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A numerical method based on the radial basis functions for solving nonlinear two-dimensional Volterra integral equations of the second kind on non-rectangular domains

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Abstract. In this investigation, a numerical method for solving nonlinear two-dimensional Volterra integral equations is presented. This method uses radial basis functions (RBFs) constructed on scattered points as a basis in the discrete collocation method. Therefore, the method does not need any background mesh or cell structure of the domain. All the integrals that appear in this method are approximated by the composite Gauss-Legendre integration formula. This method transforms the source problem into a system of nonlinear algebraic equations. Error analysis is presented for this method. Finally, numerical examples are included to show the validity and efficiency of this technique.

Keywords: Radial basis functions, nonlinear two-dimensional Volterra integral equations, meshless method, non-rectangular domains
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1 Introduction

Consider the following two-dimensional nonlinear Volterra integral equations of the second kind

$$u(x,t)-\sum_{l=1}^{r}\int_{a_l}^{t}\int_{\alpha_{1,l}(z)}^{x}K(x,t,y,z,u(y,z))\mathrm{d}y\mathrm{d}z=f(x,t),\ (x,t)\in D, \tag{1}$$

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where f and K are known functions, $u(x,t)$ is the unknown function to be determined and $D \subset \mathbb{R}^2$ is two-dimensional non-rectangular shape as

$$D = \cup_{l=1}^{r} \{(y,z) \in \mathbb{R}^2 : \alpha_{1,l}(z) \leq y \leq \alpha_{2,l}(z) \text{ and } a_l \leq z \leq b_l\}, \tag{2}$$

in which $a_l, b_l \in \mathbb{R}$, $\alpha_{1,l}(z)$ and $\alpha_{2,l}(z)$ are continuous functions of z.

Two-dimensional integral equations are used widely in various fields of science and engineering. For example, a class of telegraph equation is equivalent to a two-dimensional Volterra integral equation [16], the problem of prediction water structure around molecules [21], electromagnetic problem [31], asexual population problem in a changing environment [33], etc.

Obtaining an exact solution for two-dimensional Volterra integral equations (2-D VIEs) is difficult. Therefore it is necessary to find an efficient numerical method to solve this kind of problem. Several methods have been proposed for the numerical solutions of 2-D VIEs. The Chebysheve polynomials as the basis in the collocation method [6], differential transformation method [17, 34], reproducing kernel function method [15], hybrid of block-pulse functions and Taylor series method [22, 23], hat function method [23], two-dimensional rationalized Haar(RH) functions [7], degenerate method [1], fast collocation method [10], piecewise polynomial projection methods [3], Nystrom method [4], the Gauss product quadrature rules [8] and extrapolation method [26] have been utilized to obtain an approximate solution of 2-DVIEs. Also, the existence and uniqueness of the solution of 2-DVIEs are proved in [26].

To apply collocation and Galerkin or Nystrom methods for a two-dimensional integral equation on a non-rectangular region, the solution region must be divided into non-overlapping triangular fragments. Additionally, the integral equation must be discretized based on approximation and numerical integration over the segments [5, 18]. Therefore, to avoid triangulation and mesh refinement methods based on scattered data approximation (meshless methods) can be used to approximate or interpolate a function without mesh generation on the domain. Dehghan *et al.* in [2] have applied radial basis functions (RBFs) method to solve the second kind of two-dimensional nonlinear Fredholm integral equations on regions of non-rectangular . In recent years, meshless methods have been utilized in various areas such as artificial intelligence, computer graphics, image processing, and sampling theory. Among these methods, the RBFs method has become recognized as a powerful tool for scattered data interpolation. The primary advantage of RBFs is that they involve a single independent variable regardless of the problem's dimensions [9, 35]. The only geometric properties used in RBF approximation are pairwise distances between points, which are easy to compute in any number of space dimensions, making working in higher dimensions no more difficult [13]. For more information on RBFs and their implementations see [13, 35] and the references therein. In recent years, the focus of RBF applications has shifted from scattered data interpolation to the numerical solution of partial different equations (PDEs). For more information on using RBFs in PDEs, readers can refer to [11, 19, 20, 28, 29].

In this paper, we present a numerical method based on the RBFs method for the solution of a nonlinear 2-DVIEs of the second kind, such that the region of the solution is non-rectangular. In this method, we use the interpolating extension of RBFs which can be applied in scattered nodal points on the domain D to approximation of the unknown function u in the discrete collocation method. To approximate the integrals arised in the discrete collocation method, we use the double Gauss-Legendre (DGL) quadrature formula. So, the solution of the nonlinear Volterra integral equation reduces to the solution of a system of nonlinear algebraic equations. We also discuss the error bound and the rate of convergence for the proposed method.

The outline of this paper is as follows: In Section 2, we review some basic formulations and properties of the RBFs. In Section 3, we present a computational method for solving Volterra integral equations (1) in two-dimensional cases utilizing the RBF approximation. In Section 4, the error analysis for the new method is provided. Numerical examples are given in Section 5. Finally, we conclude the article in Section 6.

2 Radial basis functions

To introduce our method, we present an outline of RBFs in this section. We first quote the following definitions from [9, 35].

Definition 1. *A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial provided there exists a univariate function $\phi : [0,\infty) \to \mathbb{R}$ such that*

$$\Phi(\mathbf{x}) = \phi(\| \mathbf{x} \|), \qquad \forall \mathbf{x} \in \mathbb{R}^d,$$

where $\| \cdot \|$ is some norm in $\mathbb{R}^d$, usually Euclidean norm.

Definition 2. *A real-value continuous even function Φ is called conditionally positive definite of order m on $\mathbb{R}^d$ if for all sets $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \subseteq \mathbb{R}^d$ of distinct points, and all vectors $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N]^T \in \mathbb{R}^N$ satisfying $\sum_{i=1}^N \lambda_i P(\mathbf{x}_i) = 0$ for all polynomials of degree less than m the quadratic for*

$$\sum_{i=1}^N \sum_{i=1}^N \lambda_i \lambda_j \Phi(\mathbf{x}_i - \mathbf{x}_j),$$

is positive.

Let $\beta = \{P_1, P_2, \ldots, P_m\}$ form a basis for the $M = \binom{d+m-1}{m-1}$-dimensional linear space $\Pi_{m-1}^d(D)$ of total of degree less than or equal to $m-1$ in d variables on the set D.

