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Application of compact local integrated RBFs technique to solve fourth-order time-fractional diffusion-wave system

Mostafa Abbaszadeh[†], **AliReza Bagheri Salec**[‡*], **Alaa Salim Jebur**[‡]

[†]*Department of Applied Mathematics, Faculty of Mathematics and Computer Sciences, Amirkabir University of Technology (Tehran Polytechnic), No. 424, Hafez Ave., 15914 Tehran, Iran*
[‡]*Department of Mathematics, Faculty of Basic Scince, University of Qom, Alghadir Blvd., Qom, Iran*
Email(s): m.abbaszadeh@aut.ac.ir, r-bagheri@qom.ac.ir, z.alaa30@gmail.com

Abstract. The main aim of the current paper is to apply the compact local integrated RBFs technique to the numerical solution of the fourth-order time-fractional diffusion-wave system. A finite difference formula is employed to obtain a time-discrete scheme. The stability and convergence rate of the semi-discrete plan are proved by the energy method. A new unknown variable is defined to obtain a second-order system of PDEs. Then, the compact local integrated radial basis functions (RBFs) is used to approximate the spatial derivative. The utilized numerical method is a truly meshless technique. The numerical approach put forth is genuinely meshless, allowing for the utilization of irregular physical domains in obtaining numerical solutions.

Keywords: Time fractional PDEs, integrated RBFs, radial basis functions (RBFs), stability, convergence.
AMS Subject Classification 2010: 65M12, 65N12, 65N30, 65N40.

1 Introduction

In recent years, there has been a growing interest in fractional calculus, as evidenced by numerous studies [31, 32]. The field of fractional differential equations has garnered increasing attention due to its diverse applications in various scientific and engineering domains [9]. Phenomena in fluid mechanics, viscoelasticity, chemistry, physics, finance, and other disciplines find successful description through models employing mathematical tools from fractional calculusspecifically, the theory of derivatives and integrals of fractional order. Notable applications are highlighted in works such as Oldham and Spanier's book [30], Podlubny's book [32], Bagley and Trovik [6]. Despite numerous theoretical analyses [11,40], obtaining explicit analytic solutions for most fractional differential equations remains a challenge. Consequently, many authors have turned to numerical solution strategies, emphasizing convergence and sta-

*Corresponding author
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bility analysis [7–9,20,35,36,41,43]. Liu has contributed significantly to the finite difference method for solving fractional differential equations [21–23]. There are various definitions for a fractional derivative of order $\alpha > 0$ [30, 31], with the Riemann-Liouville and Caputo being the two most commonly used. The key distinction between these definitions lies in the order of evaluation [29].

Definition 1. *The left fractional integral of function $f \in H^1([a,b])$ with order $\alpha > 0$ is defined as*

$$ {}_a\mathscr{D}_t^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s. $$

Definition 2. *The left and right Caputo fractional derivative of function $f \in H^1([a,b])$ with order $\alpha > 0$ are defined, respectively, as*

$$ {}_a^C\mathscr{D}_t^{\alpha} f(x) = {}_aD_t^{-(m-\alpha)} \left[f^{(m)}(x) \right] = \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-s)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s, $$

$$ {}_x^C\mathscr{D}_b^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (s-x)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s. $$

In this paper, we consider the multi-dimensional time-fractional fourth-order diffusion-wave equation

$$ {}_a^C\mathscr{D}_t^{\alpha} u(x,y,t) + \Delta(\Delta u(x,y,t)) = f(x,y,t), \quad x,y \in \Omega, \quad 0 \le t \le T, \qquad 1 < \alpha < 2, \tag{1} $$

with initial and boundary conditions

$$ u(x,y,0) = \psi(x,y), \qquad \frac{\partial u(x,y,0)}{\partial t} = \phi(x,y), \quad x,y \in \Omega, \tag{2} $$

$$ u(x,y,t) = \varphi_1(x,y,t), \quad \Delta u(x,y,t) = \varphi_2(x,y,t), \quad x,y \in \partial\Omega, \quad 0 \le t \le T. \tag{3} $$

Fractional partial differential equations (PDEs) have been a subject of investigation through both analytical and numerical techniques. In [16], the authors proposed a finite difference method for solving the fourth-order fractional diffusion-wave system. Meanwhile, the primary focus of [4,5] was to propose a general solution for a fourth-order fractional diffusionwave equation within a bounded spatial domain.

Various researchers have explored the application of non-uniform meshes to solve time-fractional PDEs. Examples include finite difference methods [3, 19], spectral Galerkin methods [17], Galerkin-Legendre spectral approximation [18], and Jacobi Spectral Galerkin methods [15]. In [28], a binary fractional reproducing kernel collocation method based on the Caputo-Fabrizio derivative was proposed for solving the time-fractional Cattaneo equation. Additionally, [42] employed a computationally effective implicit difference approximation to solve the time fractional diffusion equation.

Combining finite difference techniques with spectral collocation methods, [27] presented a numerical solution for semilinear time fractional convection-reaction-diffusion equations with time delay. The objective of [34] was to propose a numerical solution for the space-time variable fractional order advection-dispersion equation using radial basis functions. In [26], the authors studied the combination of the Sinc and Gaussian radial basis functions (GRBF) to develop numerical methods for time-space fractional diffusion equations with the Riesz fractional derivative.

On the numerical side, the global Integrated Radial Basis Function (IRBF) technique was explored in [33]. Experiments in [24,25,33] indicated that the results of IRBFs methods were more accurate than classic RBFs collocation approaches. [38,39] introduced a compact local IRBFs approach based on three points in each stencil to solve differential equations. The convection-diffusion equation was addressed in [37] through a combination of the Alternating Direction Implicit (ADI) method and Compact Local IRBF (CLIRBF) approximations. Other applications of the IRBF method for solving various PDEs can be found in [1,2,10,12–14].

The rest of this article is structured as follows: The time-discrete scheme is introduced in Section 2. In Section 3, the numerical methodology, including spatial discretization by IRBF method is proposed. Section 4 is devoted to the results of numerical simulations. Finally, this article is ended with a brief conclusion in Section 5.

2 Time-discrete formulation

Now, let $t_n = ndt$ for $n = 0,1,2,\ldots,N_T$ and $dt = T/N_T$

$$u^{n-\frac{1}{2}} = \frac{1}{2}\left(u^n + u^{n-1}\right), \quad \delta_t u^{n-\frac{1}{2}} = \frac{1}{dt}\left(u^n - u^{n-1}\right),$$

where $u^n = u(x,y,t_n)$. In this paper, the capital and small letters denote the approximate and exact solutions, respectively. The following lemma is needed to discrete the fractional derivative.

