$ for any initial vector value $x^0 \in \mathbb{R}^n$ if and only if the spectral radius $\rho(M^{-1}N) < 1$, where $T = M^{-1}N$ is called the iteration matrix. There are some several iterative methods for solving Eq. (1) based on Eq. (2), for instance $Jacobi$, $Gauss - Seidel$, SOR, etc (see [4, 18, 24, 25, 26, 27, 28, 31, 33] and the references therein). For Eq. (1) a preconditioner P transforms the system to

$$PAx = Pb, \quad P \in \mathbb{R}^{n\times n}. \tag{3}$$

Furthermore, it can be transformed to

$$AFy = b, \quad x = Fy, \tag{4}$$

where P and F are linear operators, called left and right preconditioners respectively. Therefore, we have

$$x^{(i+1)} = M_P^{-1}N_Px^{(i)} + M_P^{-1}Pb, \quad i = 0, 1, \ldots,$$

where $PA = M_P - N_P$ and M_P is nonsingular. Moreover,

$$y^{(i+1)} = M_F^{-1}N_Fy^{(i)} + M_F^{-1}b, \quad i = 0, 1, \ldots,$$

where $AF = M_F - N_F$ and M_F is nonsingular. The purpose of preconditioning is to change the matrix of the system, in order to accelerate the convergence of iterative solvers. To improve the convergence rate of a basic iterative method, various models of preconditioning systems have been proposed (see e.g., [1, 2, 3, 7, 16, 29]). In the literature, various authors have suggested different models of $(I+S)$-type preconditioner for the above mentioned problem. In [15], Milaszewicz presented the preconditioner $(I + S')$, where the elements of the first column below the diagonal of A are eliminated. Gunawardena, Jain and Snyder considered [10] a modification of Jacobi and Gauss-Seidel methods and reported that the convergence rate of the Gauss-Seidel method using the preconditioner

$$P = I + S, \tag{5}$$

is superior to that of the standard Gauss-Seidel method, where $S = (s_{ij})_{n\times n}$ with

$$s_{ij} = \begin{cases} -a_{ij}, & \text{for } \ j = i+1, i = 1, 2, \ldots, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Inspiring from the same idea, Kohno et al. [13] proposed an extended modification of Jacobi and Gauss-Seidel methods. Their preconditioner is $(I + S_\alpha)$ where

$$(S_\alpha)_{ij} = \begin{cases} -\alpha_i a_{ij}, & \text{for } \ j = i+1, 0 \leq \alpha_i \leq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

In [30] Usui et al. proposed to adopt

$$P = I + U \ (\text{or } I + L), \tag{7}$$

as the preconditioner, where $U(L)$ is strictly upper (strictly lower) triangular part of the matrix A. They have obtained excellent convergence rate compared with that of by other iterative methods.

Kotakemori et al. [14] used

$$\tilde{P} = I + S_{\max}, \tag{8}$$

where $S_{\max}$ is

$$(S_{\max})_{ij} = \begin{cases} -a_{i,V_i}, & \text{for } \ i = 1, 2, \ldots, n-1, j > i, \\ 0, & \text{Otherwise,} \end{cases} \tag{9}$$

and,

$$V_i = \min \ j \in \{j | \max_j |a_{ij}|\} \text{ for } i < n.$$

Also, Harano and Niki [12] considered the preconditioner

$$P = I + (1+\gamma)(L+U), \tag{10}$$

where $U(L)$ is strictly upper (strictly lower) triangular of matrix A and γ is a small positive number. Furthermore, some more preconditioners presented in the literature can be found in [5, 6, 17, 20, 21, 22, 23, 32, 34]. In this article we propose a new preconditioner of $(I + S)$-type.

2 Prerequisite

We begin with some basic notation and preliminary results which we refer to later. For more details (see e.g. [4] and [31]).