To approximate of function $u(\mathbf{x})$, using the RBF $\Phi(\mathbf{x}) = \phi(\| \mathbf{x} \|)$ from a given set of distinict nodal points $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ in $D \in \mathbb{R}^d$, the approximation is of the form

$$u(\mathbf{x}) \approx p_{N,M} u(\mathbf{x}) = \sum_{k=1}^N \gamma_k \phi(\| \mathbf{x} - \mathbf{x}_k \|) + \sum_{l=1}^M d_l P_l(\mathbf{x}), \qquad \mathbf{x} \in D. \tag{3}$$

Some well-known RBFs are listed in the Table1. These RBFs have a parameter called shape parameter. This parameter in the RBF TPS is shown by $K \in \mathbb{N}$ and the other ones are shown by $c > 0$. There is a lot of experimental evidence showing that the accuracy of RBFs depends heavily on the choice of the shape parameter. The problem of how to select an appropriate value for the shape parameter has been a hot topic both from the theoretical and application points of view [14, 27, 30, 32].

Remark 1. *Note that $P_{M,N} : C(D) \to C_N^M(D)$ is the collocation projection operator on the collocation points $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_N\}$, where the subspace $C_N^M(D) = span\{\Phi_1, \ldots, \Phi_N, P_1, \ldots, P_M\} \subseteq C(D)$ has finite dimensions. We call the basis functions $\Phi_k(\mathbf{x}) = \phi(\|\mathbf{x} - \mathbf{x}_k\|), k = 1, 2, 3, \ldots, N$ as shape functions of the RBF interpolation corresponding to nodal points $\{\mathbf{x}_k\}_{k=1}^N$.*

Table 1: Some well-known radial basis functions.

Name of RBF	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2), c > 0$
Multiquadrics (MQ)	$\phi(r) = (r^2 + c^2)^{\frac{1}{2}}, c > 0$
Inverse multiquadrics (IMQ)	$\phi(r) = (r^2 + c^2)^{\frac{-1}{2}}, c > 0$
Thin plate splines (TPS)	$(-1)^{k+1} r^{2k} \log(r), k \in \mathbb{N}$

Since enforcing the interpolation conditions $P_{N,M}u(\mathbf{x}_i) = u(\mathbf{x}_i)$, $i = 1,2,3,\ldots,N$, leads to a system of N linear equations in $N+M$ unknowns γ_k and d_l, one usually adds the M additional conditions

$$\sum_{k=1}^{N} \gamma_k P_l(\mathbf{x}_k) = 0, \quad l = 1,\ldots,M, \tag{4}$$

to ensure a unique solution [13]. In general, solving the interpolation problem based on the extended expansion (3) amounts to solving a system of linear equations of the form

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}, \tag{5}$$

with $A_{j,k} = \Phi_k(\mathbf{x}_j)$, $j,k = 1,2,\ldots,N$, $P_{j,k} = P_k(\mathbf{x}_j)$, $j = 1,2,\ldots,N$, $k = 1,2,\ldots,M$, $\mathbf{c} = [c_1,c_2,c_3,\ldots,c_N]^T$, $\mathbf{d} = [d_1,d_2,d_3,\ldots,c_N]^T$, $\mathbf{u} = [u(\mathbf{x}_1),u(\mathbf{x}_2),u(\mathbf{x}_3),\ldots,u(\mathbf{x}_N)]$, and 0 is a zero vector of lenght M.

Lemma 1. *Let $\{x_1,x_2,\ldots,x_N\}$ be a distinict nodal points, as well as $\{P_1,P_2,\ldots,P_N\}$ be a basis for $\Pi^d_{m-1}(D)$. If $N = M$ in RBF approximation method by (3), then $\mathbf{x}_r = 0$, for $r = 1,2,\ldots,N$. In this case, the RBF approximation is reduced to polynomial interpolation.*

Proof. The matrix form of Eq. (5) for the case $N = M$ is

$$\begin{pmatrix} 0 & (P^T)^{-1} \\ P^{-1} & -P^{-1}A(P^T)^{-1} \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{6}$$

thus, $c = 0$ and $d = P^{-1}u$. □

3 Description of the method

In this section, we solve Eq.(1) using RBF methods. To estimate the unknown function $u(x,t)$ by the RBF interpolation methods, we need an RBF Φ and N nodes distributed throughout the domain D, such as $X = \{(x_1,t_1),(x_2,t_2),\ldots,(x_N,t_N)\}$. Then, we estimate the unknown function $u(x,t)$ by the RBF interpolation method as

$$u_{\mathcal{N}}(x,t) = \sum_{r=1}^{N} \gamma_r \Phi_i(x,t) + \sum_{j=1}^{M} d_r P_j(x,t), \tag{7}$$

where $r = 1,2,\ldots,N$ and $\mathscr{N} = N+M$ as well as

$$\Phi(x,t) = \phi(\sqrt{(x-x_i)^2+(t-t_i)^2}).$$

Then Eq.(1) becomes as follows

$$\mathscr{R}_{\mathscr{N}} = u_{\mathscr{N}}(x,t) - f(x,t) - \sum_{l=1}^{r}\int_{a_l}^{t}\int_{\alpha_{1,l}(z)}^{x} K(x,t,y,z,u_{\mathscr{N}}(y,z))\mathrm{d}y\mathrm{d}z. \tag{8}$$

By setting (8) to be zero at N collocation nodes, (x_i,t_i), $i = 1,2,\ldots,N$, we have

$$\mathscr{R}_{\mathscr{N}} = u_{\mathscr{N}}(x_i,t_i) - f(x_i,t_i) - \sum_{l=1}^{r}\int_{a_l}^{t_i}\int_{\alpha_{1,l}(z)}^{x_i} K(x_i,t_i,y,z,u_{\mathscr{N}}(y,z))\mathrm{d}y\mathrm{d}z, \tag{9}$$

The additional conditions due to (4) are written as

$$\sum_{k=1}^{N}\gamma_k P_l(x_k,t_k), \quad l = 1,2,\ldots,M. \tag{10}$$

Note that to simplify the computations, we use the set of scattered points as the set of collocation points.

To approximate integrals in Eq.(9), we use a Gauss-Legendre integration scheme. To start the estimate of the integrals, we let D be a normal domain with a sufficiently smooth boundary as

$$D = \cup_{l=1}^{r}\{(y,z)\in\mathbb{R}^2 : \alpha_{1,l}(z)\le y\le\alpha_{2,l}(z) \text{ and } a_l\le z\le b_l\}, \tag{11}$$

where $a_l, b_l \in \mathbb{R}$ and $\alpha_{1,l}(z), \alpha_{2,l}(z)$ are continuous function. Then we have

$$\sum_{l=1}^{r}\int_{a_l}^{t_i}\int_{\alpha_{1,l}(z)}^{x_i} K(x_i,t_i,y,z,u_{\mathscr{N}}(y,z))\mathrm{d}y\mathrm{d}z = \sum_{p=1}^{\nu_N}\sum_{k=1}^{\nu_N}\left[\sum_{l=1}^{r} w_p w_k \hat{K}(x_i,t_i,S_k(\hat{z}_p),z_p,u_{\mathscr{N}}(S_k(\hat{z}_p),z_p))\right], \tag{12}$$

