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Numerical treatment for a multiscale nonlinear system of singularly perturbed differential equations of convection-diffusion type

Manikandan Mariappan*

Department of Mathematics, School of Engineering, Presidency University, Bengaluru - 560 064, Karnataka, India
Email(s): manimaths89@yahoo.com

Abstract. In this article, a multiscale nonlinear system of singularly perturbed differential equations of convection-diffusion type is considered. A numerical technique combined with the continuation method is constructed to obtain the numerical computations. The newly developed numerical method is shown to be first order convergent uniformly with respect to the perturbation parameter.

Keywords: Multiscale nonlinear system of singularly perturbed differential equations, boundary layers, finite difference scheme, Shishkin mesh, the continuation method, parameter-uniform convergence.
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1 Introduction

Due to the rapid changes in the solutions of Singular Perturbation Problems (SPPs), classical numerical methods are not suitable for them [11]. As mentioned in [5] and [10], even the non-classical numerical methods for SPPs render their uselessness whenever conditions are imposed on the magnitude of the perturbation parameters or artificial conditions are imposed on the problems.

System of singularly perturbed nonlinear differential equations arise in various fields, for instance, in catalytic reaction theory [2] and control systems [7, 14]. For a nonlinear system of Singularly Perturbed Differential Equations (SPDEs) of Reaction-Diffusion (RD) type, various numerical methods are constructed in [4, 8, 9, 12].

The *Navier-Stokes* equations, a system of four nonlinear partial differential equations of Convection-Diffusion (CD) type, describes the dynamics of fluid and gas. This system exhibits the singularly perturbed nature when the magnitude of the convective terms is much larger than that of the diffusion terms [11].

*Corresponding author
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In the *Navier-Stokes* equations not all the leading term in each equation is multiplied by a parameter. Thus all the singular perturbation parameters associated with a system of DEs need not have small magnitudes. In [9], a higher order numerical method is developed for a second order nonlinear system of two SPDEs of RD type in which the highest order derivative term in the first equation alone is multiplied by a small parameter ε. Article [1] deals with the construction of a numerical method for a nonlinear system of two SPDEs such that in the system, along with the nonlinear terms, the first equation has only the first derivative of the first component and the second equation has only the second derivative of the second component.

Construction of robust, layer-resolving and parameter-uniform numerical method for a nonlinear system of SPDEs of CD type is quite complicated. Recently, in the article [10], a parameter-uniform and robust numerical method is developed for a nonlinear system of SPDEs of CD type with two different perturbation parameters. It should be noted that for a nonlinear system of SPDEs of CD type no parameter-uniform, layer-resolving and robust numerical method is available in the literature except [10].

It is important to observe that the nonlinear system considered in the present article cannot be resolved by using the numerical method reported in [10] where the condition $0 < \varepsilon_1 < \varepsilon_2 < 1$ is assumed. Thus it is not possible to include the choice $\varepsilon_2 = 1$ in [10]. Moreover the solution pattern for the nonlinear system considered in the present article and that of the system in [10] are totally different. Hence a new technique is required to resolve the nonlinear system considered in the present article.

Induced by the *Navier-Stokes* equations as mentioned in [1, 9–11], in the present article, a nonlinear system of SPDEs of CD type in which the highest order derivative term in the first equation alone is multiplied by a small parameter is considered.

It is worth observing that no condition is imposed on the magnitude of the perturbation parameter occurring in the nonlinear system considered in the present article and also no artificial condition is imposed on the problem either for theoretical or for computational purposes. Hence the numerical method developed in this article is robust, layer-resolving and parameter-uniform.

The following norms for any vector function $\overrightarrow{\psi}$ on $[0,1]$ are introduced: $\| \overrightarrow{\psi}(t) \| = \max_i |\psi_i(t)|$ and $\| \overrightarrow{\psi} \| = \sup_{t\in[0,1]} \| \overrightarrow{\psi}(t) \|$. For any mesh function $\overrightarrow{\Psi}$ on $\overline{\Omega}^N = \{t_j\}_{j=0}^N$ the following discrete norms are introduced: $\| \overrightarrow{\Psi}(t_j) \| = \max_i |\Psi_i(t_j)|$ and $\| \overrightarrow{\Psi} \| = \max_{t_j\in\overline{\Omega}^N} \| \overrightarrow{\Psi}(t_j) \|$.