Lemma 1. *[35] If $f(t) \in C^2[0,t_n]$ and $1 < \alpha < 2$, then*

$$\left| \int_0^{t_n} (t_n - s)^{1-\alpha} f'(s)dt - \frac{1}{dt}\left[a_0 f(t_n) - \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k}) f(t_k) - a_{n-1} f(t_0) \right] \right| \le Cdt^{3-\alpha},$$

where

$$\begin{aligned} C &= \frac{1}{2-\alpha}\left[\frac{2-\alpha}{12} + \frac{2^{3-\alpha}}{3-\alpha} - (1+2^{1-\alpha}) \right] \max_{0\le t\le t_n} \left| f''(t) \right|, \\ a_k &= \frac{dt^{2-\alpha}}{2-\alpha}\left[(k+1)^{2-\alpha} - k^{2-\alpha} \right]. \end{aligned}$$

Let

$$w(x,y,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial v(x,y,s)}{\partial s} \frac{ds}{(t-s)^{\alpha-1}}, \tag{4}$$

where

$$v(x,y,t) = \frac{\partial u(x,y,t)}{\partial t}, \tag{5}$$

From Eq. (1) at the point (x,y,t_n), we have

$${}_a^C\mathscr{D}_t^\alpha u(\boldsymbol{x},t_{n-\frac{1}{2}}) + \Delta(\Delta u(\boldsymbol{x},t_{n-\frac{1}{2}})) = f(\boldsymbol{x},t_{n-\frac{1}{2}}), \tag{6}$$

Now, according the above relation, the following estimations can be written

$$v^{n-\frac{1}{2}} = \delta_t u^{n-\frac{1}{2}} + (R_1)^{n-\frac{1}{2}}, \tag{7}$$

$$w^{n-\frac{1}{2}} = \Delta(\Delta u^{n+\frac{1}{2}}) + f^{n-\frac{1}{2}} + (R_2)^{n-\frac{1}{2}}, \tag{8}$$

in which for a positive constant c_1

$$\left|(R_1)^{n-\frac{1}{2}}\right| \leq c_1 dt^2 \quad , \quad \left|(R_2)^{n-\frac{1}{2}}\right| \leq c_1 \left(dt^2\right). \tag{9}$$

Assume

$$\mathscr{F}\left(u^{n-\frac{1}{2}}, q\right) = a_0 u^{n-\frac{1}{2}} - \sum_{k=1}^{n-1} \left(a_{n-k-1} - a_{n-k}\right) u^{k-\frac{1}{2}} - a_{n-1} q,$$

thus

$$w^n = \frac{1}{\Gamma(2-\alpha)} \int_0^{t_n} \frac{\partial v(x,t)}{\partial t} \frac{dt}{(t_n - t)^{\alpha - 1}} = \frac{1}{\Gamma(2-\alpha)} \frac{1}{dt} \mathscr{F}\left(v^n, \phi\right) + \mathscr{O}(dt^{3-\alpha}).$$

Now, Lemma 1 results in

$$w^{n-\frac{1}{2}} = \frac{1}{\Gamma(2-\alpha) dt} \mathscr{F}\left(v^{n-\frac{1}{2}}, \phi\right) + (R_3)^{n-\frac{1}{2}}, \tag{10}$$

and

$$\exists\, c_2 > 0, \qquad \left|(R_3)^{n-\frac{1}{2}}\right| \leq c_2 dt^{3-\alpha}. \tag{11}$$

Substituting (7) into (10), gives

$$w^{n-\frac{1}{2}} = \frac{1}{\Gamma(2-\alpha) dt} \mathscr{F}\left(\delta_t u^{n-\frac{1}{2}}, \phi\right) + \frac{1}{\Gamma(2-\alpha) dt} \mathscr{F}\left((R_1)^{n-\frac{1}{2}}, 0\right) + (R_3)^{n-\frac{1}{2}}.$$

Inserting the obtained relations into (8), yields

$$\frac{1}{\Gamma(2-\alpha) dt} \mathscr{F}\left(\delta_t u^{n-\frac{1}{2}}, \phi\right) + \Delta(\Delta u^{n-\frac{1}{2}}) = f^{n-\frac{1}{2}} + R_\alpha^{n-\frac{1}{2}}, \qquad n \geq 1, \tag{12}$$

where according to Lemma 1 there exists a positive constant $\mathscr{C}$ such that

$$|R_\alpha^{n-\frac{1}{2}}| \leq \mathscr{C} dt^{3-\alpha}. \tag{13}$$

Omitting the small term $R_\alpha^{n-\frac{1}{2}}$ results the following semi-discrete scheme

$$\begin{cases} \dfrac{1}{\Gamma(2-\alpha) dt} \mathscr{F}\left(\delta_t U^{n-\frac{1}{2}}(x,y), \phi(x,y)\right) + \Delta(\Delta U^{n-\frac{1}{2}}(x,y)) = 0, \quad n \geq 1, \\ U^0(x,y) = \psi(x,y), \qquad x,y \in \Omega, \\ U^n(x,y) = \varphi_1(x,y,t_n), \quad \Delta U(x,y,t_n) = \varphi_2(x,y,t_n), \quad x,y \in \partial\Omega. \end{cases} \tag{14}$$

Now, we use the following change of variable to obtain a full-discrete scheme

$$Q^{n-\frac{1}{2}} = \Delta U^{n-\frac{1}{2}}(x,y),$$

then, the following system of equations will be obtained

$$\begin{cases} Q^{n-\frac{1}{2}} = \Delta U^{n-\frac{1}{2}}(x,y), \\ \dfrac{1}{\Gamma(2-\alpha)dt}\mathscr{F}\left(\delta_t U^{n-\frac{1}{2}}(x,y),\phi(x,y)\right) + \Delta Q^{n-\frac{1}{2}} = 0, \quad n \geq 1, \end{cases}$$

with the initial and boundary conditions

$$\begin{cases} U^0(x,y) = \psi(x,y), \quad Q^0(x,y) = \Delta\psi(x,y) \quad x,y \in \Omega, \\ U^n(x,y) = \varphi_1(x,y,t_n), \quad Q^n(x,y) = \varphi_2(x,y,t_n), \quad x,y \in \partial\Omega. \end{cases}$$

2.1 Error estimate of the semi-discrete scheme

To analyse the time-discrete scheme, the following lemma is needed.

Lemma 2. *[35] For any* $\mathrm{D} = \{\mathrm{D}_1, \mathrm{D}_2, \ldots\}$ *and q, we obtain*

$$\sum_{n=1}^{N}\left[a_0\mathrm{D}_n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\mathrm{D}_k - a_{n-1}q\right]\mathrm{D}_n \geq \frac{t_N^{1-\alpha}}{2}dt\sum_{n=1}^{N}\mathrm{D}_n^2 - \frac{t_N^{2-\alpha}}{2(2-\alpha)}q^2,, \quad N = 1,2,\ldots,$$

where a_l are defined in Lemma 1.