and

$$S_k(\hat{z}_p) = \frac{x_i-\alpha_{1,l}(z_p)}{2}\theta_k + \frac{x_i+\alpha_{1,l}(z_p)}{2},$$
$$z_p = \frac{t_i-a_l}{2}\chi_p + \frac{t_i+a_l}{2},$$

where $\{\theta_k\}$, $\{\chi_p\}$ are the coefficients and $\{w_k\}$, $\{w_p\}$ are the weights that relatived in the ν_N-point Gauss-Legendre quadrature formula in the interval $[-1,1]$. Therefore

$$\begin{aligned}&\hat{K}(x_i,t_i,S_k(\hat{z}_p),z_p,u_{\mathscr{N}}(S_k(\hat{z}_p),z_p))\\ &= (\frac{t_i-a_l}{2})(\frac{x_i-\alpha_1(z)}{2})K(x_i,t_i,\frac{x_i-\alpha_{1,l}(z_p)}{2}\theta_k+\frac{x_i+\alpha_{1,l}(z_p)}{2},\frac{t_i-a_l}{2}\chi_p+\frac{t_i+a_l}{2},u_{\mathscr{N}}(S_k(\hat{z}_p),z_p)).\end{aligned}$$

Using the numerical integration (12), we obtain the following system

$$\mathscr{R}_{\mathscr{N}}(x_i,t_i) = u_{\mathscr{N}}(x_i,t_i) - f_l(x_i,t_i) - \sum_{p=1}^{\nu_N}\sum_{k=1}^{\nu_N}\left[\sum_{l=1}^{r} w_p w_k \hat{K}(x_i,t_i,S_k(\hat{z}_p),z_p,u_{\mathscr{N}}(S_k(\hat{z}_p),z_p))\right], \tag{13}$$

for $i = 1, \ldots, \mathscr{N}$. The additional conditions due to (4) are written as

$$\sum_{k=1}^{\mathscr{N}} \gamma_k P_l(x_k) = 0, \quad l = 1, 2, \ldots, M. \tag{14}$$

According to Eq.(13) and Eq.(14), we obtain the following nonlinear algebraic system

$$\begin{cases} \mathscr{R}_{\mathscr{N}}(x_i, t_i) = 0, & i = 1, \ldots, \mathscr{N}, \\ \sum_{k=1}^{\mathscr{N}} \gamma_k P_l(x_k) = 0, & l = 1, 2, \ldots, M. \end{cases} \tag{15}$$

4 Error and convergence analysis

First we present the outline of modifications of some definitions and theorems as already quoted in [2,24]. In the error analysis, we restrict ourselves to the regions satisfying an interior cone condition defined as follows.

Definition 3. *A set $D \subset \mathbb{R}^d$ is said to satisfy an interior cone condition if there exists an angle $\theta \in (\theta, \pi/2)$ and a radius $r > 0$ such that for every $x \in D$ a unit vector $\varepsilon(x)$ exists such that the cone*

$$C(x, \varepsilon(x), \theta, r) = \{x + \lambda y : y \in \mathbb{R}^d, \| y \|_2 = 1, y^T \varepsilon(x) \geq \cos(\theta), \lambda \in [0, r]\}, \tag{16}$$

is contained in D.

Definition 4. *The fill distance of a set of points $X = \{x_1, \ldots, x_N\} \subseteq D$ for a bounded domain D is defined by*

$$h_{X,D} = \sup_{x \in D} \min_{1 \leq j \leq N} \| x - x_j \| .$$

Definition 5. *The set X is said to be quasi-uniform with respect to a constant $c > 0$ if $q_X \leq h_{X,D} \leq c q_X$.*

Definition 6. *Let H be a real Hilbert space of functions $u : D \to \mathbb{R}$. A function $K : D \times D \to \mathbb{R}$ is called reproducing kernel for H if*

1. *$K(x, .) \in$ for all $x \in D$,*
2. *$u(x) = \langle u, K(, x) \rangle_H$ for all $u \in H$and all $x \in D$.*

In the special case, we assume that $K(x, y) = \phi(xy)$, $x, y \in \mathbb{R}^d$ where ϕ is an RBF. In [35] it is shown that K is a symmetric reproducing kernel and also

$$H_\phi(D) = span\{\Phi(\cdot - y) : y \in D\}$$

is the space with an associated bilinear form

$$\left\langle \sum_{j=1}^{N} c_j \Phi(\cdot - x_j), \sum_{k=1}^{N} d_k \Phi(\cdot - y_k) \right\rangle_\Phi = \sum_{j=1}^{N} \sum_{k=1}^{N} c_j d_k \Phi(x_j, y_k), \tag{17}$$

provided $x_j, y_k \in D$.

Theorem 1. *If Φ is a radial basis and strictly positive definite function, then the bilinear form $\langle . \rangle_\Phi$ defines an inner product on $H_\Phi(D)$. Furthermore, $H_\Phi(D)$ is a pre-Hilbert space with reproducing Φ.*

Proof. See [35, Chapter 10, Theorem 10.7]. □

Definition 7. *The native space $N_\Phi(D)$ of ϕ is defined to be the completion of $H_\Phi(D)$ with respect to the Φ-norm $\|\cdot\|_\Phi$ so that $\|u\|_\Phi = \|u\|_{N_\Phi(D)}$ for all $u \in N_\Phi(D)$.*

To obtain the error bound for approximating u by strictly conditionlly positive definite function, defined in (3), we present the following theorem [35, pp. 181–183].

Theorem 2. *Let $D \subseteq \mathbb{R}^d$ be open and bounded, satisfying an interior cone condition. Also, suppose that Φ is conditionally positive definite with respect to $\prod_{m1}^d(D)$ and the reproducing kernel $K = \Phi(\cdot,\cdot)$ is in $C^{2k}(D \times D)$. Suppose the interpolant to $u \in N_\Phi(D)$ is based on the m-unisolvent set $X = \{x_1, \ldots, x_N\} \subseteq D$. If derivatives of Φ of order $2k$ are continuous on $\bar{D} \times \bar{D}$, then there exist constants $h_0, C > 0$ such that*

$$\|u - P_{\mathscr{N}} u\|_{L(D)}^\infty \leq C C_\Phi^{1/2} h_{X,D}^k \|u\|_{N_\Phi(D)}, \tag{18}$$

provided $h_{X,D} \leq h_0$. The number C_Φ is defined by

$$C_\Phi = \max_{\beta=2k} \|D^\beta \Phi\|_{L^\infty(B(0,2c_2 h_{X,D}))}, \tag{19}$$

and is independent of x.

In the sequel, we give the following remark obtained from Theorem 2 in [2].

