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Unconditionally stable finite element method for the variable-order fractional Schrödinger equation with Mittag-Leffler kernel

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Abstract. The Schrdinger equation with variable-order fractional operator is a challenging problem to be solved numerically. In this study, an implicit fully discrete continuous Galerkin finite element method is developed to tackle this equation while the fractional operator is expressed with a nonsingular Mittag-Leffler kernel. To begin with, the finite difference scheme known as the L1 formula is employed to discretize the temporal term. Next, the continuous Galerkin method is used for spatial discretization. This combination ensures accuracy and stability of the numerical approximation. Our next step is to conduct a stability and error analysis of the proposed scheme. Finally, some numerical results are carried out to validate the theoretical analysis.

Keywords: Variable-order fractional equation, Schrödinger equation, finite element method, stability, error estimate.
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1 Introduction

The Schrödinger equations (SEs) have applications in quantum mechanics, optics, and fluid dynamics [8] and describe the optical soliton in fiber [3, 23, 53, 59]. The fractional SEs depict fractal phenomena in quantum mechanics [40]. Laskin [34, 35] developed SEs into space fractional (SFSEs) by extending the Brownian path integral to the Lévy flights. The applications of SFSEs in optics can be found in [39]. Naber [50] showed the equivalence of the standard SE with the time-dependent Hamiltonian and the fractional SEs.

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This paper is concerned with the following time-fractional SE

$$i{}^{HH}_{0}\mathrm{D}^{\alpha(t)}\Psi + \Delta\Psi + \Psi = 0, \quad \text{in } \Omega \times (0, T], \tag{1}$$

subject to initial condition $\Psi(x,y,0) = \Psi_0(x,y)$ and homogeneous Dirichlet boundary conditions, where ${}^{HH}_{0}\mathrm{D}^{\alpha(t)}$ is the nonsingular variable-order (VO) of order $\alpha(t)$ defined in [30] by Heydari and Hosseininia (from now on we call it HH derivative), Ω is a bounded domain in $\mathbb{R}^2$, with piecewise smooth boundary $\partial\Omega$ and the final time T is a fixed and positive real number.

In the past decades, many numerical methods such as finite difference methods [19, 20, 55], continuous [1, 9, 10, 13] and discontinuous Galerkin finite element methods [4, 18, 42, 43, 58], meshless methods [54], spectral methods [21, 22], kernel-based methods [45–47], cardinal approach [29] and operational matrix [5, 32, 52] constructed for solving the fractional ordinary differential equations (FODEs) [41], fractional partial differential equations [14], integro-differential equations [11, 44] and fractional optimal problems [27]. Particularly, the results of the numerical study of the standard and fractional SEs have been presented by many authors [2, 15, 16]. The main aim of [49] is to use the compact boundary value method for the solution of SEs. The authors of [12] proposed the MLPG method for solving n-coupled nonlinear SEs. Karamali and Mohammadi [31] utilized a combination of the Laplace transform and compact finite difference scheme with matrix transformation technique for the solution of time-space fractional linear SEs. Heydari and Atagana [25] utilized a cardinal approach for the solution of VO-SE with the Atangana-Baleanu-Caputo operator. Heydari in [24] proposed an efficient method for solving Klein–Gordon–SEs with distributed-order fractional operator. Bhrawy and Zaky employed a collocation method for multi-dimensional space-time VO-SEs in [6]. Wei et al. [56, 57] developed discontinuous Galerkin FE methods for time-fractional Schrödinger and coupled SEs. Mohebbi et al. [48] utilized a meshless technique for the solution of the time-fractional SEs arising in quantum mechanics. Heydari et al. numerically investigated a fractal-fractional coupled Schrödinger-Boussinesq (CSB) system in [28]. Well-posedness, conservation, and convergence properties of Galerkin FE methods for the fractional SEs with the Riesz-space fractional derivative are presented by Li et al. in [37, 38]. Li et al. established linearized and mass-energy preserving FE methods for the coupled fractional SEs in [36, 60]. Finite element method has a significant advantage over finite difference and spectral methods as it can be applied to complex computational domains. Also, due to the presence of various generalizations, it is possible to enhance its computational accuracy. In this paper, a combination of temporal discretization using the finite difference scheme and spatial discretization using the Galerkin FE method is used for solving Eq. (1).

The paper is organized as follows: Section 2 includes some definitions and lemmas. In Section 3 we construct an implicit fully discrete FE method (L1-FEM) for solving Eq. (1). In Section 4, some theoretical results such as the unconditional stability and error estimate are proven. Some test problems are presented to validate the analytical results in Section 5 and finally, we conclude this paper in Section 6.

2 Preliminaries

This section provides some basic definitions and lemmas that are used in the rest of the paper.

Definition 1. *The two-parameter Mittag-Leffler function is defined by [51]*

$$\mathrm{E}_{\vartheta_1,\vartheta_2}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(\vartheta_1 j+\vartheta_2)},\qquad \vartheta_1,\vartheta_2>0,\qquad \vartheta_1,\vartheta_2\in\mathbb{R},t\in\mathbb{C}. \tag{2}$$

A special case of this function with important roles in fractional calculus is the one-parameter Mittag-Leffler (M-L) function $\mathrm{E}_{\vartheta}(t)=\mathrm{E}_{\vartheta,1}(t)$.

In this study, we consider the VO fractional derivative with the nonsingular kernel as the HH fractional derivative [30] which is expressed below.