To analyse the stability of the proposed numerical scheme, we examine the homogeneous version of Eq. (14) as follows:

$$\begin{cases} \dfrac{1}{\Gamma(2-\alpha)dt}\mathscr{F}\left(\delta_t U^{n-\frac{1}{2}}(x,y),\phi(x,y)\right) + \Delta(\Delta U^{n-\frac{1}{2}}(x,y)) = 0, \quad n \geq 1, \\ U^0(x,y) = \psi(x,y), \qquad x,y \in \Omega, \\ U^n(x,y) = 0, \quad \Delta U(x,y,t_n) = 0, \quad x,y \in \partial\Omega. \end{cases} \tag{15}$$

Thus, the variational weak form of problem (15) is to find $U_h^{n-\frac{1}{2}} \in H_0^2(\Omega)$ such that

$$\frac{1}{\Gamma(2-\alpha)dt}\left\langle \mathscr{F}\left(\delta_t U^{n-\frac{1}{2}},\phi\right),\xi\right\rangle + \left\langle \Delta U^{n-\frac{1}{2}},\Delta\xi\right\rangle = 0, \quad \forall\, \xi \in H_0^2(\Omega). \tag{16}$$

Theorem 1. *The finite difference scheme (15) is stable.*

Proof. Let $\overline{U}^n$ be the approximate value of U^n. Then, we introduce

$$\frac{1}{\Gamma(2-\alpha)dt}\left\langle \mathscr{F}\left(\delta_t\Theta^{n-\frac{1}{2}},0\right),\xi\right\rangle = -\left\langle \Delta\Theta^{n-\frac{1}{2}},\Delta\xi\right\rangle, \quad \forall\, \xi \in H_0^2(\Omega). \tag{17}$$

where $\Theta^n = U^n - \bar{U}^n$. Assume $\xi = \delta_t\Theta^{n-\frac{1}{2}}$ gives

$$\frac{1}{\Gamma(2-\alpha)dt}\left\langle \mathscr{F}\left(\delta_t\Theta^{n-\frac{1}{2}},0\right),\delta_t\Theta^{n-\frac{1}{2}}\right\rangle = -\left\langle \Delta\Theta^{n-\frac{1}{2}},\Delta\delta_t\Theta^{n-\frac{1}{2}}\right\rangle, \quad \forall\, \xi \in H_0^2(\Omega). \tag{18}$$

Multiply (17) by dt and summing it from $n=1$ to m, arrive at

$$\frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\langle F\left(\delta_t\Theta^{n-\frac{1}{2}},0\right),\delta_t\Theta^{n-\frac{1}{2}}\right\rangle = -dt\sum_{n=1}^{m}\left\langle \Delta\Theta^{n-\frac{1}{2}},\Delta\delta_t\Theta^{n-\frac{1}{2}}\right\rangle. \tag{19}$$

The Schwarz inequality and Lemma 2, yield

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\langle F\left(\delta_t\Theta^{n-\frac{1}{2}},0\right),\delta_t\Theta^{n-\frac{1}{2}}\right\rangle\\
=\ & \frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\{a_0\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|^2_{L^2(\Omega)}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left(\delta_t\Theta^{k-\frac{1}{2}},\delta_t\Theta^{n-\frac{1}{2}}\right)\right\}\\
\geq\ & \frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\{a_0\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|^2_{L^2(\Omega)}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left\|\delta_t\Theta^{k-\frac{1}{2}}\right\|_{L^2(\Omega)}\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|_{L^2(\Omega)}\right\}\\
=\ & \frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\{a_0\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|_{L^2(\Omega)}-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\left\|\delta_t\Theta^{k-\frac{1}{2}}\right\|_{L^2(\Omega)}\right\}\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|_{L^2(\Omega)}\\
\geq\ & \frac{t_m^{1-\alpha}dt}{2\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{20}$$

Now, for another term, we have

$$\begin{aligned}
-dt\sum_{n=1}^{m}\int_\Omega(\Delta\Theta^{n-\frac{1}{2}})(\delta_t\Delta\Theta^{n-\frac{1}{2}})dx &= -dt\sum_{n=1}^{m}\int_\Omega\left(\frac{\Delta\Theta^n+\Delta\Theta^{n-1}}{2}\right)\left(\frac{\Delta\Theta^n-\Delta\Theta^{n-1}}{dt}\right)dx\\
&= -\frac{1}{2}\sum_{n=1}^{m}\left\{\int_\Omega\left[(\Delta\Theta^n)^2-\left(\Delta\Theta^{n-1}\right)^2\right]dx\right\}\\
&= -\frac{1}{2}\sum_{n=1}^{m}\left\{\|\Delta\Theta^n\|^2_{L^2(\Omega)}-\left\|\Delta\Theta^{n-1}\right\|^2_{L^2(\Omega)}\right\}
\end{aligned} \tag{21}$$

The use of relations (20)-(21) in Eq. (19), yields

$$\begin{aligned}
\frac{t_m^{1-\alpha}dt}{2\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|^2_{L^2(\Omega)} &\leq \frac{1}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\langle F\left(\delta_t\Theta^{n-\frac{1}{2}},0\right),\delta_t\Theta^{n-\frac{1}{2}}\right\rangle\\
&= -dt\sum_{n=1}^{m}\left\langle\Delta\Theta^{n-\frac{1}{2}},\Delta\delta_t\Theta^{n-\frac{1}{2}}\right\rangle\\
&= -\frac{1}{2}\sum_{n=1}^{m}\left\{\|\Delta\Theta^n\|^2_{L^2(\Omega)}-\left\|\Delta\Theta^{n-1}\right\|^2_{L^2(\Omega)}\right\}.
\end{aligned}$$

Now, we can get

$$\frac{t_m^{1-\alpha}dt}{\Gamma(2-\alpha)}\sum_{n=1}^{m}\left\|\delta_t\Theta^{n-\frac{1}{2}}\right\|^2_{L^2(\Omega)}+\|\Delta\Theta^m\|^2_{L^2(\Omega)}\leq\left\|\Delta\Theta^0\right\|^2_{L^2(\Omega)}. \tag{22}$$

Changing index m to n, results in $\|\Delta\Theta^n\|_{L^2(\Omega)} \leq \left\|\Delta\Theta^0\right\|_{L^2(\Omega)}$, which it completes the proof. □

Theorem 2. *Let u^n and U^n be solutions of Eqs.* (12) *and* (14)*, respectively. Thus, numerical scheme* (14) *is convergent with convergence order* $3-\alpha$.