Remark 2. *From Theorem 2, we can conclude*

$$\|u - u_{\mathscr{N}}\|_\infty \leq \mu \|u\|_{N_{\Phi(D)}}, \tag{20}$$

where

1. *for Gaussian $\Phi(x) = e^{\alpha \|x\|_2^2}$, $\alpha > 0$, we get for some positive constant c that $\mu = e^{-\frac{c|\log h_{X,D}|}{h_{X,D}}}$.*
2. *for (inverse) multiquadrics $\Phi(x) = (\|x\|_\alpha)^\beta$, > 0, $\beta < 0$ or $\beta > 0$ and $\beta \notin \mathbb{N}$, we have $\mu = \exp^{-\frac{-c}{h_{X,D}}}$.*
3. *for thin plate splines $\Phi(x) = (-1)^{k+1} \|x\|_2^{2k}$, $k \in \mathbb{N}$ we get $\mu = C h_{X,D}^k$.*

To discuss the error and rate of convergence of the proposed method, we define the integral operator as follows:

$$\mathscr{K} = \sum_{l=1}^r \int_{a_l}^t \int_{\alpha_{1,l}(z)}^x K(x,t,y,z,u(y,z)) \mathrm{d}y \mathrm{d}z, \tag{21}$$

where $(x,t) \in D$. Then we can rewrite (3) in the operation form as follows

$$u = f + \mathscr{K}. \tag{22}$$

Therefore, we can show the Eq. (9) is as follows

$$u_{\mathcal{N}} = P_{\mathcal{N}} f + P_{\mathcal{N}} \mathcal{K}. \tag{23}$$

Now, suppose that

$$\mathcal{K}_{\mathcal{N}} = \sum_{p=1}^{\nu_N} \sum_{k=1}^{\nu_N} \left[\sum_{l=1}^{r} w_p w_k \hat{K}(x,t,\hat{y}_p, S_k(\hat{y}_p), u_{\mathcal{N}}(\hat{y}_p, S_k(\hat{y}_p))) \right].$$

Thus the operator form of (13) becomes as follows

$$\hat{u}_{\mathcal{N}} = P_{\mathcal{N}} \mathcal{K}_{\mathcal{N}}. \tag{24}$$

Now Eqs. (22), (23) and (24) can be written as follows

$$u = f + \mathcal{K}[u], \tag{25}$$
$$u_{\mathcal{N}} = P_{\mathcal{N}} f + P_{\mathcal{N}} \mathcal{K}[u_{\mathcal{N}}] \tag{26}$$

and

$$\hat{u}_{\mathcal{N}} = P_{\mathcal{N}} f + P_{\mathcal{N}} \mathcal{K}_{\mathcal{N}}[\hat{u}_{\mathcal{N}}], \tag{27}$$

respectively. The iterated discrete collocation method for solving Eq. (25) is defined by

$$\bar{u}_{\mathcal{N}} = f + \mathcal{K}_{\mathcal{N}}[\hat{u}_{\mathcal{N}}]. \tag{28}$$

From Eqs. (27) and (4), we find $P_{\mathcal{N}} \bar{u}_{\mathcal{N}} = \hat{u}_{\mathcal{N}}$, then $\bar{u}_{\mathcal{N}}$ satisfies

$$\bar{u}_{\mathcal{N}} = f + \mathcal{N}[P_{\mathcal{N}} \bar{u}_{\mathcal{N}}]. \tag{29}$$

Define $\mathcal{T}[u] = f + \mathcal{K}[u]$, $\mathcal{T}_{\mathcal{N}}[u] = P_{\mathcal{N}} f + P_{\mathcal{N}} \mathcal{K}[u]$, $\hat{\mathcal{T}}_{\mathcal{N}}[u] = P_{\mathcal{N}} f + P_{\mathcal{N}} \mathcal{K}_{\mathcal{N}}[\hat{u}]$ and $\bar{\mathcal{T}}_{\mathcal{N}}[u] = f + \mathcal{K}_{\mathcal{N}}[P_{\mathcal{N}} \bar{u}_{\mathcal{N}}]$, then Eqs. (25), (26), (27), (29) can be written as follows

$$u = \mathcal{T}[u], \tag{30}$$
$$u_{\mathcal{N}} = \mathcal{T}_{\mathcal{N}}[u_{\mathcal{N}}], \tag{31}$$
$$\hat{u}_{\mathcal{N}} = \hat{\mathcal{T}}_{\mathcal{N}}[\hat{u}_{\mathcal{N}}], \tag{32}$$

and

$$\bar{u}_{\mathcal{N}} = \bar{\mathcal{T}}_{\mathcal{N}}[\bar{u}_{\mathcal{N}}],$$

respectively. As given in [2,5], we first give an analysis for the iterated discrete collocation solution, and then the convergence of the discrete RBFs-collocation solution will follow from

$$u - \hat{u}_{\mathcal{N}} = (u - P_{\mathcal{N}}) + P_{\mathcal{N}}(u - \bar{u}_{\mathcal{N}}). \tag{33}$$

To continue our discussion, we outline the following hypotheses on $\mathcal{T}$ and $\mathcal{T}_{\mathcal{N}}$ as already given in [2,5]. Here, we assume that $D = [a,b]^2$.

H1 . $\mathscr{T}$ and $\mathscr{T}_{\mathscr{N}}, \mathscr{N} \geq 1$, are completely continuous nonlinear operators on $C(D)$ into $C(D)$.

H2 . $\mathscr{T}_{\mathscr{N}}$, $\mathscr{N} \geq 1$, is a collectively compact family on $C(D)$, i.e., for every bounded set $B \subseteq C(D)$,the closure of the set $\bigcup_{n=1}^{\infty} \mathscr{T}_{\mathscr{N}}$ is compact in $C(D)$.

H3 . $\mathscr{T}_{\mathscr{N}}$ is pointwise convergent to $\mathscr{T}$ on $C(D)$, i.e., $\mathscr{T}_{\mathscr{N}}[u] \to \mathscr{T}[u], u \in C(D)$.

H4 . At each point of $C(D), \{\mathscr{T}_{\mathscr{N}}\}$is an equicontinuous family.

H5 . For a given fixed point u_0 of $\mathscr{T}$, assume $\mathscr{T}$ and $\mathscr{T}_{\mathscr{N}}, \mathscr{N} \geq 1$, are twice Frechet differential on $B(u_0, r) = \{u : \|u - u_0\| \leq r, r \geq 1\}$ with $\|\mathscr{T}''[u]\|, \|\mathscr{T}''_{\mathscr{N}}[u]\| \leq \alpha, \mathscr{N} \geq 1, u \in B(u_0, r)$ where α is a positive constant.

Remark 3. *It is important to note that the hypotheses* **H1** ,**H2** *and* **H4** *can be proved using Arzela-Ascoli theorem under certain conditions, therefore, the hypotheses mentioned above can be reduced to two hypotheses.*

Now, to use (33), we first give the following theorem from [2] about the iterated discrete collocation method.