Definition 1. A real $n \times n$ matrix $A = (a_{ij})$ is called:
(i) Z-matrix if for any $i \neq j$, $a_{i,j} \leq 0$.
(ii) M-matrix, if A is nonsingular, and $A^{-1} \geq 0$.
(iii) H-matrix if and only if $< A >= (m_{i,j}) \in \mathbb{R}^{n \times n}$ is an M-matrix, where

$$m_{i,i} = |a_{i,i}|; \ m_{i,j} = -|a_{i,j}|, \ i \neq j.$$

Definition 2. Let A be a real matrix. The splitting $A = M - N$ is called:
(i) convergent if $\rho(M^{-1}N) < 1$.
(ii) regular if $M^{-1} \geq 0$ and $N \geq 0$.
(iii) weak regular if $M^{-1}N \geq 0$ and $N \geq 0$.
(iv) M-splitting if M is a nonsingular M-matrix and $N \geq 0$.
Clearly, a regular splitting is weak regular.

Lemma 1. *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is M-matrix.*

Lemma 2. *Let A be a Z-matrix. Then A is M-matrix if and only if there is a positive vector x such that $Ax > 0$.*

Lemma 3. *Assume that A and B are Z-matrices and A is an M-matrix. If $A \leq B$ then B is also an M-matrix.*

Lemma 4. *Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A, where $A^{-1} \geq 0$. If $M_1^{-1} \geq M_2^{-1}$, then $\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2)$.*

3 The new preconditioner and its theoretical analysis

Consider a model of $(I + S)$-type preconditioner, say the preconditioner presented by Gunawardena et al. [10] . Then, the new preconditioner is proposed follows

$$(I + K) = (I + S)\{(I - S) + (L + U)(I + S)\}. \tag{11}$$

Theorem 1. *Let A be a Z-matrix, then $(I + K)$ is nonnegative. Moreover,*

$$(I + K) \geq (I + S).$$

Proof. By Eq. (11) we have

$$\begin{aligned}
(I + K) &= (I + S)\{I - S + L + U + US + LS\} \\
&= (I + S)\{I + L + \underbrace{(U - S)}_{\geq 0} + \underbrace{(U + L)S}_{\geq 0}\} \\
&= \underbrace{(I + S)}_{\geq 0} + \underbrace{(I + S)\{L + (U - S) + (U + L)S\}}_{\geq 0},
\end{aligned}$$

and the proof is completed. □

In what follows, we prove that under certain conditions, the preconditioned matrix is Z-matrix.

Let A be a Z-matrix and consider $\hat{A} = (I+S)A$, then the preconditioned matrix is as follows

$$\hat{A} = (I+S)A = (I-L-U) + S - SL - SU.$$

Then

$$\hat{A} = \hat{D} - \hat{L} - \hat{U},$$

where

$$\hat{D} = (I - D_1),\ \hat{L} = (L - L_1),\ \hat{U} = ((U-S) + U_1 + SU), \tag{12}$$

in which D_1, L_1 and U_1 are respectively, the diagonal, strictly lower and strictly upper triangular parts of

$$SL = D_1 + L_1 + U_1 \geq 0. \tag{13}$$

Now, for $\bar{A} = (I+K)A$, we have

$$\begin{aligned}
\bar{A} &= (I+S)A + \underbrace{(I+S)(-S)A}_{-S(I+S)A} + U(I+S)A + SU(I+S)A \\
&\quad +(I+S)L(I+S)A \\
&= (I+S)A + (I+S)L(I+S)A + ((U-S)+SU)(I+S)A \\
&= (I + (I+S)L + \tilde{U})(I+S)A,
\end{aligned}$$

where $\tilde{U} = ((U-S)+SU)$. Thus, by Eq.(12) and Eq.(13) we obtain

$$\begin{aligned}
(I+K)A &= (I + L + D_1 + L_1 + U_1 + \tilde{U})(I+S)A \\
&= ((I-D_1) + \hat{L} + (U_1 + \tilde{U}))(\hat{D} - \hat{L} - \hat{U}).
\end{aligned}$$