Proof. The proof is similar to Theorem 1. □

3 Compact local IRBF method

In the compact local IRBF method, we assume

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \sum_{j=1}^{N} \Upsilon_j^{[x]} \Psi_j^{[x]}(x,y), \tag{23}$$

where $\{\Upsilon_j^{[x]}\}_{j=1}^N$ are unknown coefficients and $\{\Psi_j^{[x]}\}_{j=1}^N$ are the radial basis functions (RBFs). Now, to approximate the first-order derivative, we have

$$\frac{\partial u(x,y)}{\partial x} = \sum_{j=1}^{N} \Upsilon_j^{[x]} \Phi_j^{[x]}(x,y) + w_1^{[x]}(y), \tag{24}$$

$$u^{[x]}(x,y) = \sum_{j=1}^{N} \Upsilon_j^{[x]} \overline{\Phi}^{[x]}(x,y) + x w_1^{[x]}(y) + w_2^{[x]}(y), \tag{25}$$

where

$$\overline{\Phi}^{[x]}(x,y) = \int \Phi_j^{[x]}(x,y)dx, \qquad \Phi_j^{[x]}(x,y) = \int \Psi_j^{[x]}(x,y)dx,$$

and also $w_1^{[x]}(y)$ and $w_2^{[x]}(y)$ are the constant of integration. Similarly, we can compute

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \sum_{j=1}^{N} \Upsilon_j^{[y]} \Psi_j^{[y]}(x,y), \tag{26}$$

$$\frac{\partial u(x,y)}{\partial y} = \sum_{j=1}^{N} \Upsilon_j^{[y]} \Phi_j^{[y]}(x,y) + w_1^{[y]}(y), \tag{27}$$

$$u^{[y]}(x,y) = \sum_{j=1}^{N} \Upsilon_j^{[y]} \overline{\Phi}^{[y]}(x,y) + y w_1^{[y]}(y) + w_2^{[y]}(y), \tag{28}$$

where

$$\overline{\Phi}^{[y]}(x,y) = \int \Phi_j^{[y]}(x,y)dy, \qquad \Phi_j^{[y]}(x,y) = \int \Psi_j^{[y]}(x,y)dy.$$

Let

$$\begin{bmatrix} x,y_3 & x,y_6 & x,y_9 \\ x,y_2 & x,y_5 & x,y_8 \\ x,y_1 & x,y_4 & x,y_7 \end{bmatrix},$$

be a computational stencil with center $x, y_5 = x, y_c$. Using the collocation idea for (23)-(25), yields

$$\begin{aligned} \widehat{\frac{\partial^2 u(x,y)}{\partial x^2}} &= \Psi^{[x]}\Xi^{[x]}, \\ \widehat{\frac{\partial u(x,y)}{\partial x}} &= \Phi^{[x]}\Xi^{[x]}, \\ \widehat{u^{[x]}(x,y)} &= \overline{\Phi}^{[x]}\Xi^{[x]}, \end{aligned} \tag{29}$$

where

$$\Xi^{[x]} = \left(\Upsilon_1^{[x]},\ldots,\Upsilon_9^{[x]}, w_1^{[x]}(y_1), w_1^{[x]}(y_2), w_1^{[x]}(y_3), w_2^{[x]}(y_1), w_2^{[x]}(y_2), w_2^{[x]}(y_3)\right)^T,$$

$$\Psi^{[x]} = \begin{bmatrix} \psi_1^{[x]}(x,y_1) & \psi_2^{[x]}(x,y_1) & \ldots & \psi_9^{[x]}(x,y_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_1^{[x]}(x,y_2) & \psi_2^{[x]}(x,y_2) & \ldots & \psi_9^{[x]}(x,y_2) & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_1^{[x]}(x,y_3) & \psi_2^{[x]}(x,y_3) & \ldots & \psi_9^{[x]}(x,y_3) & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ \psi_1^{[x]}(x,y_8) & \psi_2^{[x]}(x,y_8) & \ldots & \psi_9^{[x]}(x,y_8) & 0 & 0 & 0 & 0 & 0 & 0 \\ \psi_1^{[x]}(x,y_9) & \psi_2^{[x]}(x,y_9) & \ldots & \psi_9^{[x]}(x,y_9) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 15},$$

$$\Phi^{[x]} = \begin{bmatrix} \phi_1^{[x]}(x,y_1) & \phi_2^{[x]}(x,y_1) & \ldots & \phi_9^{[x]}(x,y_1) & 1 & 0 & 0 & 0 & 0 & 0 \\ \phi_1^{[x]}(x,y_2) & \phi_2^{[x]}(x,y_2) & \ldots & \phi_9^{[x]}(x,y_2) & 0 & 1 & 0 & 0 & 0 & 0 \\ \phi_1^{[x]}(x,y_3) & \phi_2^{[x]}(x,y_3) & \ldots & \phi_9^{[x]}(x,y_3) & 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \phi_1^{[x]}(x,y_8) & \phi_2^{[x]}(x,y_8) & \ldots & \phi_9^{[x]}(x,y_8) & 0 & 1 & 0 & 0 & 0 & 0 \\ \phi_1^{[x]}(x,y_9) & \phi_2^{[x]}(x,y_9) & \ldots & \phi_9^{[x]}(x,y_9) & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{9\times 15},$$

$$\overline{\Phi}^{[x]} = \begin{bmatrix} \overline{\phi}_1^{[x]}(x,y_1) & \overline{\phi}_2^{[x]}(x,y_1) & \ldots & \overline{\phi}_9^{[x]}(x,y_1) & x_1 & 0 & 0 & 1 & 0 & 0 \\ \overline{\phi}_1^{[x]}(x,y_2) & \overline{\phi}_2^{[x]}(x,y_2) & \ldots & \overline{\phi}_9^{[x]}(x,y_2) & 0 & x_2 & 0 & 0 & 1 & 0 \\ \overline{\phi}_1^{[x]}(x,y_3) & \overline{\phi}_2^{[x]}(x,y_3) & \ldots & \overline{\phi}_9^{[x]}(x,y_3) & 0 & 0 & x_3 & 0 & 0 & 1 \\ \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ \overline{\phi}_1^{[x]}(x,y_8) & \overline{\phi}_2^{[x]}(x,y_8) & \ldots & \overline{\phi}_9^{[x]}(x,y_8) & 0 & x_8 & 0 & 0 & 1 & 0 \\ \overline{\phi}_1^{[x]}(x,y_9) & \overline{\phi}_2^{[x]}(x,y_9) & \ldots & \overline{\phi}_9^{[x]}(x,y_9) & 0 & 0 & x_9 & 0 & 0 & 1 \end{bmatrix}_{9\times 15}.$$