Theorem 3. *Suppose* **H1** $-$ **H5**. *Let u_0 be a solution of Eq.(4) or a fixed point of $\mathscr{T}$, and assume that 1 is not an eigenvalue of $\mathscr{T}'[u_0]$, where $\mathscr{T}'[u_0]$ denotes the Frechet derivative of $\mathscr{T}$ at u_0. If* **H5** *is satisfied on $B(u_0, r) \subseteq C(D)$, then u_0 is an isolated solution of Eq.(4), of nonzero index. Moreover, there are $\varepsilon > 0$ and $M_1 > 0$ such that for every $N > M_1$, $\tilde{\mathscr{T}}_{\mathscr{N}}$ has a unique fixed point $\bar{u}_{\mathscr{N}} i\ n\ B(u_0, \varepsilon)$, where $\tilde{\mathscr{T}}_{\mathscr{N}}[u] = \mathscr{T}_{\mathscr{N}}[P_{\mathscr{N}} u]$. Also, there is a constant $\gamma_1 > 0$ such that*

$$\|\bar{u}_{\mathscr{N}} - u_0\|_{L^\infty(D)} \leq \gamma_1 \|\mathscr{T}[u_0] - \mathscr{T}_{\mathscr{N}}[P_{\mathscr{N}} u_0]\|_{L^\infty(D)}, N \geq M_1. \tag{34}$$

This gives a bound on the rate of convergence of the iterated solution $\bar{u}_{\mathscr{N}}$ to u_0.

Finally, we give the following theorem from [2], with some modifications, to complete our discussion.

Theorem 4. *Having in mind the assumptions of Theorems 2 and 3, suppose that $u_0 \in N_\Phi(D)$ is the exact solution of Eq.(4) and the proposed method has been installed on the quasi-uniform set $X = \{x_i\}_{i=1}^N \subseteq D$. Then there exist $\hat{\varepsilon}, \hat{M} > 0$ such that for every $N > \hat{M}$, the method has a unique solution $\hat{u}_{\mathscr{N}} \in B(u_0, \hat{\varepsilon})$. Also, we have*

$$\|\hat{u}_{\mathscr{N}} - u_0\|_{L^\infty(D)} \to 0 \text{ as } \mathscr{N} \to \infty, \tag{35}$$

furthermore, we find

$$\|\hat{u}_{\mathscr{N}} - u_0\|_{L^\infty(D)} \leq (1 + C_p \gamma_1 \gamma_2) C C_\Phi^{\frac{1}{2}} h_{X,D}^k \|u_0\|_{N_\Phi(D)} + C_p \gamma_1 \|\mathscr{T}[u_0] - \mathscr{T}_{\mathscr{N}}[u_0]\|_{L^\infty(D)}, \tag{36}$$

where $\gamma_1, \gamma_2, C, C_\Phi$ and C_p constants.

Proof. From Theorem 3 , there exist $\varepsilon, M1 > 0$ such that for every $N \geq M1$, Eq. (32) has a unique discrete iterated solution and furthermore

$$\begin{aligned} \|\bar{u}_{\mathscr{N}} - u_0\|_{L^\infty(D)} &\leq \gamma_1 \|\mathscr{T}[u_0] - \mathscr{T}_{\mathscr{N}}[P_{\mathscr{N}} u_0]\|_{L^\infty(D)} \\ &\leq \gamma_1 (\|\mathscr{T}[u_0] - \mathscr{T}_{\mathscr{N}}[u_0]\|_{L^\infty(D)} + \|\mathscr{T}[u_0] - \mathscr{T}_{\mathscr{N}}[P_{\mathscr{N}} u_0]\|_{L^\infty(D)}). \end{aligned} \tag{37}$$

Using the hypotheses **H1** and **H4**, we find

$$\|\mathscr{T}[u_0]-\mathscr{T}_{\mathscr{N}}[P_{\mathscr{N}}u_0]\|_{L^\infty(D)}\leq\gamma_2\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}, \tag{38}$$

where $\gamma_2=\sup_{N\geq1}\|\mathscr{T}_{\mathscr{N}}\|_{L^\infty(D)}<\infty$. From (37) and (38), we obtain

$$\|\bar{u}_{\mathscr{N}}-u_0\|_{L^\infty(D)}\leq\gamma_1(\|\mathscr{T}[u_0]-\mathscr{T}_{\mathscr{N}}[u_0]\|_{L^\infty(D)}+\gamma_2\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}). \tag{39}$$

Using the decomposition (33), by setting $u=u_0$and $\hat{u}_N=P_{\mathscr{N}}\bar{u}_{\mathscr{N}}$, we find

$$\|\hat{u}_{\mathscr{N}}-u_0\|_{L^\infty(D)}\leq\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}+\|P_{\mathscr{N}}\|\|u_0-\bar{u}_{\mathscr{N}}\|_{L^\infty(D)}. \tag{40}$$

Utilizing Theorem 3 for $u_0\in N_\Phi(D)$, since, $h_{X,D}\to0$ as $N\to\infty$ (justified by the quasi-uniform condition on X), we find $\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}\to0$, in other words, there exists $M_2>0$ such that for every $N>M_2$ we have $\|P_{\mathscr{N}}u_0-u_0\|_{L^\infty(D)}<\bar{\varepsilon}$. Therefore, from the principle of the uniform boundedness [5, page 517, Theorem A.3], we conclude

$$C_p=\sup_{\mathscr{N}\geq1}\|P_{\mathscr{N}}\|<\infty. \tag{41}$$

From (40) and (41) we obtain

$$\|\hat{u}_{\mathscr{N}}-u_0\|_{L^\infty(D)}\leq\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}+C_p\|u_0-\bar{u}_{\mathscr{N}}\|_{L^\infty(D)}<\bar{\varepsilon}+C_p\varepsilon. \tag{42}$$

Therefore by choosing $\bar{M}=\max M_1,M_2$ and $\hat{\varepsilon}=\bar{\varepsilon}+C_p\varepsilon$, we find $\hat{u}_{\mathscr{N}}\in B(u_0,\hat{\varepsilon})$ for $N>\hat{M}$.