In this article C denotes a positive constant which is free from t, ε and N.

2 The nonlinear system

Consider the following nonlinear system of SPDEs of CD type

$$\begin{aligned} \varepsilon_1 u_1''(t) + a_1(t)u_1'(t) - f_1(t, u_1(t), u_2(t)) &= 0, \\ \varepsilon_2 u_2''(t) + a_2(t)u_2'(t) - f_2(t, u_1(t), u_2(t)) &= 0 \text{ on } \Omega = (0,1), \end{aligned} \tag{1}$$

$$\text{with } u_1(0) = u_{01},\, u_2(0) = u_{02},\, u_1(1) = u_{11} \text{ and } u_2(1) = u_{12}. \tag{2}$$

Here u_{01}, u_{02}, u_{11} and u_{12} are known constants and $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. For all $t \in \overline{\Omega} = [0,1]$ and for $i = 1,2$, $f_i(t, u_1(t), u_2(t)) \in C^3(\overline{\Omega} \times \mathbb{R}^2)$ and $a_i(t) \in C^3(\overline{\Omega})$. It is assumed that for $i = 1,2$ and for all $t \in \overline{\Omega}$,

$a_i(t) \geq \alpha > 0$ and for all $(t, u_1(t), u_2(t)) \in \overline{\Omega} \times \mathbb{R}^2$,

$$\frac{\partial f_i(t, u_1(t), u_2(t))}{\partial u_j} \leq 0, \; i, j = 1, 2, \; i \neq j \tag{3}$$

and

$$\min_{\substack{t \in \overline{\Omega} \\ i=1,2}} \left(\frac{\partial f_i(t, u_1(t), u_2(t))}{\partial u_1} + \frac{\partial f_i(t, u_1(t), u_2(t))}{\partial u_2} \right) \geq \beta > 0. \tag{4}$$

With the above conditions, a unique solution $(u_1(t),\; u_2(t))$ of (1)-(2) such that $u_i(t) \in C^3(\overline{\Omega})$, $i = 1, 2$, can be ensured by the implicit function theorem.

As explained in [9], there are four possible cases based on the magnitudes of ε_1 and ε_2. One of the cases is discussed in [10] and the other unique case $0 < \varepsilon_1 < \varepsilon_2 = 1$ is considered in the present article elaborately. Based on the assumption $0 < \varepsilon_1 < \varepsilon_2 = 1$ problem (1)-(2) is rewritten as

$$\begin{aligned} \varepsilon u_1''(t) + a_1(t) u_1'(t) - f_1(t, u_1(t), u_2(t)) &= 0, \\ u_2''(t) + a_2(t) u_2'(t) - f_2(t, u_1(t), u_2(t)) &= 0 \text{ on } \Omega, \end{aligned} \tag{5}$$

$$\text{with } \; u_1(0) = u_{01}, u_2(0) = u_{02}, u_1(1) = u_{11} \text{ and } u_2(1) = u_{12}. \tag{6}$$

In operator form, problem (5)-(6) can be written as

$$\overrightarrow{\mathbb{T}} \overrightarrow{u}(t) = E \overrightarrow{u}''(t) + A(t) \overrightarrow{u}'(t) - \overrightarrow{f}(t, \overrightarrow{u}) = \overrightarrow{0} \; \text{ on } \Omega, \tag{7}$$

$$\text{with } \; \overrightarrow{u}(0) = \overrightarrow{u}_0 \text{ and } \overrightarrow{u}(1) = \overrightarrow{u}_1, \tag{8}$$

where $\overrightarrow{u}_0 = (u_{01}, u_{02})^T$ and $\overrightarrow{u}_1 = (u_{11}, u_{12})^T$. For all $t \in \overline{\Omega}$, $\overrightarrow{u}(t) = (u_1(t),\; u_2(t))^T$, $\overrightarrow{f}(t, \overrightarrow{u}) = (f_1(t, u_1, u_2), f_2(t, u_1,\; u_2))^T$, $E = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}$ and $A(t) = \begin{bmatrix} a_1(t) & 0 \\ 0 & a_2(t) \end{bmatrix}$.