Definition 2. *For a differentiable function $\Psi(t)$ on $\mathbb{R}$ and a continuous function $\alpha(t):[0,\infty)\to(0,1)$, the HH VO fractional derivative of order $\alpha(t)$ for function $\Psi(t)$ is defined by*

$$ {}^{HH}_{0}\mathrm{D}^{\alpha(t)}\Psi(t)=\frac{1}{1-\alpha(t)}\int_0^t \mathrm{E}_1\left(\frac{-\alpha(t)(t-s)}{1-\alpha(t)}\right)\frac{d\Psi(s)}{ds}\,ds,\qquad t>0. \tag{3}$$

Denote $t_n=n\Delta t$, where Δt is the temporal step length. We use the $L1$ approximation of the fractional derivative of order $\alpha(t)$ $(0<\alpha(t)<1)$ for function $\Psi(t)$ presented in [17,26]:

$$\tilde{\mathrm{D}}^{\alpha(t)}_{n+1}\Psi(t)=\rho_0^{(n+1)}\Psi^{n+1}-\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\Psi^j-\rho_n^{(n+1)}\Psi^0, \tag{4}$$

such that $\Psi^{n+1}=\Psi(t_{n+1})$ and

$$\rho_j^{(n+1)}=\frac{1}{1-\alpha^{n+1}}\left[(j+1)\mathrm{E}_{1,2}\left(-\frac{(j+1)\Delta t\,\alpha^{n+1}}{1-\alpha^{n+1}}\right)-j\mathrm{E}_{1,2}\left(-\frac{j\Delta t\,\alpha^{n+1}}{1-\alpha^{n+1}}\right)\right],\qquad 0\le j\le n. \tag{5}$$

Lemma 1. *The coefficients $\rho_j^{(n+1)}$, $j\ge 0$ are strictly monotonically decreasing with respect to j.*

Proof. Since

$$\frac{d\rho_j^{(n+1)}}{dj}=\frac{1}{1-\alpha^{n+1}}\mathrm{E}_1(-c(j+1))-\frac{1}{1-\alpha^{n+1}}\mathrm{E}_1(-cj)\ge 0, \tag{6}$$

for any $0<\alpha(t)<1$, the coefficients $\rho_j^{(n+1)}$, $j\ge 0$ are strictly monotonically decreasing with respect to j. □

Lemma 2. *([17]) Let us assume that ${}^{HH}_{0}\mathrm{D}^{\alpha(t)}\Psi(t)$ is approximated by $\tilde{\mathrm{D}}^{\alpha(t)}_{n+1}\Psi(t)$ and $\Psi\in C^2[0,t_{n+1}]$, then we have*

$$\begin{aligned}
&{}^{HH}_{0}\mathrm{D}^{\alpha(t)}\Psi(t)\big|_{t_{n+1}}-\tilde{\mathrm{D}}^{\alpha(t)}_{n+1}\Psi(t)=\\
&\Delta t^2\sum_{i=0}^{n}\frac{\Psi''(\xi^{i+1})}{2}\Big(2\left[(n-i+1)^2\mathrm{E}_{1,3}\left(\frac{-\alpha^{n+1}(n-i+1)\Delta t}{1-\alpha^{n+1}}\right)-(n-i)^2\mathrm{E}_{1,3}\left(\frac{-\alpha^{n+1}(n-i)\Delta t}{1-\alpha^{n+1}}\right)\right]\\
&\qquad-\left[(n-i+1)\mathrm{E}_{1,2}\left(\frac{-\alpha^{n+1}(n-i+1)\Delta t}{1-\alpha^{n+1}}\right)+(n-i)\mathrm{E}_{1,2}\left(\frac{-\alpha^{n+1}(n-i)\Delta t}{1-\alpha^{n+1}}\right)\right]\Big).
\end{aligned} \tag{7}$$

3 The FEM for the VOF-SE with M-L kernel

This section focuses on constructing the fully-discrete finite element method for Eq. (1). We give the scheme for a coupled system of equations, instead of a complex equation. Decomposing the complex functions $\Psi(x,y,t)$ into their real and imaginary parts and defining $\Psi(x,y,t) = P(x,y,t) + iQ(x,y,t)$ in Eq. (1), give the following coupled system of equations:

$$
{}_{0}^{HH}\mathrm{D}^{\alpha(t)}P + \Delta Q + Q = 0, \tag{8a}
$$

$$
{}_{0}^{HH}\mathrm{D}^{\alpha(t)}Q - \Delta P - P = 0. \tag{8b}
$$

In the following parts, we aim to present the numerical scheme for the solution of system (8).

3.1 Temporal discretization method

First, utilizing scheme (4), we obtain the time-discrete system as follows

$$
\tilde{\mathrm{D}}_{n+1}^{\alpha(t)}P + \Delta Q^{n+1} + Q^{n+1} = 0, \tag{9a}
$$

$$
\tilde{\mathrm{D}}_{n+1}^{\alpha(t)}Q - \Delta P^{n+1} - P^{n+1} = 0. \tag{9b}
$$

Note that using (4) in combination with the FE method leads to an implicit method that is accurate to second-order in both space and time.

3.2 Spatial discretization method

Let Ω_h be a family of triangulation of Ω indexed by the maximum diameter of the elements, h. We define the L^2 inner product and corresponding induced norm as

$$
(v,w) = \int_{\Omega} vw\,d\boldsymbol{x}, \qquad \|v\|^2 = (v,v). \tag{10}
$$

The variational formulations of system (9) is given as: Find P^{n+1} and $Q^{n+1} \in H_0^1(\Omega)$, satisfying

$$
\rho_0^{(n+1)}\left(P^{n+1},v\right) - \left(\nabla Q^{n+1},\nabla v\right) + \left(Q^{n+1},v\right) = \sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)} - \rho_{n+1-j}^{(n+1)}\right)\left(P^j,v\right) + \rho_n^{(n+1)}\left(P^0,v\right),
$$

$$
\rho_0^{(n+1)}\left(Q^{n+1},w\right) + \left(\nabla P^{n+1},\nabla w\right) - \left(P^{n+1},w\right) = \sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)} - \rho_{n+1-j}^{(n+1)}\right)\left(Q^j,w\right) + \rho_n^{(n+1)}\left(Q^0,w\right),
$$

for all v and $w \in H_0^1(\Omega)$. We introduce the FE space, a subspace of $H_0^1(\Omega)$, as

$$
V_h := \left\{ v \in V := C(\bar{\Omega}) \cap H_0^1(\Omega) : \ v\Big|_E \in P_k(E), \ \forall E \in \Omega_h \right\}, \tag{11}
$$

where $P_k(E)$ denotes the set of polynomials whose degrees are no more than k. Here, we use the space of continuous piecewise linear functions vanishing on $\partial\Omega$.