Now, substituting (39) in (40) and simplifying it, yields

$$\|\hat{u}_{\mathscr{N}}-u_0\|_{L^\infty(D)}\leq(1+\gamma_1\gamma_2\|P_N\|)\|u_0-P_{\mathscr{N}}u_0\|_{L^\infty(D)}+\gamma_1\|P_{\mathscr{N}}\|\|\mathscr{T}[u_0]-\mathscr{T}[u_0]\|_{L^\infty(D)}. \tag{43}$$

Then, by applying the error bound of theorem 3, (43) yields

$$\|\hat{u}_{\mathscr{N}}-u_0\|_{L^\infty(D)}\leq(1+C_p\gamma_1\gamma_2)CC_\Phi^{\frac{1}{2}}h_{X,D}^k\|u_0\|_{N_{\Phi(D)}}+\gamma_1C_p\|\mathscr{T}[u_0]-\mathscr{T}[u_0]\|_{L^\infty(D)}. \tag{44}$$

This gives a bound on the rate of convergence of the $\hat{u}_{\mathscr{N}}tou_0$. To establish (35), since $h_{X,D}\to0$as $N\to\infty$(then $\mathscr{N}\to\infty$) and using the hypothesis **H3**, we find that $\mathscr{T}_{\mathscr{N}}[u_0]\to\mathscr{T}[u_0]$as $\mathscr{N}\to\infty$, then from (44) we see that $\|\hat{u}_{\mathscr{N}}u_0\|_{L^\infty(D)}to0$as $\mathscr{N}\to\infty$. □

5 Numerical examples

In this section, we solve some examples, involving nonlinear integral equations on the non-rectangular domain with this method. We use Gaussian (GA), multiquadric (MQ), and inverse multiquadric (IMQ) by adding the polynomials $\{1,x,t\}$ in this computaions. Since there is no alternative method for these types of equations, we compare the exact solution with the solution given by above mentioned method. We report the absolute error which is defined by following norms

$$\|E_n(x,t)\|_2=\left(\int_D|u(x,t)-\hat{u}(x,t)|^2\right)^{\frac{1}{2}},\quad\|E_n(x,t)\|_\infty=\max_{(x,t)\in D}\{|u(x,t)-\hat{u}(x,t)|\},$$

where $u(x,t)$ and $\hat{u}(x,t)$ are the exact and the approximate solution respectively. All calculations are performed in Mathematica 12 software with the Digits 20 and run on a Pentium IV PC Laptop with a 2.10 GHz CPU and 4 GB RAM. The "FindRoot" command is used to solve the nonlinear system of algebraic equations that employs floating-point arithmetic. we choose the zero vector of length $N+M$ as our initial guess, i.e., $\hat{u}(0) = [0,0,\ldots,0]^t$. To approximate the integrals in this method for solving three examples, we use the 10-point double Gauss-Legendre quadrature formula and we put the value of shape parameter $c = 0.1\sqrt{N}$ that is a medial value and obtained by trial and error. In the following, we present three examples, the kernels and exact solutions of these examples are selected from [12, 25, 26].

Example 1. Consider the following Volterra integral equation of the second kind

$$u(x,t) - \sum_{l=1}^{r} \int_{a_l}^{t} \int_{\alpha_{1,l}(z)}^{x} u^2(y,z)\mathrm{d}y\mathrm{d}z = f(x,t),\ (x,t) \in D, \tag{45}$$

where $u(x,t) = x^2 + t^2$ [25]. The region D which is the star domain and the distribution of nodes are depicted in Fig. 1. The results for Eq. (45) are shown in Table 2 and the absolute error for $N = 22$ via GA and MQ radial basis functions are graphically shown in Figs. 2 and 3, respectively. Here we can separate the domain D as $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned}
D_1 &= \{(x,t) \in \mathbb{R}^2 : 0 < x < 2/5, (1/4)x + 1/2 < t < (1/4)x + 1/2\},\\
D_2 &= \{(x,t) \in \mathbb{R}^2 : 2/5 < x < 1/2, (-4)x + 2 < t < (4)x - 1\},\\
D_3 &= \{(x,t) \in \mathbb{R}^2 : 1/2 < x < 3/5, 4x - 2 < t < -4x + 3\},\\
D_4 &= \{(x,t) \in \mathbb{R}^2 : 3/5 < x < 1, (1/4)x + 1/4 < t < (-1/4)x + 3/4\}.
\end{aligned}$$

Table 2: Numerical results of Example 1.

N	c	$\|e\|_\infty$ with GA
9	0.3	5.15478×10^{-2}
14	0.3741166	1.000446×10^{-4}
18	0.424264	3.44224×10^{-5}
22	0.469042	5.22231×10^{-5}
		$\|e\|_\infty$ with MQ
9	0.3	6.98218×10^{-3}
14	0.374166	2.45592×10^{-4}
18	0.424264	6.10213×10^{-1}
22	0.469042	1.68197×10^{-2}

Example 2. Consider the following Volterra integral equation of the second kind

$$u(x,t) - \int_0^t \int_{\frac{z}{10}}^{x} \cos(z) u^2(y,z)\mathrm{d}y\mathrm{d}z = f(x,t),\ (x,t) \in D, \tag{46}$$