Therefore

$$\bar{A} = (I+K)A = \bar{D} - \bar{L} - \bar{U},$$

where

$$\begin{aligned}
\bar{D} &= (I+D_1)\hat{D} - D_2 - D_3, \\
\bar{L} &= (I+D_1)\hat{L} - \hat{L}\hat{D} + (\hat{L})^2 + L_2 + L_3, \\
\bar{U} &= (I+D_1)\hat{U} + U_2 + (U_1+\tilde{U})\hat{D} + (U_1+\tilde{U})\hat{U},
\end{aligned}$$

and

$$\begin{aligned} \hat{L}\hat{U} &= D_2 + L_2 + U_2 \geq 0, \\ (U_1 + \tilde{U})\hat{L} &= D_3 + L_3 + U_3 \geq 0. \end{aligned}$$

Moreover, since

$$\begin{aligned} \bar{L} &= (I + D_1)\hat{L} - \hat{L}\hat{D} + (\hat{L})^2 + L_2 + L_3 = (2D_1)\hat{L} + (\hat{L})^2 + L_2 + L_3, \\ \bar{U} &= (I + D_1)\hat{U} + U_2 + (U_1 + \tilde{U})\hat{D} + (U_1 + \tilde{U})\hat{U} = (2D_1)\hat{U} + U_2 + (\hat{U})^2, \end{aligned}$$

we get

$$\bar{L}, \bar{U} \geq 0.$$

Therefore we proved the following result.

Theorem 2. *Let A be Z-matrix, then $\bar{A} = (I + K)A$ is also Z-matrix.*

Next, we consider the AOR preconditioned method and show that under some conditions the AOR method with $\bar{A} = (I + K)A$ is better than the unpreconditioned AOR method.

The AOR iterative method to solve Eq. (1) is written as (see [11])

$$x^{(i+1)} = L_{r,w}x^{(i)} + (I - rL)^{-1}wb, \; i = 0, 1, \ldots, \tag{14}$$

with the iteration matrix

$$L_{r,w} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU], \tag{15}$$

where (w, r) are real parameters with $w \neq 0$. Then the iterative matrix with preconditioner of Eq.(11) defined as

$$\bar{L}_{r,w} = (\bar{D} - r\bar{L})^{-1}[(1 - w)\bar{D} + (w - r)\bar{L} + w\bar{U}]. \tag{16}$$

Theorem 3. *Let $L_{r,w}$ and $\bar{L}_{r,w}$ be the iterations matrices given b Eq. (15) and Eq. (16), respectively. If A is an M-matrix, then we have*

$$\rho(\bar{L}_{r,w}) \leq \rho(L_{r,w}) < 1.$$

Proof. Since $A = M_{r,w} - N_{r,w}$ is M-splitting, where

$$M_{r,w} = \frac{1}{w}(I - rL), \; N_{r,w} = \frac{1}{w}[(1 - w)I + (w - r)L + wU].$$

then by Lemma 1 $\rho(L_{r,w}) < 1$. On the other hand, since A is M-matrix it is easy to see that $\bar{A}$ is also M-matrix (by Lemma 2). Now, it follows that the entries of its diagonal are positive. Thus

$$N_{r,w} = \frac{1}{w}[(1 - w)\bar{D} + (w - r)\bar{L} + w\bar{U}] \geq 0,$$

$$\bar{A} \leq \bar{M} = \frac{1}{w}(\bar{D} - r\bar{L}).$$

Then by Lemma 3 $\bar{M}$ is also an M-matrix and therefore $\bar{A}$ is a regular splitting. Moreover, it can be shown that $\bar{D} \leq I$ and$\bar{L} \geq L$. Thus, $\bar{M}^{-1} \geq M^{-1}$ and finally by Lemma 4 the proof is completed. □