Now, suppose that $\left(P_h^{n+1}, Q_h^{n+1}\right) \in V_h \times V_h$ is the approximations of $(P(\cdot,t_{n+1}), Q(\cdot,t_{n+1})) \in H_0^1(\Omega) \times H_0^1(\Omega)$, respectively. Then, the implicit fully discrete finite element method is defined as: Find $\left(P_h^{n+1}, Q_h^{n+1}\right) \in V_h \times V_h$, such that

$$\begin{aligned}
&\rho_0^{(n+1)}\left(P_h^{n+1}, v_h\right) - \left(\nabla Q_h^{n+1}, \nabla v_h\right) + \left(Q_h^{n+1}, v_h\right) \\
&\qquad = \sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)} - \rho_{n+1-j}^{(n+1)}\right)\left(P_h^{j}, v_h\right) + \rho_n^{(n+1)}\left(P_h^0, v_h\right), \\
&\rho_0^{(n+1)}\left(Q_h^{n+1}, w_h\right) + \left(\nabla P_h^{n+1}, \nabla w_h\right) - \left(P_h^{n+1}, w_h\right) \\
&\qquad = \sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)} - \rho_{n+1-j}^{(n+1)}\right)\left(Q_h^{j}, w_h\right) + \rho_n^{(n+1)}\left(Q_h^0, w_h\right),
\end{aligned} \tag{12}$$

for all v_h and $w_h \in V_h \times V_h$.

4 Theoretical analysis

This section focuses on the stability analysis and error estimate of the proposed scheme.

4.1 Stability analysis

Here, we derive the unconditional stability results of the L1-FEM.

Theorem 1. *Let $(P_h^{n+1}, Q_h^{n+1}) \in V_h \times V_h$ be the approximate solution of system (8). Then for boundary conditions considered in (1), the fully discrete finite element scheme (12) is unconditionally stable, and the numerical solution $(P_h^{n+1}, Q_h^{n+1}) \in V_h \times V_h$ satisfies*

$$\|P_h^m\|^2 + \|Q_h^m\|^2 \le \|P_h^0\|^2 + \|Q_h^0\|^2, \qquad m = 1, 2, \ldots, N. \tag{13}$$

Proof. Setting $v = P_h^{n+1}$ and $w = Q_h^{n+1}$ in Eq. (12) and summing up the equations, one obtains

$$\begin{aligned}
\rho_0^{(n+1)}\left(\|P_h^{n+1}\|^2 + \|Q_h^{n+1}\|^2\right) = &\sum_{i=1}^{n}\left(\rho_{n-i}^{(n+1)} - \rho_{n+1-i}^{(n+1)}\right)\left(P_h^i, P_h^{n+1}\right) + \rho_n^{(n+1)}\left(P_h^0, P_h^{n+1}\right) \\
&+ \sum_{i=1}^{n}\left(\rho_{n-i}^{(n+1)} - \rho_{n+1-i}^{(n+1)}\right)\left(Q_h^i, Q_h^{n+1}\right) + \rho_n^{(n+1)}\left(Q_h^0, Q_h^{n+1}\right).
\end{aligned} \tag{14}$$

If $n = 0$, then

$$\rho_0^{(1)}\left(\|P_h^1\|^2 + \|Q_h^1\|^2\right) = \rho_0^{(1)}\left(P_h^0, P_h^1\right) + \rho_0^{(1)}\left(Q_h^0, Q_h^{n+1}\right), \tag{15}$$

and Youngs inequalities gives

$$\|P_h^1\|^2 + \|Q_h^1\|^2 \le \|P_h^0\|^2 + \|Q_h^0\|^2.$$

Now, suppose that for $m = 0, 1, \ldots, K$,

$$\|P_h^m\|^2 + \|Q_h^m\|^2 \le \|P_h^0\|^2 + \|Q_h^0\|^2. \tag{16}$$

If in (14), we take $n = K$, we arrive at

$$\begin{aligned}\rho_0^{(K+1)}\left(\|P_h^{K+1}\|^2+\|Q_h^{K+1}\|^2\right) \le \frac{1}{2}\sum_{i=1}^{K}\left(\rho_{K-i}^{(K+1)}-\rho_{K+1-i}^{(K+1)}\right)\left(\|P_h^{i}\|^2+\|P_h^{K+1}\|^2+\|Q_h^{i}\|^2+\|Q_h^{K+1}\|^2\right)\\ +\frac{1}{2}\rho_K^{(K+1)}\left(\|P_h^{0}\|^2+\|P_h^{K+1}\|^2+\|Q_h^{0}\|^2+\|Q_h^{K+1}\|^2\right).\end{aligned} \tag{17}$$

Using Eq. (16) in Eq. (17) yields

$$\begin{aligned}\rho_0^{(K+1)}\left(\|P_h^{K+1}\|^2+\|Q_h^{K+1}\|^2\right) &\le \frac{1}{2}\left[\sum_{i=1}^{K}\left(\rho_{K-i}^{(K+1)}-\rho_{K+1-i}^{(K+1)}\right)+\rho_K^{(K+1)}\right]\left(\|P_h^{K+1}\|^2+\|Q_h^{K+1}\|^2\right)\\ &+\frac{1}{2}\left[\sum_{i=1}^{K}\left(\rho_{K-i}^{(K+1)}-\rho_{K+1-i}^{(K+1)}\right)+\rho_K^{(K+1)}\right]\left(\|P_h^{0}\|^2+\|Q_h^{0}\|^2\right)\\ &=\frac{1}{2}\rho_0^{(K+1)}\left(\|P_h^{K+1}\|^2+\|Q_h^{K+1}\|^2\right)+\frac{1}{2}\rho_0^{(K+1)}\left(\|P_h^{0}\|^2+\|Q_h^{0}\|^2\right),\end{aligned} \tag{18}$$

which concludes the desired result $\|P_h^{K+1}\|^2+\|Q_h^{K+1}\|^2 \le \|P_h^{0}\|^2+\|Q_h^{0}\|^2$. □