A reduced problem (obtained by putting $\varepsilon = 0$) of (7)-(8) is as follows:

$$\overrightarrow{\mathbb{T}}_0 \overrightarrow{v}_0(t) = E_0 \overrightarrow{v}_0''(t) + A(t) \overrightarrow{v}_0'(t) - \overrightarrow{f}(t, \overrightarrow{v}_0) = \overrightarrow{0} \; \text{ on } \; \Omega, \tag{9}$$

$$\text{with } v_{02}(0) = u_{02} \text{ and } \overrightarrow{v}_0(1) = \overrightarrow{u}_1, \tag{10}$$

where $E_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. As above, the implicit function theorem ensures a unique solution $\overrightarrow{v}_0(t)$ to (9)-(10). Moreover,

$$|v_{0i}^{(k)}(t)| \leq C \; \text{ for } \; i = 1, 2, \; k = 0, 1, 2, 3 \; \text{ and } t \in \overline{\Omega}. \tag{11}$$

3 Theoretical results

Let $\overrightarrow{\mathbb{T}}'$ be a linear operator such that

$$\overrightarrow{\mathbb{T}}'\overrightarrow{\phi}(t) = E\,\overrightarrow{\phi}''(t) + A(t)\,\overrightarrow{\phi}'(t) - P(t)\,\overrightarrow{\phi}(t),$$

where $P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}$ with $p_{ij}(t) \leq 0$, for $i, j = 1,2,\ i \neq j$ and

$$\min_{t \in \overline{\Omega}} \left(p_{11}(t) + p_{12}(t),\ p_{21}(t) + p_{22}(t)\right) > 0.$$

Theorem 1. *Let $\overrightarrow{\psi}$ be any vector function such that $\overrightarrow{\psi}(0) \geq \overrightarrow{0}$, $\overrightarrow{\psi}(1) \geq \overrightarrow{0}$ and $\overrightarrow{\mathbb{T}}'\overrightarrow{\psi} \leq \overrightarrow{0}$ on Ω, then $\overrightarrow{\psi} \geq \overrightarrow{0}$ on $\overline{\Omega}$.*

Proof. Let i^* and t^* be such that $\psi_{i^*}(t^*) = \min\limits_{i,t} \psi_i(t)$ and suppose $\psi_{i^*}(t^*) < 0$. Then $t^* \notin \{0,1\}$. Further,

$$\overrightarrow{\mathbb{T}}'\overrightarrow{\psi}(t^*) = \begin{bmatrix} \varepsilon\,\psi_1''(t^*) + a_1(t^*)\psi_1'(t^*) - p_{11}(t^*)\psi_1(t^*) - p_{12}(t^*)\psi_2(t^*) \\ \psi_2''(t^*) + a_2(t^*)\psi_2'(t^*) - p_{21}(t^*)\psi_1(t^*) - p_{22}(t^*)\psi_2(t^*) \end{bmatrix}.$$

Thus,

$$(\overrightarrow{\mathbb{T}}'\overrightarrow{\psi})_{i^*}(t^*) \geq \begin{cases} \varepsilon\,\psi_1''(t^*) + a_1(t^*)\psi_1'(t^*) - (p_{11}(t^*) + p_{12}(t^*))\,\psi_1(t^*), & \text{if } i^* = 1, \\ \\ \psi_2''(t^*) + a_2(t^*)\psi_2'(t^*) - (p_{21}(t^*) + p_{22}(t^*))\,\psi_2(t^*), & \text{if } i^* = 2. \end{cases}$$

Using the properties of $A(t)$ and $P(t)$, $(\overrightarrow{\mathbb{T}}'\overrightarrow{\psi})_{i^*}(t^*) > 0$, which is a contradiction. Hence $\overrightarrow{\psi} \geq \overrightarrow{0}$ on $\overline{\Omega}$. □