Theorem 4. *Let $L_{r,w}$ and $\bar{L}_{r,w}$ be the iteration matrices given by Eq. (15) and Eq. (16), respectively. If A is an H-matrix, then for $0 \leq r \leq w \leq 1$, $w \neq 0$ and $r \neq 1$, we have*

$$\rho(\bar{L}_{r,w}) \leq \rho(<\bar{L}_{r,w}>) \leq \rho(<L_{r,w}>) < 1.$$

Proof. Let A be an H-matrix. Then $<A>$ is M-matrix. Therefore by Theorem 3

$$\rho(<\bar{L}_{r,w}>) \leq \rho(<L_{r,w}>) < 1.$$

By definition of preconditioned AOR we have

$$\begin{aligned}
|\bar{L}_{r,w}| &= |(\bar{D} - r\bar{L})^{-1}[(1-w)\bar{D} + (w-r)\bar{L} + w\bar{U}]| \\
&= |(\bar{D}(I - r\bar{D}^{-1}\bar{L})^{-1}[\bar{D}\{(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}\}]| \\
&= |((I - r\bar{D}^{-1}\bar{L})^{-1}[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}]| \\
&= |((I + r\bar{D}^{-1}\bar{L} + (r\bar{D}^{-1}\bar{L})^2 + \cdots) \\
&\quad +[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}]| \\
&= |((r\bar{D}^{-1}\bar{L} + (r\bar{D}^{-1}\bar{L})^2 + \cdots)[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}] \\
&\quad +[(1-w)I + (w-r)\bar{D}^{-1}\bar{L} + w\bar{D}^{-1}\bar{U}]| \\
&\leq (\{r|\bar{D}^{-1}\bar{L}| + (r|\bar{D}^{-1}\bar{L}|)^2 + \cdots) \\
&\quad \times[(1-w)I + (w-r)|\bar{D}^{-1}\bar{L}| + w|\bar{D}^{-1}\bar{U}|]) \\
&\quad +[(1-w)I + (w-r)|\bar{D}^{-1}\bar{L}| + w|\bar{D}^{-1}\bar{U}|] \\
&= (I - r|\bar{D}^{-1}\bar{L}|)^{-1}[(1-w)I + (w-r)|\bar{D}^{-1}\bar{L}| + w|\bar{D}^{-1}\bar{U}|]) \\
&= <\bar{L}_{r,w}> .
\end{aligned}$$

Then

$$\rho(|\bar{L}_{r,w}|) \leq \rho(<\bar{L}_{r,w}>).$$

Therefore we have

$$\rho(\bar{L}_{r,w}) \leq \rho(|\bar{L}_{r,w}|) \leq \rho(<\bar{L}_{r,w}>),$$

which completes the proof. □

Theorem 5. *Let $\bar{L}_{r_1,w_1}$ and $\bar{L}_{r_2,w_2}$ $(0 \le r_i \le w_i \le 1; i = 1, 2.)$ be the iteration matrices of preconditioned AOR methods, with different parameters. If A be an M-matrix, $w_1 \le w_2$ and $\frac{r_1}{w_1} \le \frac{r_2}{w_2}$, then we have*

$$\rho(\bar{L}_{r_2,w_2}) \le \rho(\bar{L}_{r_1,w_1}) < 1.$$

Proof. By Theorem 4, we have

$$\rho(\bar{L}_{r_1,w_1}) < 1.$$

Moreover, since A is M-matrix, $\bar{A}$ is also an M-matrix. Thus similar to Theorem 3, we can see the splittings

$$M_{r_1,w_1} = \frac{1}{w_1}(I - r_1 L), \quad N_{r_1,w_1} = \frac{1}{w_1}[(1 - w_1)I + (w_1 - r_1)L + w_1 U],$$

$$M_{r_2,w_2} = \frac{1}{w_2}(I - r_2 L), \quad N_{r_2,w_2} = \frac{1}{w_2}[(1 - w_2)I + (w_2 - r_2)L + w_2 U].$$

are regular and

$$\bar{L}_{r_1,w_1} = M^{-1}_{r_1,w_1} N_{r_1,w_1}, \qquad \bar{L}_{r_2,w_2} = M^{-1}_{r_2,w_2} N_{r_2,w_2},$$