4.2 Error estimate

Here, we consider Eq. (1) with the variational formulations that is given as: Find P^{n+1} and $Q^{n+1} \in H_0^1(\Omega)$, satisfying

$$\begin{aligned}&\rho_0^{(n+1)}\left(P^{n+1},v\right)-\left(\nabla Q^{n+1},\nabla v\right)+\left(Q^{n+1},v\right)\\ &\qquad=\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(P^{j},v\right)+\rho_n^{(n+1)}\left(P^{0},v\right)+\left(\mathscr{R}_1,v\right),\\ &\rho_0^{(n+1)}\left(Q^{n+1},w\right)+\left(\nabla P^{n+1},\nabla w\right)-\left(P^{n+1},w\right)\\ &\qquad=\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(Q^{j},w\right)+\rho_n^{(n+1)}\left(Q^{0},w\right)+\left(\mathscr{R}_2,w\right),\end{aligned} \tag{19}$$

for all v and $w \in H_0^1(\Omega)$ where $\mathscr{R}_j = \mathscr{O}\left(\Delta t^{\delta}\right)$ results from (7) for $j = 1,2$. On the other hand, the FEM for solving (19) will be in the following way

$$\begin{aligned}&\rho_0^{(n+1)}\left(P_h^{n+1},v\right)-\left(\nabla Q_h^{n+1},\nabla v\right)+\left(Q_h^{n+1},v\right)\\ &\qquad=\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(P_h^{j},v\right)+\rho_n^{(n+1)}\left(P_h^{0},v\right),\\ &\rho_0^{(n+1)}\left(Q_h^{n+1},w\right)+\left(\nabla P_h^{n+1},\nabla w\right)-\left(P_h^{n+1},w\right)\\ &\qquad=\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(Q_h^{j},w\right)+\rho_n^{(n+1)}\left(Q_h^{0},w\right),\end{aligned} \tag{20}$$

with $P_h^0 = \mathscr{R}_h P_0$ and $Q_h^0 = \mathscr{R}_h Q_0$, where $\mathscr{R}_h : H_0^1 \to V_h$ is the Ritz projection introduced by

$$\left(\nabla\left(U-\mathscr{R}_h U\right),\nabla v_h\right)=0, \qquad \forall v_h \in V_h, \tag{21}$$

which satisfies the following property

$$\|U-\mathbb{R}_h U\|+h\|\nabla(U-\mathbb{R}_h U)\| \leq Ch^2, \qquad \forall U \in H_0^1 \cap H^2. \tag{22}$$

More details can be found in [7, 33].

Theorem 2. *Suppose $\Psi(\cdot,\cdot,t_n)=P(\cdot,\cdot,t_n)+iQ(\cdot,\cdot,t_n)$ be the solution of (8) and P_h^{n+1} and Q_h^{n+1} be the approximations of $P(\cdot,\cdot,t_n)$ and $Q(\cdot,\cdot,t_n)$ by the L1-FEM (20), respectively. Then, we have the following error estimate*

$$\|P(\cdot,\cdot,t_n)-P_h^{n+1}\|+\|Q(\cdot,\cdot,t_n)-Q_h^{n+1}\| \leq C\left(h^2+\Delta t^{\delta}\right), \tag{23}$$

where C is a positive constant and independent of h.

Proof. We denote

$$e_P = P-P_h = (\mathscr{R}_h P - P_h)-(\mathscr{R}_h P - P) := \eta_P - \xi_P, \tag{24a}$$
$$e_Q = Q-Q_h = (\mathscr{R}_h Q - Q_h)-(\mathscr{R}_h Q - Q) := \eta_Q - \xi_Q. \tag{24b}$$

and subtract (20) from the variational formulation of (19) to get the following error equations

$$\begin{aligned}\rho_0^{(n+1)}\left(e_P^{n+1},v\right)-\left(\nabla e_Q^{n+1},\nabla v\right)+\left(e_Q^{n+1},v\right)&\\ =\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(e_P^j,v\right)+\rho_n^{(n+1)}\left(e_P^0,v\right)+\mathscr{R}_1,&\end{aligned} \tag{25a}$$

$$\begin{aligned}\rho_0^{(n+1)}\left(e_Q^{n+1},w\right)+\left(\nabla e_P^{n+1},\nabla w\right)-\left(e_P^{n+1},w\right)&\\ =\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(e_Q^j,w\right)+\rho_n^{(n+1)}\left(e_Q^0,w\right)+\mathscr{R}_2.&\end{aligned} \tag{25b}$$

Substituting (24) in Eqs. (25) and summing up the equations, we have

$$\begin{aligned}&\rho_0^{(n+1)}\left(\eta_P^{n+1},v\right)-\left(\nabla\eta_Q^{n+1},\nabla v\right)+\left(\eta_Q^{n+1},v\right)+\rho_0^{(n+1)}\left(\eta_Q^{n+1},w\right)+\left(\nabla\eta_P^{n+1},\nabla w\right)-\left(\eta_P^{n+1},w\right)\\
&\quad=\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\eta_P^j,v\right)+\rho_n^{(n+1)}\left(\eta_P^0,v\right)\\
&\quad\quad+\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\eta_Q^j,w\right)+\rho_n^{(n+1)}\left(\eta_Q^0,w\right)\\
&\quad\quad+\rho_0^{(n+1)}\left(\xi_P^{n+1},v\right)-\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\xi_P^j,v\right)-\rho_n^{(n+1)}\left(\xi_P^0,v\right)\\
&\quad\quad+\rho_0^{(n+1)}\left(\xi_Q^{n+1},w\right)-\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\xi_Q^j,w\right)-\rho_n^{(n+1)}\left(\xi_Q^0,w\right)\\
&\quad\quad-\left(\nabla\xi_Q^{n+1},\nabla v\right)+\left(\xi_Q^{n+1},v\right)+\left(\nabla\xi_P^{n+1},\nabla w\right)-\left(\xi_P^{n+1},w\right)+\left(\mathscr{R}_1,v\right)+\left(\mathscr{R}_2,w\right).\end{aligned} \tag{26}$$