Furthermore, for variable y, we can get

$$\begin{aligned}\frac{\partial^2 \widehat{u(x,y)}}{\partial y^2} &= \Psi^{[y]}\Xi^{[y]},\\ \frac{\partial \widehat{u(x,y)}}{\partial y} &= \Phi^{[y]}\Xi^{[y]},\\ \widehat{u^{[y]}(x,y)} &= \overline{\Phi}^{[y]}\Xi^{[y]},\end{aligned} \tag{30}$$

where

$$\Xi^{[y]} = \left(\Upsilon_1^{[y]},\ldots,\Upsilon_9^{[y]}, w_1^{[y]}(x_1), w_1^{[y]}(x_4), w_1^{[y]}(x_7), w_2^{[y]}(x_1), w_2^{[y]}(x_4), w_2^{[y]}(x_7)\right)^T,$$

$$\Psi^{[y]} = \begin{bmatrix} \psi_1^{[y]}(x,y_1) & \psi_2^{[y]}(x,y_1) & \ldots & \psi_9^{[y]}(x,y_1) & 0 & 0 & 0 & 0 & 0 & 0\\ \psi_1^{[y]}(x,y_2) & \psi_2^{[y]}(x,y_2) & \ldots & \psi_9^{[y]}(x,y_2) & 0 & 0 & 0 & 0 & 0 & 0\\ \psi_1^{[y]}(x,y_3) & \psi_2^{[y]}(x,y_3) & \ldots & \psi_9^{[y]}(x,y_3) & 0 & 0 & 0 & 0 & 0 & 0\\ \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots\\ \psi_1^{[y]}(x,y_8) & \psi_2^{[y]}(x,y_8) & \ldots & \psi_9^{[y]}(x,y_8) & 0 & 0 & 0 & 0 & 0 & 0\\ \psi_1^{[y]}(x,y_9) & \psi_2^{[y]}(x,y_9) & \ldots & \psi_9^{[y]}(x,y_9) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 15},$$

$$\Phi^{[y]} = \begin{bmatrix} \phi_1^{[y]}(x,y_1) & \phi_2^{[y]}(x,y_1) & \ldots & \phi_9^{[y]}(x,y_1) & 1 & 0 & 0 & 0 & 0 & 0\\ \phi_1^{[y]}(x,y_2) & \phi_2^{[y]}(x,y_2) & \ldots & \phi_9^{[y]}(x,y_2) & 1 & 0 & 0 & 0 & 0 & 0\\ \phi_1^{[y]}(x,y_3) & \phi_2^{[y]}(x,y_3) & \ldots & \phi_9^{[y]}(x,y_3) & 1 & 0 & 0 & 0 & 0 & 0\\ \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\ \phi_1^{[y]}(x,y_8) & \phi_2^{[y]}(x,y_8) & \ldots & \phi_9^{[y]}(x,y_8) & 0 & 0 & 1 & 0 & 0 & 0\\ \phi_1^{[y]}(x,y_9) & \phi_2^{[y]}(x,y_9) & \ldots & \phi_9^{[y]}(x,y_9) & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}_{9\times 15},$$

$$\overline{\Phi}^{[y]} = \begin{bmatrix} \overline{\phi}_1^{[y]}(x,y_1) & \overline{\phi}_2^{[y]}(x,y_1) & \ldots & \overline{\phi}_9^{[y]}(x,y_1) & y_1 & 0 & 0 & 1 & 0 & 0\\ \overline{\phi}_1^{[y]}(x,y_2) & \overline{\phi}_2^{[y]}(x,y_2) & \ldots & \overline{\phi}_9^{[y]}(x,y_2) & y_2 & 0 & 0 & 1 & 0 & 0\\ \overline{\phi}_1^{[y]}(x,y_3) & \overline{\phi}_2^{[y]}(x,y_3) & \ldots & \overline{\phi}_9^{[y]}(x,y_3) & 1 & 0 & 0 & 1 & 0 & 0\\ \vdots & \ldots & \vdots & \ldots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots\\ \overline{\phi}_1^{[y]}(x,y_8) & \overline{\phi}_2^{[y]}(x,y_8) & \ldots & \overline{\phi}_9^{[y]}(x,y_8) & 0 & 0 & y_8 & 0 & 0 & 1\\ \overline{\phi}_1^{[y]}(x,y_9) & \overline{\phi}_2^{[y]}(x,y_9) & \ldots & \overline{\phi}_9^{[y]}(x,y_9) & 0 & 0 & y_9 & 0 & 0 & 1 \end{bmatrix}_{9\times 15}.$$

Now, form the above relation we can conclude

$$\begin{pmatrix} \widehat{u} \\ \widehat{o} \end{pmatrix} = \underbrace{\begin{bmatrix} \overline{\Phi}^{[x]} & \mathscr{O} \\ \overline{\Phi}^{[x]} & -\overline{\Phi}^{[y]} \end{bmatrix}}_{\mathscr{N}} \begin{pmatrix} \Xi^{[x]} \\ \Xi^{[y]} \end{pmatrix}, \tag{31}$$

where $\widehat{o} = \texttt{zeros}(9)$ (command of MATLAB software). Solving Eq. (31), results

$$\begin{pmatrix} \Xi^{[x]} \\ \Xi^{[y]} \end{pmatrix} = \mathscr{N}^{-1} \begin{pmatrix} \widehat{u} \\ \widehat{o} \end{pmatrix},$$

or

$$\Xi^{[x]} = \mathscr{N}_{[x]}^{-1} (\widehat{u}, \widehat{o})^T, \tag{32}$$

$$\Xi^{[y]} = \mathscr{N}_{[y]}^{-1} (\widehat{u}, \widehat{o})^T. \tag{33}$$

Substituting $x, y = x, y_c$ in Eqs. (32), (33), (29) and (30), gives

$$\begin{aligned}
\frac{\partial^2 u(x,y_c)}{\partial x^2} &= \left[\Psi_1^{[x]}(x,y_c), \Psi_2^{[x]}(x,y_c), \ldots, \Psi_8^{[x]}(x,y_c), \Psi_9^{[x]}(x,y_c), 0,0,0,0,0,0\right] \mathscr{N}_{[x]}^{-1} (\widehat{u}, \widehat{o})^T, \\
\frac{\partial u(x,y_c)}{\partial x} &= \left[\Phi_1^{[x]}(x,y_c), \Phi_2^{[x]}(x,y_c), \ldots, \Phi_8^{[x]}(x,y_c), \Phi_9^{[x]}(x,y_c), 0,1,0,0,0,0\right] \mathscr{N}_{[x]}^{-1} (\widehat{u}, \widehat{o})^T, \\
\frac{\partial^2 u(x,y_c)}{\partial y^2} &= \left[\Psi_1^{[y]}(x,y_c), \Psi_2^{[y]}(x,y_c), \ldots, \Psi_8^{[y]}(x,y_c), \Psi_9^{[y]}(x,y_c), 0,0,0,0,0,0\right] \mathscr{N}_{[y]}^{-1} (\widehat{u}, \widehat{o})^T, \\
\frac{\partial u(x,y_c)}{\partial y} &= \left[\Phi_1^{[y]}(x,y_c), \Phi_2^{[y]}(x,y_c), \ldots, \Phi_8^{[y]}(x,y_c), \Phi_9^{[y]}(x,y_c), 0,1,0,0,0,0\right] \mathscr{N}_{[y]}^{-1} (\widehat{u}, \widehat{o})^T.
\end{aligned}$$