Decompose the solution $\overrightarrow{u}$ of (7)-(8) into a smooth component $\overrightarrow{v}$ and a singular component $\overrightarrow{w}$ such that $\overrightarrow{u} = \overrightarrow{v} + \overrightarrow{w}$ where $\overrightarrow{v}$ and $\overrightarrow{w}$ are the solutions of the following problems

$$E\,\overrightarrow{v}''(t) + A(t)\,\overrightarrow{v}'(t) - \overrightarrow{f}(t, \overrightarrow{v}) = \overrightarrow{0} \quad \text{on} \quad \Omega, \tag{12}$$

$$\text{with } \overrightarrow{v}(0) \text{ being suitably chosen and } \overrightarrow{v}(1) = \overrightarrow{u}_1 \tag{13}$$

and

$$E\,\overrightarrow{w}''(t) + A(t)\,\overrightarrow{w}'(t) - \overrightarrow{f}(t, \overrightarrow{v} + \overrightarrow{w}) + \overrightarrow{f}(t, \overrightarrow{v}) = \overrightarrow{0} \quad \text{on} \quad \Omega, \tag{14}$$

$$\text{with } \overrightarrow{w}(0) = \overrightarrow{u}_0 - \overrightarrow{v}(0) \text{ and } \overrightarrow{w}(1) = \overrightarrow{0}. \tag{15}$$

3.1 Bounds on vec$v(t)$ and its derivatives

Theorem 2. *For all $t \in \overline{\Omega}$, $|v_1^{(k)}(t)| \leq C$, $k = 0,1,2$, $|v_1^{(3)}(t)| \leq C\varepsilon^{-1}$, $|v_2^{(l)}(t)| \leq C$, $l = 0,1,2,3$.*

Proof. From (12) and (9),

$$\varepsilon v_1''(t) + a_1(t)\left(v_1' - v_{01}'\right)(t) - b_{11}(t)\left(v_1 - v_{01}\right)(t) - b_{12}(t)\left(v_2 - v_{02}\right)(t) = 0 \tag{16}$$

and

$$v_2''(t) + a_2(t)\left(v_2' - v_{02}'\right)(t) - b_{21}(t)\left(v_1 - v_{01}\right)(t) - b_{22}(t)\left(v_2 - v_{02}\right)(t) = 0, \tag{17}$$

where $b_{ij}(t) = \dfrac{\partial f_i(t, \vec{\mu}(t))}{\partial u_j}$ are intermediate values.

Equations (16) and (17) can be written together as

$$\vec{\mathbb{T}}'\vec{v}(t) = E\,\vec{v}''(t) + A(t)\vec{v}'(t) - B(t)\vec{v}(t) = A(t)\vec{v}_0'(t) - B(t)\vec{v}_0(t) = \vec{g}(t), \tag{18}$$

where $B(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix}$. For convenience, $\vec{v}(t)$ is decomposed as

$$\vec{v}(t) = \vec{y}_0(t) + \varepsilon\,\vec{y}_1(t) + \varepsilon^2\vec{y}_2(t), \tag{19}$$

where $\vec{y}_0(t)$ is the solution of

$$\begin{aligned} a_1(t)y_{01}'(t) - b_{11}(t)y_{01}(t) - b_{12}(t)y_{02}(t) &= g_1(t), \\ y_{02}''(t) + a_2(t)y_{02}'(t) - b_{21}(t)y_{01}(t) - b_{22}(t)y_{02}(t) &= g_2(t), \\ y_{02}(0) = u_{02},\; y_{01}(1) = u_{11},\; y_{02}(1) &= u_{12}, \end{aligned} \tag{20}$$

$\vec{y}_1(t)$ is the solution of

$$\begin{aligned} a_1(t)y_{11}'(t) - b_{11}(t)y_{11}(t) - b_{12}(t)y_{12}(t) &= -y_{01}''(t), \\ y_{12}''(t) + a_2(t)y_{12}'(t) - b_{21}(t)y_{11}(t) - b_{22}(t)y_{12}(t) &= 0, \\ y_{12}(0) = 0,\; y_{11}(1) = 0,\; y_{12}(1) &= 0 \end{aligned} \tag{21}$$