On the other hand,

$$\begin{aligned} M_{r_2,w_2} - M_{r_1,w_1} &= \frac{1}{w_2}(I - r_2 L) - \frac{1}{w_1}(I - r_1 L) \\ &= (\frac{w_1 - w_2}{w_1 w_2})I - (\frac{r_2 w_1 - r_1 w_2}{w_1 w_2})L. \end{aligned}$$

Now, since $w_1 \le w_2$, then

$$\frac{w_1 - w_2}{w_1 w_2} \le 0.$$

Furthermore, since $\frac{r_1}{w_1} \le \frac{r_2}{w_2}$ we have

$$\frac{r_2 w_1 - r_1 w_2}{w_1 w_2} \ge 0,$$

which gives

$$M_{r_2,w_2} \le M_{r_1,w_1}.$$

Since M_{r_1,w_1} and M_{r_2,w_2} are M-matrices, then

$$M^{-1}_{r_1,w_1} \le M^{-1}_{r_2,w_2}.$$

Therefore by Lemma 4 the proof is complete. □

Remark 1. *In Eq. (15) by choosing special parameters, the similar results can be obtained. For example:*
(i) $Jacobi\ method\ for\ w = 1\ and\ r = 0.$
(ii) $JOR\ (Jacobioverrelaxation)\ method\ for\ r = 0.$
(iii) $Gauss - Seidel\ method\ for\ r = w = 1.$
(iv) $SOR\ method\ for\ r = w.$

4 Numerical experiments

In this section, we give some examples to illustrate the results obtained in previous sections. First, a simple numerical experiment is carried out to investigate the validity of the proposed method. The convergence behaviors of iterative methods are illustrated by comparing the spectral radii of the corresponding iteration matrices for a small sized dense system.

Example 1. The coefficient matrix A of Eq. (1) is given by

$$A = \begin{pmatrix} 1 & -0.2 & -0.023 & -0.18 & -0.27 & -0.31 & -0.1 \\ -0.1 & 1 & -0.31 & -0.18 & -0.07 & -0.1 & -0.2 \\ -0.01 & -0.1 & 1 & -0.1 & -0.2 & -0.17 & -0.0098 \\ -0.021 & -0.2 & -0.03 & 1 & -0.3 & -0.01 & -0.1 \\ -0.01 & -0.014 & -0.09 & -0.3 & 1 & -0.1 & -0.1 \\ -0.02 & -0.023 & -0.1 & -0.27 & -0.3 & 1 & -0.1 \\ -0.18 & -0.0081 & -0.1 & -0.19 & -0.1 & -0.2 & 1 \end{pmatrix}.$$

In the Table 1, we reported the spectral radius of the corresponding iteration matrix with different parameters w and r. We denoted the spectral radius of the AOR method by ρ. Furthermore, $\hat{\rho}$ and $\tilde{\rho}$ are spectral radius of the iteration matrix with preconditioners Eq. (8) and Eq. (11), respectively. From Table 1, we can see that the preconditioned iterative methods are superior to the basic iterative methods. The table has also shown that the preconditioned iterative methods associated with $(I+K)$ is the best. Furthermore, based on Theorem 5, we can see that in Table 1, the best convergence rate belongs to the preconditioned Gauss-Seidel method ($r = w = 1$).