Assembling the local matrices, results in

$$\frac{\partial^2 u(x,y)}{\partial x^2} = \underbrace{\begin{bmatrix}
* & * & * & * & * & * & * & * & * & 0 & 0 & \ldots & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 & \ldots & 0 \\
0 & * & * & * & * & * & * & * & * & * & 0 & \ldots & 0 \\
\vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\
0 & \ldots & 0 & * & * & * & * & * & * & * & * & * & 0 \\
0 & \ldots & 0 & 0 & * & * & * & * & * & * & * & * & * \\
0 & \ldots & 0 & 0 & * & * & * & * & * & * & * & * & *
\end{bmatrix}}_{\mathscr{D}_{xx}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix},$$

$$\frac{\partial u(x,y)}{\partial x} = \underbrace{\begin{bmatrix}
* & * & * & * & * & * & * & * & * & 0 & 0 & \ldots & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 & \ldots & 0 \\
0 & * & * & * & * & * & * & * & * & * & 0 & \ldots & 0 \\
\vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots & \ldots & \vdots \\
0 & \ldots & 0 & * & * & * & * & * & * & * & * & * & 0 \\
0 & \ldots & 0 & 0 & * & * & * & * & * & * & * & * & * \\
0 & \ldots & 0 & 0 & * & * & * & * & * & * & * & * & *
\end{bmatrix}}_{\mathscr{D}_{x}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \ldots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix},$$

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \underbrace{\begin{bmatrix} * & * & * & * & * & * & * & * & * & 0 & 0 & \dots & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & \dots & 0 \\ 0 & * & * & * & * & * & * & * & * & * & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & * & * & * & * & * & * & * & * & * & 0 \\ 0 & \dots & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & \dots & 0 & 0 & * & * & * & * & * & * & * & * & * \end{bmatrix}}_{\mathscr{D}_{yy}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix},$$

$$\frac{\partial u(x,y)}{\partial y} = \underbrace{\begin{bmatrix} * & * & * & * & * & * & * & * & * & 0 & 0 & \dots & 0 \\ * & * & * & * & * & * & * & * & * & 0 & 0 & \dots & 0 \\ 0 & * & * & * & * & * & * & * & * & * & 0 & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & * & * & * & * & * & * & * & * & * & 0 \\ 0 & \dots & 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & \dots & 0 & 0 & * & * & * & * & * & * & * & * & * \end{bmatrix}}_{\mathscr{D}_{y}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{N-2} \\ u_{N-1} \\ u_N \end{pmatrix},$$

where $*$ is a constant number.

To approximate the first- and second-order derivatives and also the unknown function of u, we have

$$\begin{aligned}
\frac{\partial^2 u(x,y)}{\partial x^2} &= \sum_{j=1}^{N} \Upsilon_j^{[x]}(t)\Psi_j^{[x]}(x,y), \\
\frac{\partial u(x,y)}{\partial x} &= \sum_{j=1}^{N} \Upsilon_j^{[x]}(t)\Phi_j^{[x]}(x,y) + c_1^{[x]}(t), \\
u^{[x]}(x,y) &= \sum_{j=1}^{N} \Upsilon_j^{[x]}(t)\overline{\Phi}_j^{[x]}(x,y) + xc_1^{[x]}(t) + c_2^{[x]}(t), \\
\frac{\partial^2 u(x,y)}{\partial y^2} &= \sum_{j=1}^{N} \Upsilon_j^{[y]}(t)\Psi_j^{[y]}(x,y), \\
\frac{\partial u(x,y)}{\partial y} &= \sum_{j=1}^{N} \Upsilon_j^{[y]}(t)\Phi_j^{[y]}(x,y) + c_1^{[y]}(t), \\
u^{[y]}(x,y) &= \sum_{j=1}^{N} \Upsilon_j^{[y]}(t)\overline{\Phi}_j^{[y]}(x,y) + yc_1^{[y]}(t) + c_2^{[y]}(t),
\end{aligned}$$

where

$$\begin{aligned}
\overline{\Phi}_j^{[y]}(x,y) &= \int \Phi_j^{[y]}(x,y)dy, \quad \Phi_j^{[y]}(x,y) = \int \Psi_j^{[y]}(x,y)dy, \\
\overline{\Phi}_j^{[x]}(x,y) &= \int \Phi_j^{[x]}(x,y)dx, \quad \Phi_j^{[x]}(x,y) = \int \Psi_j^{[x]}(x,y)dx,
\end{aligned}$$

and $c_1(t)$ and $c_2(t)$. Thus

$$U_{xx} := \frac{\partial^2 u(x,y)}{\partial x^2} = \mathbf{D}_{xx}U, \quad U_x := \frac{\partial u(x,y)}{\partial x} = \mathbf{D}_x U,$$

$$U_{yy} := \frac{\partial^2 u(x,y)}{\partial y^2} = \mathbf{D}_{yy}U, \quad U_y := \frac{\partial u(x,y)}{\partial y} = \mathbf{D}_y U.$$

According to the semi-discrete scheme

$$\begin{cases} Q^{n-\frac{1}{2}} = \Delta U^{n-\frac{1}{2}}(x,y), \\ \frac{1}{\Gamma(2-\alpha)dt}\mathscr{F}\left(\delta_t U^{n-\frac{1}{2}}(x,y), \phi(x,y)\right) + \Delta Q^{n-\frac{1}{2}} = f^{n-\frac{1}{2}}(x,y), \quad n \geq 1, \end{cases} \tag{34}$$

the following algebraic system of equations can be obtained

$$\boldsymbol{A}\boldsymbol{X}^{\boldsymbol{n}} = \boldsymbol{B}\boldsymbol{X}^{\boldsymbol{n-1}} + \boldsymbol{F}^{\boldsymbol{n}},$$

where

$$\boldsymbol{X}^{\boldsymbol{n}} = \begin{bmatrix} \boldsymbol{U}^{\boldsymbol{n}} \\ \boldsymbol{V}^{\boldsymbol{n}} \end{bmatrix},$$

$$\boldsymbol{A} = \begin{bmatrix} -\boldsymbol{D_{xx}} - \boldsymbol{D_{yy}} & \boldsymbol{I} \\ a_0\mu \boldsymbol{I} & \frac{dt}{2}\left(\boldsymbol{D_{xx}} + \boldsymbol{D_{yy}}\right) \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{D_{xx}} + \boldsymbol{D_{yy}} & -\boldsymbol{I} \\ a_0\mu \boldsymbol{I} & -\frac{dt}{2}\left(\boldsymbol{D_{xx}} + \boldsymbol{D_{yy}}\right) \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} \mathbf{0} \\ dt\,\boldsymbol{f}^{\boldsymbol{n}}(\boldsymbol{x}) + \mu a_{n-1}\psi(\boldsymbol{x}) + \mu\sum_{k=1}^{n-1}\left(a_{n-k-1} - a_{n-k}\right)\left(\boldsymbol{U}^{\boldsymbol{k}} - \boldsymbol{U}^{\boldsymbol{k-1}}\right) \end{bmatrix}.$$

4 Numerical surveys

We checked the accuracy and stability of the method presented by performing the mentioned method for different numbers of the distributed nodes or time steps. Our computations are performed utilizing MATLAB 2020b software on an Intel Core i7 machine with 32 GB of memory. Furthermore, the computational order of the proposed numerical procedure will be calculated by the following relation

$$\mathrm{C}_1\mathrm{-order} = \frac{\log_{10}\left(\frac{\mathrm{E}_\infty(N,\tau_1)}{\mathrm{E}_\infty(N,\tau_2)}\right)}{\log_{10}\left(\frac{\tau_1}{\tau_2}\right)},$$

where E_∞ is norm of error infinity.