and $\vec{y}_2(t)$ is the solution of

$$\begin{aligned} \varepsilon y_{21}''(t) + a_1(t)y_{21}'(t) - b_{11}(t)y_{21}(t) - b_{12}(t)y_{22}(t) &= -y_{11}''(t), \\ y_{22}''(t) + a_2(t)y_{22}'(t) - b_{21}(t)y_{21}(t) - b_{22}(t)y_{22}(t) &= 0, \\ y_{21}(0) = q,\; y_{22}(0) = 0,\; y_{21}(1) = 0,\; y_{22}(1) &= 0, \end{aligned} \tag{22}$$

where q is chosen such that $|q| \le C$. From (20) and (21), for $0 \le k \le 3$ and for all $t \in \overline{\Omega}$,

$$||\vec{y}_0^{(k)}(t)|| \le C \text{ and } ||\vec{y}_1^{(k)}(t)|| \le C. \tag{23}$$

Using Theorem 1 and the bound on y_{11}'', for all $t \in \overline{\Omega}$,

$$||\vec{y}_2(t)|| \le C \text{ and } |y_{22}^{(k)}(t)| \le C,\; k = 1, 2.$$

Using the mean value theorem, for all $t \in \overline{\Omega}$, $|y_{21}'(t)| \le C\varepsilon^{-1}$.

Differentiating the second equation of (22) with respect to t once and using the bound of y_{21}', $|y_{22}^{(3)}(t)| \le C\varepsilon^{-1}$. From the first equation of (22), for all $t \in \overline{\Omega}$, $|y_{21}''(t)| \le C\varepsilon^{-2}$. Differentiating the first equation of (22) with respect to t once and using the bounds of other components, for all $t \in \overline{\Omega}$, $|y_{22}^{(3)}(t)| \le C\varepsilon^{-3}$. Finally the required bounds on $v_1(t), v_2(t)$ and their derivatives follow from the above established bounds of $\vec{y}_0(t)$, $\vec{y}_1(t)$, $\vec{y}_2(t)$ and their derivatives. □

3.2 Bounds on vec$w(t)$ and its derivatives

For all $t \in \overline{\Omega}$, let $\mathbb{B}(t) = e^{-\alpha t/\varepsilon}$.

Theorem 3. *For all $t \in \overline{\Omega}$,*

$$|w_1(t)| \le C\varepsilon + C\mathbb{B}(t) + C\varepsilon^2(1-\mathbb{B}(t)), \quad |w_2(t)| \le C\varepsilon + C\varepsilon^2(1-\mathbb{B}(t)),$$
$$|w_1^{(k)}(t)| \le C\varepsilon^{-k}\mathbb{B}(t),\ k=1,2,3, \quad |w_2'(t)| \le C\varepsilon,$$
$$|w_2''(t)| \le C\varepsilon + C\mathbb{B}(t),\ |w_2^{(3)}(t)| \le C\varepsilon^{-1}\mathbb{B}(t).$$

Proof. Using (14), we have

$$\varepsilon w_1''(t) + a_1(t)w_1'(t) - c_{11}(t)w_1(t) - c_{12}(t)w_2(t) = 0 \tag{24}$$

and

$$w_2''(t) + a_2(t)w_2'(t) - c_{21}(t)w_1(t) - c_{22}(t)w_2(t) = 0, \tag{25}$$

where $c_{ij}(t) = \dfrac{\partial f_i(t, \overrightarrow{\gamma}(t))}{\partial u_j}$ are intermediate values. Equations (24) and (25) can be written together as

$$\overrightarrow{\mathbb{T}}'\overrightarrow{w}(t) = E\overrightarrow{w}''(t) + A(t)\overrightarrow{w}'(t) - C(t)\overrightarrow{w}(t) = \overrightarrow{0}, \tag{26}$$

where $C(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{bmatrix}$.