Taking the test functions $v=\eta_P^{n+1}$ and $w=\eta_Q^{n+1}$ in (26) and using the properties (21), one gets

$$\begin{aligned}
\rho_0^{(n+1)}\|\eta_P^{n+1}\|^2+\rho_0^{(n+1)}\|\eta_Q^{n+1}\|^2 &= \sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\eta_P^j,\eta_P^{n+1}\right)+\rho_n^{(n+1)}\left(\eta_P^0,\eta_P^{n+1}\right)\\
&+\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\eta_Q^j,\eta_Q^{n+1}\right)+\rho_n^{(n+1)}\left(\eta_Q^0,\eta_Q^{n+1}\right)\\
&+\rho_0^{(n+1)}\left(\xi_P^{n+1},\eta_P^{n+1}\right)+\rho_0^{(n+1)}\left(\xi_Q^{n+1},\eta_Q^{n+1}\right)\\
&-\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\xi_P^j,\eta_P^{n+1}\right)-\rho_n^{(n+1)}\left(\xi_P^0,\eta_P^{n+1}\right)\\
&-\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\xi_Q^j,\eta_Q^{n+1}\right)-\rho_n^{(n+1)}\left(\xi_Q^0,\eta_Q^{n+1}\right)\\
&+\left(\xi_Q^{n+1},\eta_P^{n+1}\right)-\left(\xi_P^{n+1},\eta_Q^{n+1}\right)+\left(\mathscr{R}_1,\eta_P^{n+1}\right)+\left(\mathscr{R}_2,\eta_Q^{n+1}\right).
\end{aligned}\tag{27}$$

Since, the coefficients $\rho_j^{(n+1)}$, $j\geq 0$ are strictly monotonically decreasing with respect to j, we can apply the Cauchy-Schwarz inequality in (27) to obtain the following result:

$$\begin{aligned}
&\|\eta_P^{n+1}\|^2+\|\eta_Q^{n+1}\|^2\\
&\leq\left[\frac{\rho_n^{(n+1)}}{\rho_0^{(n+1)}}\left(\|\eta_P^0\|+\|\xi_P^0\|\right)+\frac{1}{\rho_0^{(n+1)}}\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\|\eta_P^j\|+\|\xi_P^j\|\right)+\|\xi_P^{n+1}\|+\mathscr{O}(\Delta t^{\delta})\right]\|\eta_P^{n+1}\|\\
&+\left[\frac{\rho_n^{(n+1)}}{\rho_0^{(n+1)}}\left(\|\eta_Q^0\|+\|\xi_Q^0\|\right)+\frac{1}{\rho_0^{(n+1)}}\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\|\eta_Q^j\|+\|\xi_Q^j\|\right)+\|\xi_Q^{n+1}\|+\mathscr{O}(\Delta t^{\delta})\right]\|\eta_Q^{n+1}\|.
\end{aligned}\tag{28}$$

Young's inequality $ab\leq\varepsilon a^2+\frac{1}{4\varepsilon}b^2$, together with Holder's inequality give

$$\begin{aligned}
&\|\eta_P^{n+1}\|^2+\|\eta_Q^{n+1}\|^2\\
&\leq\frac{1}{4\varepsilon}\left[\frac{\rho_n^{(n+1)}}{\rho_0^{(n+1)}}\left(\|\eta_P^0\|+\|\xi_P^0\|\right)+\frac{1}{\rho_0^{(n+1)}}\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\|\eta_P^j\|+\|\xi_P^j\|\right)+\|\xi_P^{n+1}\|+\mathscr{O}(\Delta t^{\delta})\right]^2\\
&+\frac{1}{4\varepsilon}\left[\frac{\rho_n^{(n+1)}}{\rho_0^{(n+1)}}\left(\|\eta_Q^0\|+\|\xi_Q^0\|\right)+\frac{1}{\rho_0^{(n+1)}}\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)\left(\|\eta_Q^j\|+\|\xi_Q^j\|\right)+\|\xi_Q^{n+1}\|+\mathscr{O}(\Delta t^{\delta})\right]^2\\
&+\varepsilon\|\eta_P^{n+1}\|^2+\varepsilon\|\eta_Q^{n+1}\|^2.
\end{aligned}\tag{29}$$

When $n=0$, for a small value of ε, we have

$$\|\eta_P^1\|^2+\|\eta_Q^1\|^2\leq C\left[\|\eta_P^0\|+\|\xi_P^0\|+\|\xi_P^1\|+\mathscr{O}(\Delta t^{\delta})\right]^2+C\left[\|\eta_Q^0\|+\|\xi_Q^0\|+\|\xi_Q^1\|+\mathscr{O}(\Delta t^{\delta})\right]^2.\tag{30}$$

Thus, (30) and the property (22) yield

$$\|\eta_P^1\|^2+\|\eta_Q^1\|^2 \leq C_1\left[h^2+\mathscr{O}(\Delta t^{\delta})\right]^2.$$

Now, assume that for $m=0,1,\ldots,K$ the following inequality holds:

$$\|\eta_P^m\|^2+\|\eta_Q^m\|^2 \leq C_m\left[h^2+\mathscr{O}(\Delta t^{\delta})\right]^2.$$

Notice the fact that $\rho_n^{(n+1)}+\sum_{j=1}^{n}\left(\rho_{n-j}^{(n+1)}-\rho_{n+1-j}^{(n+1)}\right)=\rho_0^{(n+1)}$, then

$$\|\eta_P^{K+1}\|^2+\|\eta_Q^{K+1}\|^2 \leq C_{K+1}\left[h^2+\mathscr{O}(\Delta t^{\delta})\right]^2,$$

which gives the following outcome

$$\|\eta_P^{K+1}\|+\|\eta_Q^{K+1}\| \leq C\left[h^2+\mathscr{O}(\Delta t^{\delta})\right].$$