Example 2. (Application to the model of convection-diffusion equation)
Consider the three-dimensional convection-diffusion equation

$$-(u_{xx} + u_{yy} + u_{zz}) + 2u_x + u_y + u_z = f(x, y, z),$$

Table 1: Numerical results for Example 1.

w	r	ρ	$\hat{\rho}$	$\tilde{\rho}$
1.0	0.0	0.7328	0.6709	0.4257
0.9	0.4	0.7188	0.6519	0.4343
0.9	0.5	0.7058	0.6351	0.4193
0.9	0.6	0.6912	0.6160	0.4026
0.9	0.7	0.6746	0.5939	0.3837
0.9	0.8	0.6553	0.5679	0.3615
1.0	1.0	0.5604	0.4380	0.2191

on the unit cube domain $\Omega = [0,1] \times [0,1]$, with Dirichlet boundary conditions. When the seven-point finite difference discretization, for example, the centered differences to the diffusive terms, and the centered differences or the first-order upwind approximations to the convective terms, are applied to the above model, we get the system of linear equations Eq. (1) with the coefficient matrix

$$A = T_x \otimes I \otimes I + I \otimes T_y \otimes I + I \otimes I \otimes T_z,$$

where, the equidistant step-size $h = 1/(n+1)$ is used in the discretization on all of the three directions and the natural lexicographic ordering is employed to the unknowns. In addition,$\otimes$ denotes the Kronecker product, and T_x, T_y, and T_z are tridiagonal matrices given by

$$\begin{aligned} T_x &= \text{tridiag}(-\frac{2+2h}{12}, 1, -\frac{2-2h}{12}), \\ T_y = T_z &= \text{tridiag}(-\frac{2+h}{12}, 0, -\frac{2-h}{12}). \end{aligned}$$

For more details, we refer to [9, 8]. Then, we have solved the obtained $n^3 \times n^3$ linear system of equations by the GMRES (k) method, and preconditioned GMRES(k) method. In this experiment, we choose Usui *et al.*'s model $(I+L)$ and $(I+K)$ as our preconditioner. The initial approximation of $x^{(0)}$ is zero vector and we choose the right- hand side vector, such that $x = (1,1,\ldots,1)^T$ be the solution of $Ax = b$. As a stopping criterion, we choose

$$\frac{\|r_k\|_2}{\|r_0\|_2} \leq eps = 10^{-10},$$

where, $\|r_i\|_2$ is residual norm in restarted GMRES; (see [18, 19]). In Table 2, we have reported the CPU time, number of iterations (Iter) and the residual

Table 2: Numerical results for Example 2.

n	$G(20)$			PG(20)			PK(20)		
	Iter	CPU	res	Iter	CPU	res	Iter	CPU	res
3	15	0.010	5.694e-011	12	0.000	2.798e-011	8	0.000	9.996e-011
4	22	0.200	3.335e-011	18	0.010	2.343e-011	11	0.010	4.031e-011
5	31	0.040	6.088e-011	31	0.040	6.885e-011	14	0.020	1.466e-011
6	37	0.080	3.579e-011	29	0.061	8.497e-011	16	0.050	4.512e-011
7	42	0.210	6.617e-011	32	0.150	7.034e-011	18	0.110	8.415e-011
8	45	0.520	9.394e-011	39	0.431	5.405e-011	21	0.320	2.804e-011
9	53	0.982	9.257e-011	43	0.841	5.243e-011	23	0.460	4.653e-011
10	59	1.802	7.600e-011	46	1.512	5.402e-011	24	0.761	8.274e-011

norm (res) for the corresponding preconditioned GMRES methods. Here G(20) represents the restarted GMRES (20) method. The preconditioned restarted GMRES (20) method with Usui et al.'s preconditioner is denoted by PG(20), while PK(20), correspond to preconditioner (I+K). From the table, we can see that the preconditioned GMRES methods are superior to the basic GMRES method and our preconditioner is better than Usui et al.'s preconditioner.

5 Conclusion

In this paper, we have proposed a new model preconditioner from class of $(I+S)$-type based on the iterative methods. From theoretical point of view and numerical experiments, it may be concluded that the convergence behaviors of our proposed method is superior to the basic iterative methods and better than the other preconditioner of $(I+S)$-type.

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