Let $\overrightarrow{\Gamma}(t) = \begin{bmatrix} C_1\varepsilon(1-t) + C_2\mathbb{B}(t) + C_3\varepsilon^2(1-\mathbb{B}(t)) \\ C_1\varepsilon(1-t) + C_3\varepsilon^2(1-\mathbb{B}(t)) \end{bmatrix}$ and $\overrightarrow{\Lambda}^{\pm}(t) = \overrightarrow{\Gamma}(t) \pm \overrightarrow{w}(t)$. Choosing C_1, C_2 and C_3 suitably, $\overrightarrow{\Lambda}^{\pm}(0) \ge \overrightarrow{0}$, $\overrightarrow{\Lambda}^{\pm}(1) \ge \overrightarrow{0}$ and $\overrightarrow{\mathbb{T}}'\overrightarrow{\Lambda}(t) \le \overrightarrow{0}$ on Ω. Using Theorem 1, $\overrightarrow{\Lambda}^{\pm}(t) \ge \overrightarrow{0}$ on $\overline{\Omega}$. Hence the bounds on $w_1(t)$ and $w_2(t)$ hold. The bounds on the derivatives of w_1 and w_2 follow by using similar arguments in [6]. □

4 Mesh generation and the discrete problem

On $\overline{\Omega}$, a Shishkin mesh with N mesh-intervals is constructed as follows. Let $\Omega^N = \{t_j\}_{j=1}^{N-1}$ then $\overline{\Omega}^N = \{t_j\}_{j=0}^{N}$. Domain $\overline{\Omega}$ is divided into 2 sub-intervals $[0,\lambda]$ and $(\lambda,1]$ such that $\overline{\Omega} = [0,\lambda] \cup (\lambda,1]$. The parameter λ is defined to be

$$\lambda = \min\left\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\right\}.$$

From the total N mesh points, $\frac{N}{2}$ mesh points are placed uniformly on each of the sub-domains $[0,\lambda]$ and $[\lambda,1]$. Let H_1 and H_2 denote the step size in $[0,\lambda]$ and $[\lambda,1]$, respectively. Then $H_1 = \dfrac{2\lambda}{N}$ and $H_2 = \dfrac{2(1-\lambda)}{N}$. Thus,

$$t_j = \begin{cases} jH_1, & \text{for } 0 \le j \le \dfrac{N}{2}, \\ \lambda + \left(j - \dfrac{N}{2}\right)H_2, & \text{for } \dfrac{N}{2}+1 \le j \le N. \end{cases}$$

The discrete problem corresponding to (7)-(8) is defined to be

$$\overrightarrow{\mathbb{T}}^N \overrightarrow{U}(t_j) = E\,\delta^2 \overrightarrow{U}(t_j) + A(t_j) D^+ \overrightarrow{U}(t_j) - \overrightarrow{f}(t_j, \overrightarrow{U}(t_j)) = \overrightarrow{0}, \text{ for } t_j \in \Omega^N, \tag{27}$$

$$\overrightarrow{U}(t_0) = \overrightarrow{u}(t_0) \text{ and } \overrightarrow{U}(t_N) = \overrightarrow{u}(t_N). \tag{28}$$

Here

$$\delta^2 Z(t_j) = \frac{(D^+ - D^-)Z(t_j)}{\overline{h}_j}, \quad D^+ Z(t_j) = \frac{Z(t_{j+1}) - Z(t_j)}{h_{j+1}}, \quad D^- Z(t_j) = \frac{Z(t_j) - Z(t_{j-1})}{h_j},$$

$h_j = t_j - t_{j-1}$, $\overline{h}_j = \dfrac{h_{j+1} + h_j}{2}$, $\overline{h}_0 = \dfrac{h_1}{2}$ and $\overline{h}_N = \dfrac{h_N}{2}$.

5 Error in the numerical method

Let $\overrightarrow{\Upsilon}_1$ and $\overrightarrow{\Upsilon}_2$ be any two vector mesh functions such that $\overrightarrow{\Upsilon}_1(t_0) = \overrightarrow{\Upsilon}_2(t_0)$ and $\overrightarrow{\Upsilon}_1(t_N) = \overrightarrow{\Upsilon}_2(t_N)$. For $t_j \in \Omega^N$,

$$\begin{aligned}
(\overrightarrow{\mathbb{T}}^N \overrightarrow{\Upsilon}_1 - \overrightarrow{\mathbb{T}}^N \overrightarrow{\Upsilon}_2)(t_j)& \\
&= E\,\delta^2 (\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j) + A(t_j) D^+ (\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j) - \overrightarrow{f}(t_j, \overrightarrow{\Upsilon}_1(t_j)) + \overrightarrow{f}(t_j, \overrightarrow{\Upsilon}_2(t_j)) \\
&= E\,\delta^2 (\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j) + A(t_j) D^+ (\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j) - D(t_j)(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j) \\
&= (\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j),
\end{aligned} \tag{29}$$

where $D(t_j) = (d_{ik}(t_j))_{2\times 2}$, $d_{ik}(t_j) = \dfrac{\partial f_i(t_j, \overrightarrow{\zeta}(t_j))}{\partial u_k}$ are intermediate values and $(\overrightarrow{\mathbb{T}}^N)'$ is the Frechet derivative of $\overrightarrow{\mathbb{T}}^N$.