Example 1. For the one-dimensional case, we consider the following example

$${}_a^C\mathscr{D}_t^\alpha u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} = e^x\left(\Gamma(\alpha+4)\frac{t^3}{6} + t^{3+\alpha} + t\right), \quad x\in[0,1], \quad 0\le t\le T, \quad 1<\alpha<2,$$

with initial and boundary conditions

$$u(x,0)=0, \qquad \frac{\partial u(x,0)}{\partial t} = e^x, \qquad x\in[0,1],$$

$$u(0,t) = t^{3+\alpha}+t, \quad u(1,t) = e\left(t^{3+\alpha}+t\right), \quad t\ge 0,$$

$$\frac{\partial^2 u(0,t)}{\partial x^2} = t^{3+\alpha}+t, \quad \frac{\partial^2 u(1,t)}{\partial x^2} = e\left(t^{3+\alpha}+t\right), \quad t\ge 0,$$

where the exact solution is $u(x,t) = e^x\left(t^{3+\alpha}+t\right)$.

Table 1: Error computed with $N=200$ on $\Omega=[0,1]$ for Example 1.

	$\alpha=1.2$		$\alpha=1.5$		$\alpha=1.9$		CPU time(s)
τ	E_∞	*Order*	E_∞	*Order*	E_∞	*Order*	
1/10	7.73×10^{-4}	–	4.36×10^{-2}	–	3.70×10^{-2}	–	0.02
1/20	6.98×10^{-4}	0.14	1.75×10^{-2}	1.31	1.82×10^{-2}	1.02	0.8
1/40	3.18×10^{-4}	1.13	6.72×10^{-3}	1.38	8.77×10^{-3}	1.05	1.12
1/80	1.21×10^{-4}	1.39	2.50×10^{-3}	1.42	4.15×10^{-3}	1.08	3.41
1/160	4.19×10^{-5}	1.52	9.15×10^{-4}	1.45	1.95×10^{-3}	1.09	8.12
1/320	1.39×10^{-5}	1.59	3.31×10^{-4}	1.47	9.16×10^{-4}	1.09	15.4
1/640	4.43×10^{-6}	1.64	1.19×10^{-4}	1.48	4.28×10^{-4}	1.10	34.2
1/1280	1.39×10^{-6}	1.67	4.25×10^{-5}	1.48	2.00×10^{-4}	1.10	89.4
1/1280	4.26×10^{-7}	1.70	1.52×10^{-5}	1.49	9.34×10^{-5}	1.10	142.3
1/1280	1.30×10^{-7}	1.71	5.39×10^{-6}	1.49	4.36×10^{-5}	1.10	201.7

Figure 1 demonstrates the graphs of approximate solutions (left panel) and absolute errors (right panel) with 200 collocation points, $T=5$, $dt=10^{-3}$ and different fractional order α for Example 1. Table 1 presents numerical experiments and computational order obtained with $N=200$, $T=1$ and $\Omega=[0,1]$ for Example 1. The convergence rate of the proposed technique tends to $3-\alpha$. Table 1 confirms the computational and the theoretical results are close to each other.

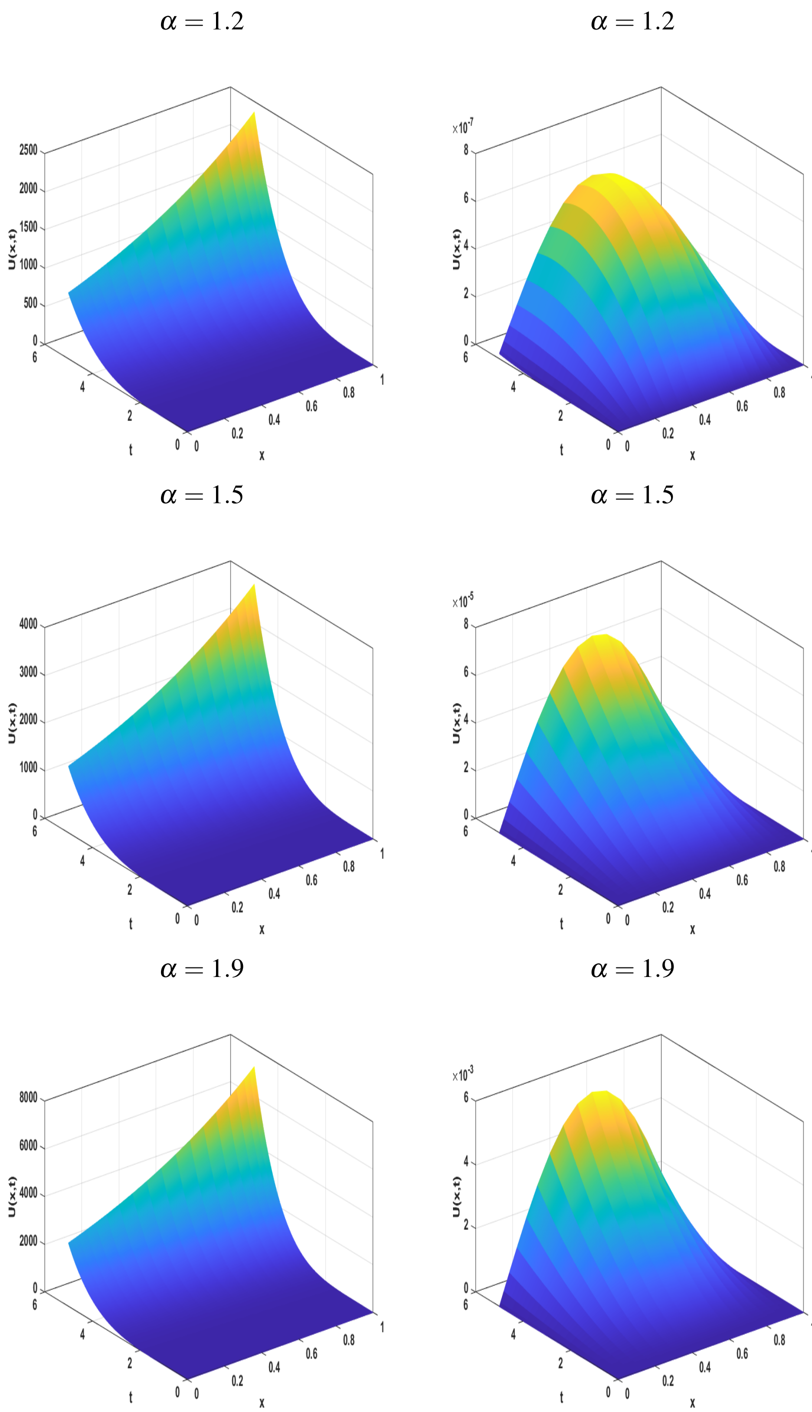

Figure 1: Approximate solution (left panel) and absolute error (right panel) with 200 collocation points, $T = 5$, $dt = 10^{-3}$ and different fractional order α for Instance1.