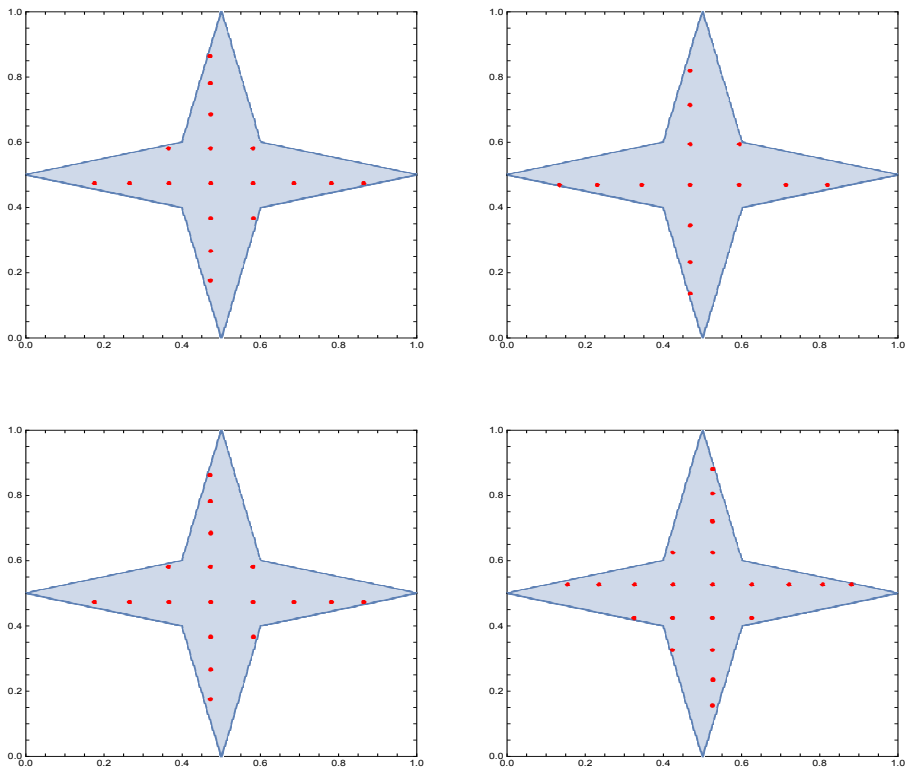

Figure 1: Node distribution ($a:9, b:14, c:18$ and 22 nodes) for Example 1

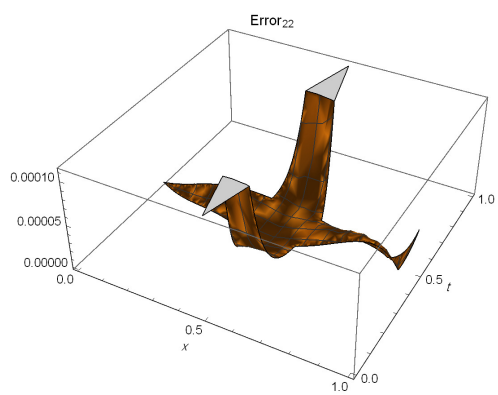

Figure 2: Absolute error for Example 1 with $N = 22$ via GA.

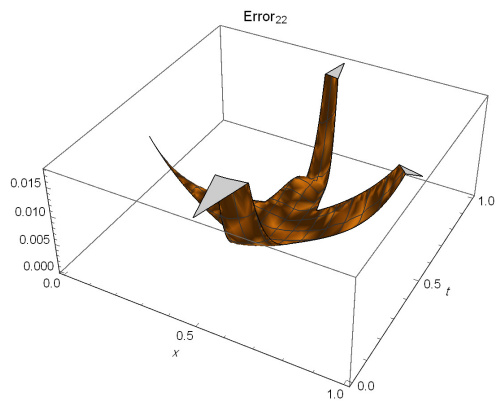

Figure 3: Absolute error for Example 1 with $N = 22$ via MQ.

where $u(x,t) = x\sin(t)$ [12]. The region D and the distribution of nodes are depicted in Fig. 4. Table 3 shows the results for Eq. (46). The absolute error for $N = 23$ via GA and IMQ radial basis functions are graphically shown in Figs. 5 and 6, respectively. Here D is the following region

$$D = \{(x,t) \in \mathbb{R}^2 : 0 < t < 1, \frac{t}{10} < x < \frac{1}{5}\exp(t)\}.$$

Table 3: Numerical results of Example 2.

N	c	$\|e\|_2$ with GA	$\|e\|_\infty$ with GA
6	0.244949	1.26352×10^{-3}	7.26535×10^{-3}
14	0.374166	2.96197×10^{-4}	2.47365×10^{-4}
18	0.424264	2.0453×10^{-5}	8.12113×10^{-5}
23	0.479583	3.09119×10^{-6}	3.26261×10^{-6}
		$\|e\|_2$ with IMQ	$\|e\|_\infty$ with IMQ
6	0.244949	1.17655×10^{-3}	9.48377×10^{-3}
14	0.374166	4.94714×10^{-5}	2.17188×10^{-4}
18	0.424264	3.7373×10^{-5}	9.29681×10^{-5}
23	0.479583	7.77711×10^{-6}	2.11557×10^{-5}

Example 3. Consider the following Volterra integral equation of the second kind

$$u(x,t) - \int_0^t \int_z^x (xy^2 + \cos(z))u^2(y,z)\mathrm{d}y\mathrm{d}z = f(x,t),\ (x,t) \in D, \tag{47}$$

where $u(x,t) = x\sin(t)$ [26]. The region D and the distribution of nodes are depicted in Fig. 4. Table 4 shows the results for Eq. (47). Here D is the following region

$$D = \{(x,t) \in \mathbb{R}^2 : 0 < t < 1, z^2 < x < z\}.$$

We compare the maximum error $\|e\|_\infty$ obatained from this method for GA, MQ, and IMQ in Fig. 8.

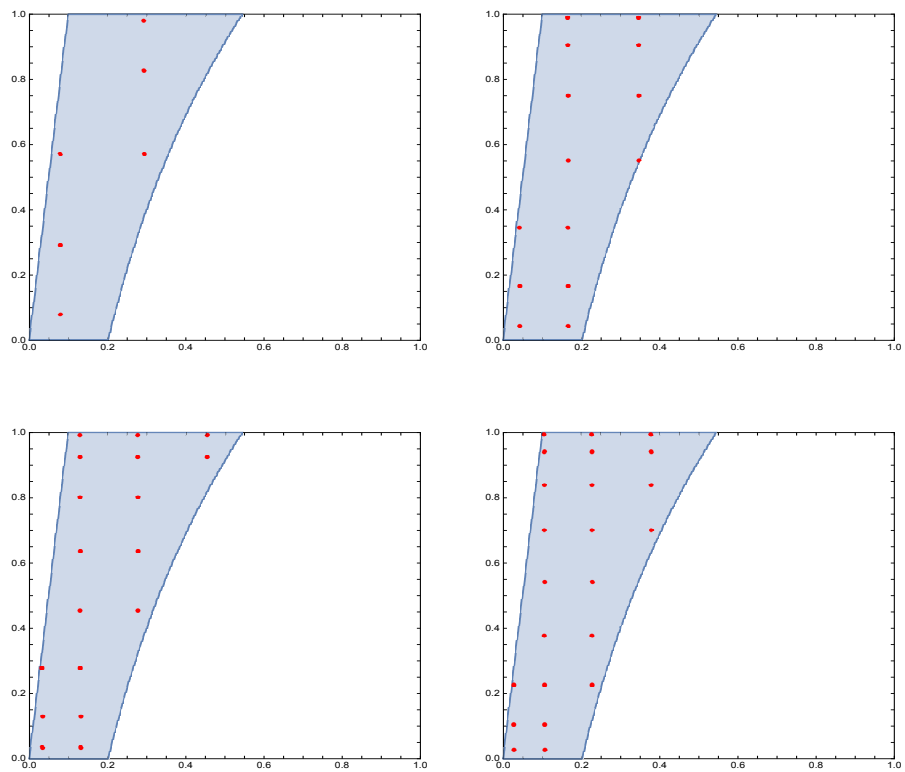

Figure 4: Node distribution ($a:6, b:14, c:18$ and 23 nodes) for Example 2.

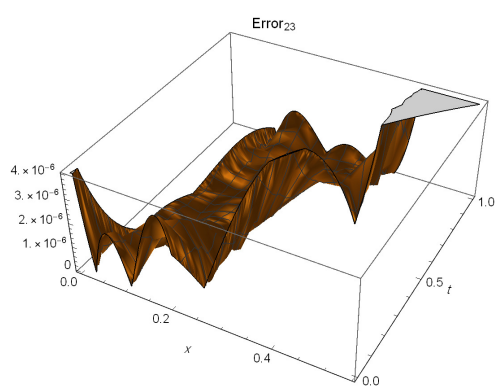

Figure 5: Absolute error for Example 1 with $N = 23$ via GA .