Theorem 4. *For any vector mesh function $\overrightarrow{\Psi}$, the inequalities $\overrightarrow{\Psi}(t_0) \geq \overrightarrow{0}$, $\overrightarrow{\Psi}(t_N) \geq \overrightarrow{0}$ and $(\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi} \leq \overrightarrow{0}$ on Ω^N imply that $\overrightarrow{\Psi} \geq \overrightarrow{0}$ on $\overline{\Omega}^N$.*

Proof. Let i^* and j^* be such that $\Psi_{i^*}(t_{j^*}) = \min\limits_{i,j} \Psi_i(t_j)$ and assume $\Psi_{i^*}(t_{j^*}) < 0$. Then $j^* \neq 0, N$. Let $t_{j^*} \in \Omega^N$. Consider

$$(\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi}(t_{j^*}) = \begin{bmatrix} \varepsilon\,\delta^2 \Psi_1(t_{j^*}) + a_1(t_{j^*}) D^+ \Psi_1(t_{j^*}) - d_{11}(t_{j^*})\Psi_1(t_{j^*}) - d_{12}(t_{j^*})\Psi_2(t_{j^*}) \\ \\ \delta^2 \Psi_2(t_{j^*}) + a_2(t_{j^*}) D^+ \Psi_2(t_{j^*}) - d_{21}(t_{j^*})\Psi_1(t_{j^*}) - d_{22}(t_{j^*})\Psi_2(t_{j^*}) \end{bmatrix}.$$

Thus,

$$((\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi})_{i^*}(t_{j^*}) \geq \begin{cases} \varepsilon\,\delta^2 \Psi_1(t_{j^*}) + a_1(t_{j^*}) D^+ \Psi_1(t_{j^*}) - (d_{11}(t_{j^*}) + d_{12}(t_{j^*}))\,\Psi_1(t_{j^*}), & \text{if } i^* = 1, \\ \\ \delta^2 \Psi_2(t_{j^*}) + a_2(t_{j^*}) D^+ \Psi_2(t_{j^*}) - (d_{21}(t_{j^*}) + d_{22}(t_{j^*}))\,\Psi_2(t_{j^*}), & \text{if } i^* = 2. \end{cases}$$

Using the properties of $A(t_j)$ and $D(t_j)$, $((\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi})_{i^*}(t_{j^*}) > 0$, a contradiction. Hence $\overrightarrow{\Psi} \geq \overrightarrow{0}$ on $\overline{\Omega}^N$. □

Theorem 5. *If $\overrightarrow{\Psi}$ is any vector mesh function on $\overline{\Omega}^N$, then for any $t_j \in \overline{\Omega}^N$*

$$||\overrightarrow{\Psi}(t_j)|| \leq \max\left\{||\overrightarrow{\Psi}(t_0)||,\ ||\overrightarrow{\Psi}(t_N)||,\ \frac{1}{\alpha}||(\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi}(t_j)||_{\Omega^N}\right\}.$$

Proof. Let $t_j \in \overline{\Omega}^N$. Consider

$$\overrightarrow{\Phi}^{\pm}(t_j) = \max\left\{||\overrightarrow{\Psi}(t_0)||,\ ||\overrightarrow{\Psi}(t_N)||,\ \frac{1}{\alpha}||(\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Psi}(t_j)||_{\Omega^N}\right\} \pm \overrightarrow{\Psi}(t_j).$$