Example 2. Now, for the two-dimensional case, we investigate

$$ {}_a^C\mathscr{D}_t^\alpha u(x,y,t) + \Delta^2 u(x,y,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} x(x-1)y(y-1), \quad (x,y) \in \Omega = [0,1]\times[0,1], $$

with $0 \le t \le T$, $1 < \alpha < 2$, and the initial and boundary conditions

$$ u(x,y,0) = 0, \qquad \frac{\partial u(x,y,0)}{\partial t} = 0, \qquad x \in \Omega, $$

$$ u(x,y,t) = 0, \quad (x,y) \in \partial\Omega, \quad t \ge 0, $$

$$ \Delta u(x,y,t) = 2x(x-1) + 2y(y-1), \quad (x,y) \in \partial\Omega, \quad t \ge 0, $$

where the exact solution is $u(x,y,t) = t^2 x(x-1)y(y-1)$.

The graph of the approximate solutions on non-rectangular physical domains with 800 distributed nodes, $T = 1$, $dt = 10^{-3}$ and $\alpha = 1.2$ are illustrated for Example 2. Table 2 shows errors and computational orders obtained with $N = 800$, $T = 1$ and different regions for Example 2. The convergence order of the present method tends to $3-\alpha$. Similar to Table 1, Table 2 acknowledges the computational and the theoretical results are close to each other.

Table 2: Error computed with $N = 800$ on $\Omega = [0,1]\times[0,1]$ for Example 2.

	$\alpha = 1.2$		$\alpha = 1.6$		$\alpha = 1.85$		CPU time(s)
τ	E_∞	*Order*	E_∞	*Order*	E_∞	*Order*	
1/10	8.26×10^{-3}	–	4.45×10^{-2}	–	1.78×10^{-1}	–	0.8
1/20	4.22×10^{-3}	0.96	1.81×10^{-2}	1.30	8.35×10^{-2}	1.09	5.3
1/40	1.67×10^{-3}	1.34	7.14×10^{-3}	1.34	3.85×10^{-2}	1.12	42.7
1/80	5.93×10^{-4}	1.49	2.78×10^{-3}	1.36	1.75×10^{-2}	1.13	98.7
1/160	1.99×10^{-4}	1.85	1.07×10^{-3}	1.38	7.95×10^{-3}	1.14	188.9
1/320	6.42×10^{-5}	1.63	4.10×10^{-4}	1.38	3.59×10^{-3}	1.14	276.4
1/640	2.02×10^{-5}	1.67	1.56×10^{-4}	1.39	1.62×10^{-3}	1.15	433.1
1/1280	6.24×10^{-6}	1.69	5.95×10^{-5}	1.39	7.32×10^{-4}	1.15	961.3

5 Conclusion

The current paper developed an efficient and simple numerical procedure to solve the one- and two-dimensional fourth-order time-fractional diffusion-wave system. First, a $(3-\alpha)$-order finite difference scheme is utilized to discrete the time derivative. Moreover, the energy method is used to analyze the stability and convergence of the proposed numerical technique. Then, the compact local integrated RBFs idea is employed to obtain differential matrices to discrete the space derivatives. The proposed numerical algorithm is tested for one- and two-dimensional examples to check its efficiency and accuracy.

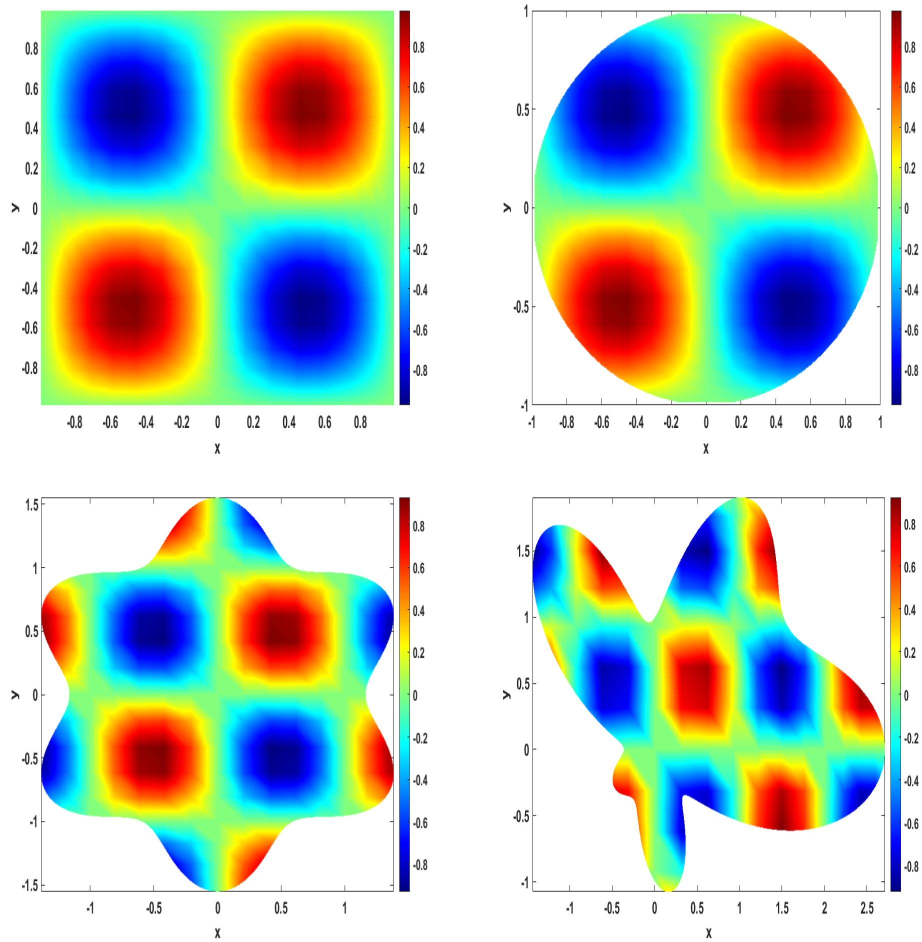

Figure 2: Approximate solution with 800 collocation points, $T = 1$, $dt = 10^{-3}$ and $\alpha = 1.2$ for Instance2.

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