The triangle inequality and (22) conclude Theorem 2. □

5 Numerical experiments

In this section, we present some experimental results demonstrating the applicability and accuracy of the proposed scheme. We employ the L1-FEM to solve time-fractional SEs. In this process, we consider the following variable-order functions [17]

$$\alpha_1(t)=\frac{2+\sin(t)}{4},\quad \alpha_2(t)=0.85-0.25e^{-t},\quad \alpha_3(t)=0.65+0.25t^3\cos(t).$$

The following cases are called under zero initial and boundary conditions. We also use the time step size $\Delta t=h$ in the proposed method. In our examples, we present the L^2 norm and convergence rate of errors that is introduced as below

$$\|e_h\|^2=\int_{\Omega}(v-v_h)^2\,d\boldsymbol{x}\,,\qquad rate=\frac{\ln\left(\frac{\|e_{h_1}\|}{\|e_{h_2}\|}\right)}{\ln\left(\frac{h_1}{h_2}\right)},\tag{31}$$

where v and v_h are the exact and numerical solutions of the problem.

Example 1. This example demonstrates the accuracy of the proposed L1-FEM for the linear HH time-fractional SE

$$i\,{}_{0}^{HH}\mathrm{D}^{\alpha(t)}\Psi+\Delta\Psi=g\left(x,y,t\right),\qquad in\ \ \Omega\times(0,T],\tag{32}$$

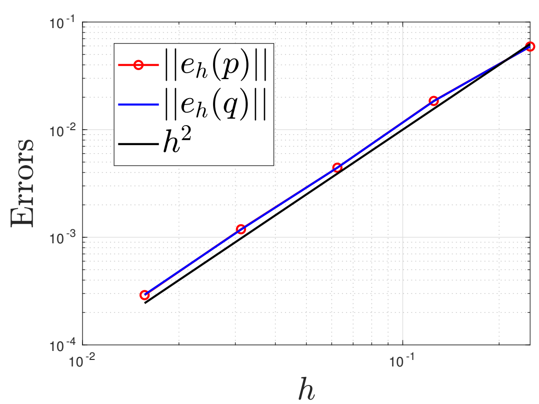
where $\Omega=[0,1]\times[0,1]$ and

$$g(x,y,t)=-\left[\frac{m!}{1-\alpha(t)}t^m E_{1,m+1}\left(-\frac{t\alpha(t)}{1-\alpha(t)}\right)\sin(2\pi x)\sin(\pi y)+5\pi^2 t^m\sin(\pi x)\sin(2\pi y)\right]$$
$$+i\left[\frac{m!}{1-\alpha(t)}t^m E_{1,m+1}\left(-\frac{t\alpha(t)}{1-\alpha(t)}\right)\sin(\pi x)\sin(2\pi y)-5\pi^2 t^m\sin(2\pi x)\sin(\pi y)\right]. \tag{33}$$

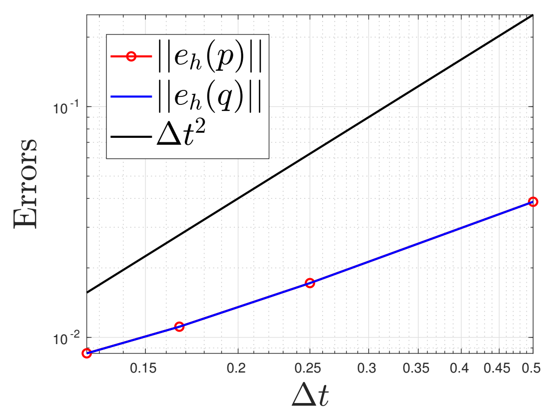
In this case, g belongs to $L^2(\Omega)$ and the exact solution is

$$\Psi(x,y,t)=t^m\left(\sin(\pi x)\sin(2\pi y)+i\sin(2\pi x)\sin(\pi y)\right). \tag{34}$$

The final time is taken $T=1$ and $m=3$.



(a) using $\Delta t=\dfrac{1}{100}$ and various h.



(b) using $h=\dfrac{1}{100}$ and various Δt.

Figure 1: Convergence plots of the scheme for variable-order functions $\alpha_1(t)$.

The L^2 errors and its convergence rate for various variable-order functions $\alpha(t)$ when $\Delta t=h$ are presented in Table 1. One can see that the order of convergence using linear piecewise continuous polynomials for real and imaginary parts gives the second order of accuracy in the L^2 norm. The corresponding CPU time is depicted in Figure 2.

Example 2. Let us consider the SE

$$i\,{}_{0}^{HH}\mathrm{D}^{\alpha(t)}\Psi+\Delta\Psi+\Psi=g(x,y,t), \qquad in\ \ \Omega\times(0,T], \tag{35}$$

such that the exact solution is $\Psi(x,y,t)=t^m\left(\sin(2\pi x)\sin(\pi y)+i\sin(\pi x)\sin(2\pi y)\right)$. Table 2 shows the errors and its convergence rate of the L1-FEM with $m=3$. The results demonstrate that the scheme's accuracy is second order in space.

Example 3. In the third example, we consider Eq. (35) subject to zero initial and BCs on a L-shaped region of Figure 3. The source terms are taken such that the exact solution be

$$\Psi(x,y,t)=t^3\left(x^2(x^2-1)y(y^2-1)+ix(x^2-1)y^2(y^2-1)\right). \tag{36}$$

The triangulation used for partitioning the computational domain is illustrated in Figure 4. Results presented in Table 3 show the second order of accuracy in space.

Table 1: Errors and convergence rate of the L1-FEM for Example 1.