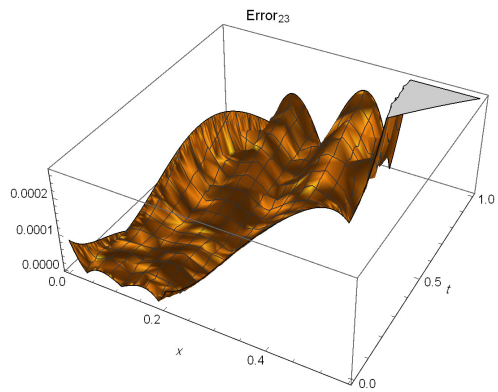

Figure 6: Absolute error for Example 2 with $N = 23$ via IMQ.

Table 4: Numerical results of Example 3.

N	c	$\|e\|_\infty$ with MQ
38	0.616441	9.14031×10^{-4}
41	0.640312	5.32874×10^{-4}
44	0.663325	4.93388×10^{-3}
51	0.714143	5.28804×10^{-4}
56	0.748331	6.87324×10^{-4}
		$\|e\|_\infty$ with IMQ
38	0.616441	7.74102×10^{-3}
41	0.640312	7.08353×10^{-3}
44	0.663325	1.28679×10^{-3}
51	0.714143	9.08813×10^{-3}
56	0.748331	7.42545×10^{-3}
		$\|e\|_\infty$ with GA
38	0.616441	4.58288×10^{-4}
41	0.640312	9.77333×10^{-5}
44	0.663325	9.94709×10^{-5}
51	0.714143	1.11194×10^{-4}
56	0.748331	1.22590×10^{-4}

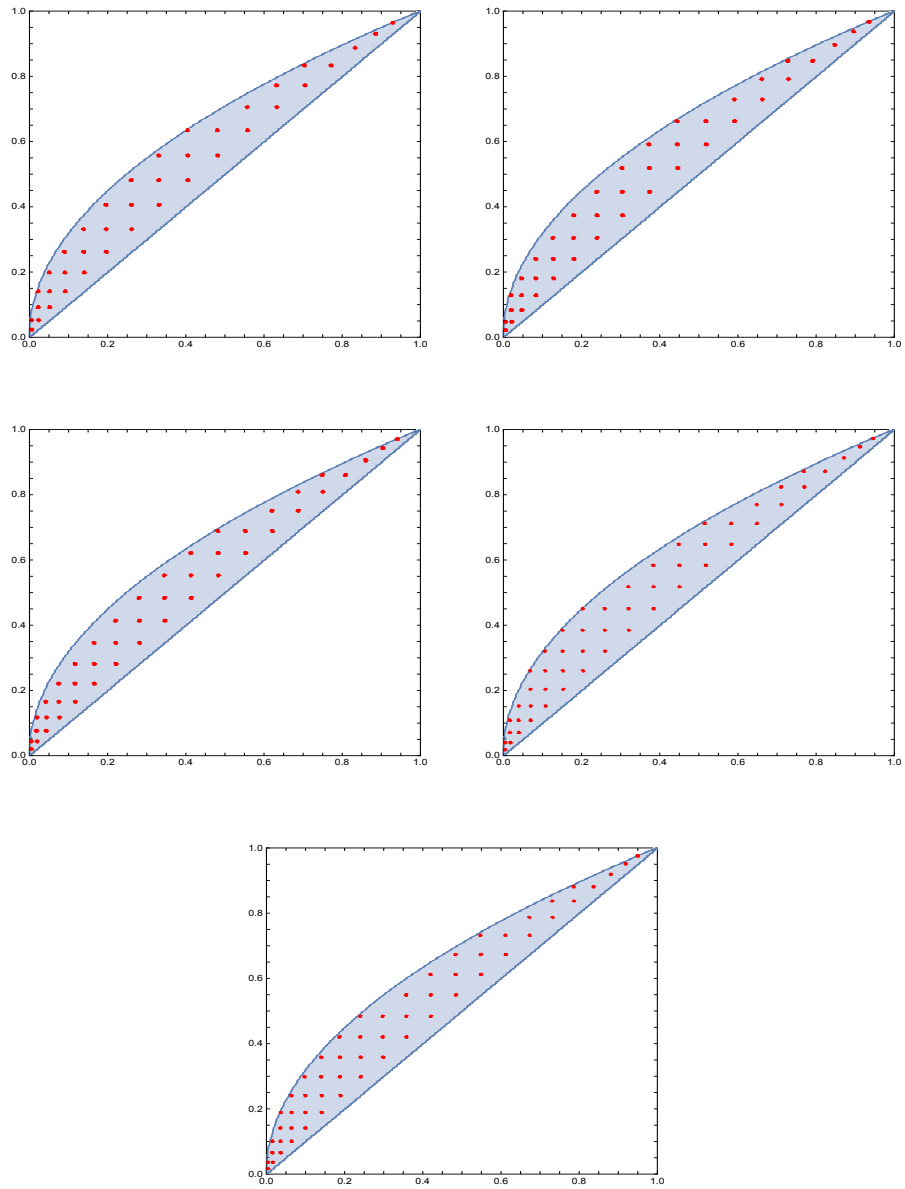

Figure 7: Node distribution ($a:38, b:41, c:44, d:51$ and $e:56$ nodes) for Example 3.

6 Conclusions

In this work, we presented an efficient and computationally attractive method to solve the class of the two-dimensional nonlinear Volterra integral equations of the second kind, which is defined on the non-rectangular domain. The method is based on the use of the radial basis functions. This approach is meshless. Therefore, this method is independent of the geometry of the domain. Error analysis is proved in the current paper. We can also expand this method to solve the two-dimensional Volterra-Fredholm integral equations and Volterra integral equations of fractional order.

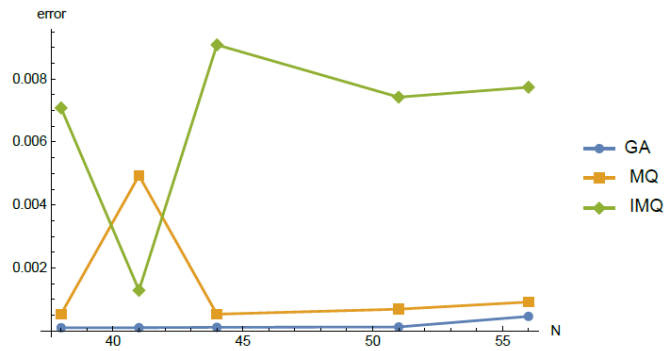

Figure 8: Maximum error for Example 3 via *GA*, *MQ*, and *IMQ*.

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