	$\Delta t = h$	$\alpha_1(t)$ $\|e_h\|$	$\alpha_1(t)$ rate	$\alpha_2(t)$ $\|e_h\|$	$\alpha_2(t)$ rate	$\alpha_3(t)$ $\|e_h\|$	$\alpha_3(t)$ rate
p	$\frac{1}{4}$	5.9207×10^{-2}	–	5.9208×10^{-2}	–	5.9204×10^{-2}	–
	$\frac{1}{8}$	1.8487×10^{-2}	1.6793	1.8487×10^{-2}	1.6793	1.8485×10^{-2}	1.6793
	$\frac{1}{16}$	4.4305×10^{-3}	2.0609	4.4301×10^{-3}	2.0611	4.4292×10^{-3}	2.0613
	$\frac{1}{32}$	1.1872×10^{-3}	1.8999	1.1872×10^{-3}	1.8998	1.1870×10^{-3}	1.8997
	$\frac{1}{64}$	2.9021×10^{-4}	2.0324	2.9022×10^{-4}	2.0324	2.9018×10^{-4}	2.0323
q	$\frac{1}{4}$	5.9215×10^{-2}	–	5.9216×10^{-2}	–	5.9211×10^{-2}	–
	$\frac{1}{8}$	1.8415×10^{-2}	1.6851	1.8415×10^{-2}	1.6851	1.8412×10^{-2}	1.6852
	$\frac{1}{16}$	4.4375×10^{-3}	2.0530	4.4381×10^{-3}	2.0528	4.4379×10^{-3}	2.0527
	$\frac{1}{32}$	1.1792×10^{-3}	1.9119	1.1793×10^{-3}	1.9121	1.1792×10^{-3}	1.9121
	$\frac{1}{64}$	2.9285×10^{-4}	2.0096	2.9286×10^{-4}	2.0096	2.9282×10^{-4}	2.0097

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6 Conclusion

In the current study, we developed an implicit fully discrete continuous Galerkin finite element method for the solution of VO time-fractional 2D Schrödinger equation with a nonsingular Mittag-Leffler kernel. We used a finite difference scheme known as the L1 formula and finite element method for the temporal and spatial discretizations, respectively. Stability analysis confirms the unconditional stability of the resulting scheme. Additionally, we provided an error estimate. The conducted numerical experiments show the applicability and accuracy of the proposed scheme. Our ongoing work focuses on extending the scheme to higher-order schemes.

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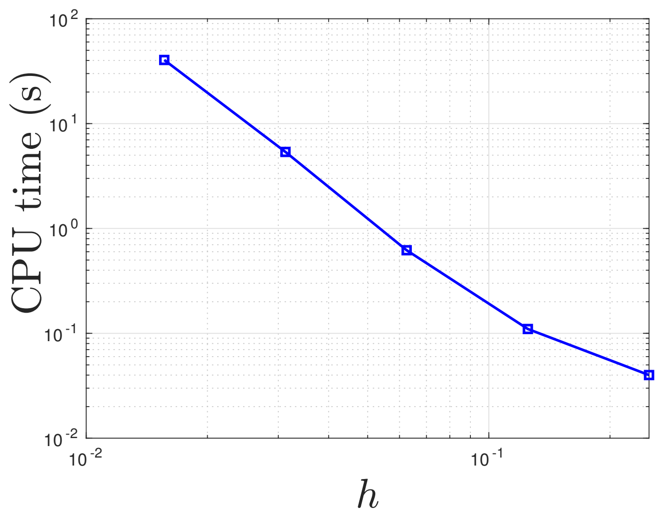

Figure 2: CPU time of implementation of the proposed scheme.

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Table 2: Errors and convergence rate of the L1-FEM for Example 2.

		$\alpha_1(t)$		$\alpha_2(t)$		$\alpha_3(t)$	
	$\Delta t = h$	$\Vert e_h \Vert$	*rate*	$\Vert e_h \Vert$	*rate*	$\Vert e_h \Vert$	*rate*
p	$\frac{1}{4}$	6.0255×10^{-2}	–	6.0256×10^{-2}	–	6.0252×10^{-2}	–
	$\frac{1}{8}$	1.8779×10^{-2}	1.6819	1.8780×10^{-2}	1.6819	1.8778×10^{-2}	1.682
	$\frac{1}{16}$	4.5140×10^{-3}	2.0567	4.5136×10^{-3}	2.0569	4.5127×10^{-3}	2.0570
	$\frac{1}{32}$	1.2014×10^{-3}	1.9096	1.2014×10^{-3}	1.9095	1.2012×10^{-3}	1.9095
	$\frac{1}{64}$	2.9855×10^{-4}	2.0087	2.9855×10^{-4}	2.0087	2.9851×10^{-4}	2.0086
q	$\frac{1}{4}$	6.0245×10^{-2}	–	6.0246×10^{-2}	–	6.0241×10^{-2}	–
	$\frac{1}{8}$	1.8835×10^{-2}	1.6775	1.8834×10^{-2}	1.6775	1.8832×10^{-2}	1.6776
	$\frac{1}{16}$	4.5294×10^{-3}	2.0560	4.5300×10^{-3}	2.0558	4.5298×10^{-3}	2.0557
	$\frac{1}{32}$	1.2115×10^{-3}	1.9025	1.2116×10^{-3}	1.9026	1.2114×10^{-3}	1.9027
	$\frac{1}{64}$	2.9597×10^{-4}	2.0333	2.9598×10^{-4}	2.0333	2.9595×10^{-4}	2.0333

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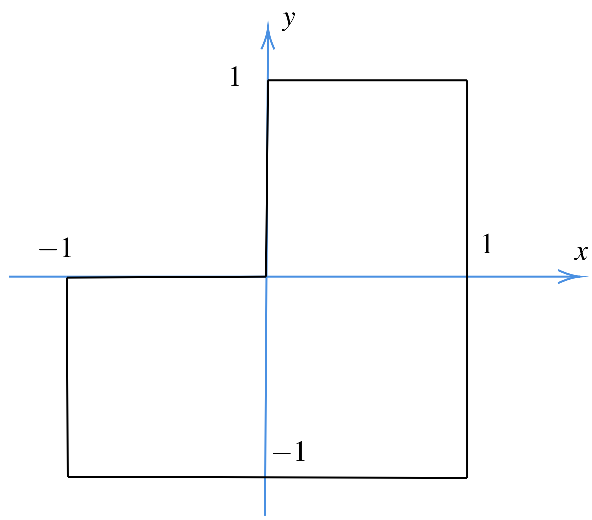

Figure 3: The L-shaped domain used in Example 3.

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Table 3: Errors and convergence rate of the L1-FEM for Example 3.

		$\alpha_1(t)$		$\alpha_2(t)$		$\alpha_3(t)$	
	$\Delta t = h$	$\|e_h\|$	*rate*	$\|e_h\|$	*rate*	$\|e_h\|$	*rate*
p	$\frac{1}{4}$	1.0068×10^{-2}	–	1.0039×10^{-2}	–	1.0015×10^{-2}	–
	$\frac{1}{8}$	2.9992×10^{-3}	1.7471	2.9919×10^{-3}	1.7464	2.9857×10^{-3}	1.7460
	$\frac{1}{16}$	9.4997×10^{-4}	1.6586	9.4769×10^{-4}	1.6586	9.4582×10^{-4}	1.6585
	$\frac{1}{32}$	2.4519×10^{-4}	1.9540	2.4464×10^{-4}	1.9538	2.4418×10^{-4}	1.9536
	$\frac{1}{64}$	6.2155×10^{-5}	1.9799	6.2040×10^{-5}	1.9794	6.1941×10^{-5}	1.9790
q	$\frac{1}{4}$	1.0596×10^{-2}	–	1.0626×10^{-2}	–	1.0641×10^{-2}	–
	$\frac{1}{8}$	3.1783×10^{-3}	1.7371	3.1864×10^{-3}	1.7376	3.1904×10^{-3}	1.7378
	$\frac{1}{16}$	9.7691×10^{-4}	1.7020	9.7913×10^{-4}	1.7024	9.8020×10^{-4}	1.7026
	$\frac{1}{32}$	2.4964×10^{-4}	1.9684	2.5015×10^{-4}	1.9687	2.5039×10^{-4}	1.9689
	$\frac{1}{64}$	6.4112×10^{-5}	1.9612	6.4232×10^{-5}	1.9614	6.4288×10^{-5}	1.9615

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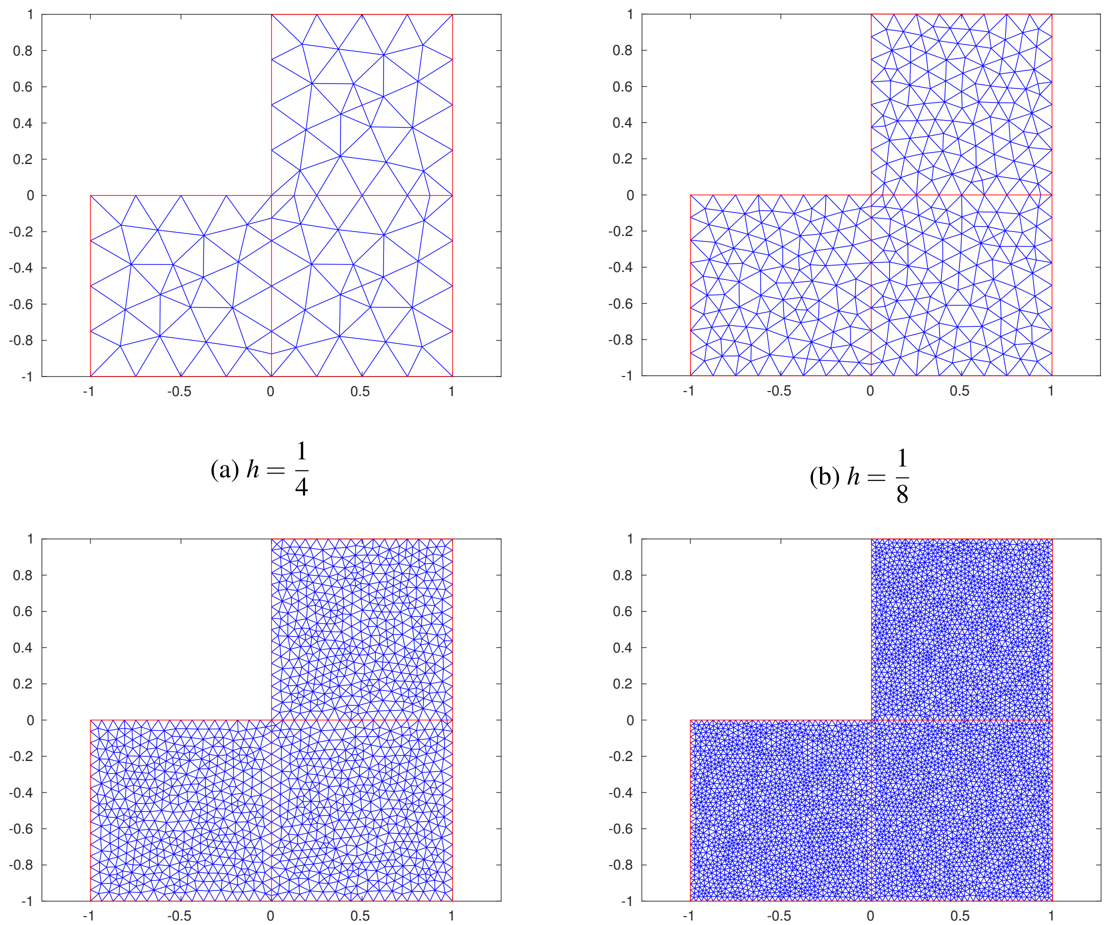

(a) $h = \frac{1}{4}$

(b) $h = \frac{1}{8}$

(c) $h = \frac{1}{16}$

(d) $h = \frac{1}{32}$

Figure 4: The triangulation used for partitioning the computational domain.

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