Then $\overrightarrow{\Phi}^{\pm}(t_j) \geq \overrightarrow{0}$ for $j = 0, N$. Using the properties of $A(t_j)$ and $D(t_j)$, $(\overrightarrow{\mathbb{T}}^N)'\overrightarrow{\Phi}^{\pm} \leq \overrightarrow{0}$ on Ω^N. Hence by Theorem 4, $\overrightarrow{\Phi}^{\pm} \geq \overrightarrow{0}$ on $\overline{\Omega}^N$. □

Using Theorem 5 with $(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)$,

$$||(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)|| \leq \max\{||(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_0)||,\ ||(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_N)||, \tfrac{1}{\alpha}||(\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)||_{\Omega^N}\}. \tag{30}$$

Since $(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_0) = \overrightarrow{0}$ and $(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_N) = \overrightarrow{0}$, (30) becomes,

$$||(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)|| \leq C||(\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)||_{\Omega^N}. \tag{31}$$

Now from (29) and (31),

$$||(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)|| \leq C||(\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Upsilon}_1 - \overrightarrow{\Upsilon}_2)(t_j)||_{\Omega^N} = C||\overrightarrow{\mathbb{T}}^N\overrightarrow{\Upsilon}_1(t_j) - \overrightarrow{\mathbb{T}}^N\overrightarrow{\Upsilon}_2(t_j)||_{\Omega^N}. \tag{32}$$

Theorem 6. *Let $\overrightarrow{u}$ be the solution of (7)-(8) and $\overrightarrow{U}$ be the solution of (27)-(28). Then for $t_j \in \overline{\Omega}^N$,*

$$||(\overrightarrow{U} - \overrightarrow{u})(t_j)|| \leq CN^{-1}\ln N. \tag{33}$$

Proof. Let $t_j \in \Omega^N$. From (32),

$$||(\overrightarrow{U} - \overrightarrow{u})(t_j)|| \leq C||(\overrightarrow{\mathbb{T}}^N\overrightarrow{U} - \overrightarrow{\mathbb{T}}^N\overrightarrow{u})(t_j)||.$$

Consider

$$\begin{aligned}
||(\overrightarrow{\mathbb{T}}^N\overrightarrow{u} - \overrightarrow{\mathbb{T}}^N\overrightarrow{U})(t_j)|| &= ||\overrightarrow{\mathbb{T}}^N\overrightarrow{u}(t_j)|| = ||(\overrightarrow{\mathbb{T}}^N\overrightarrow{u} - \overrightarrow{\mathbb{T}}\overrightarrow{u})(t_j)|| \\
&\leq E||(\delta^2\overrightarrow{u} - \overrightarrow{u}'')(t_j)|| + ||A(t_j)||\,||(D^+\overrightarrow{u} - \overrightarrow{u}')(t_j)|| \\
&\leq E\left(||(\delta^2\overrightarrow{v} - \overrightarrow{v}'')(t_j)|| + ||(\delta^2\overrightarrow{w} - \overrightarrow{w}'')(t_j)||\right) \\
&\quad + ||A(t_j)||\left(||(D^+\overrightarrow{v} - \overrightarrow{v}')(t_j)|| + ||(D^+\overrightarrow{w} - \overrightarrow{w}')(t_j)||\right).
\end{aligned}$$

As the bounds for $\overrightarrow{v}$ and $\overrightarrow{w}$ are similar to the corresponding components in [13], (33) follows by using similar procedure adopted in [13]. □

6 The continuation method

The nonlinear system of ordinary differential equations in (7)-(8) is modified to an artificial nonlinear system of partial differential equations as given below. For $(t,x) \in (0,1) \times (0,X]$,

$$\begin{aligned} -\frac{\partial u_1(t,x)}{\partial x} + \varepsilon \frac{\partial^2 u_1(t,x)}{\partial t^2} + a_1(t)\frac{\partial u_1(t,x)}{\partial t} - f_1(t, \vec{u}(t,x)) &= 0, \\ -\frac{\partial u_2(t,x)}{\partial x} + \frac{\partial^2 u_2(t,x)}{\partial t^2} + a_2(t)\frac{\partial u_2(t,x)}{\partial t} - f_2(t, \vec{u}(t,x)) &= 0, \end{aligned} \tag{34}$$

with $\vec{u}(0,x) = \vec{u}_0,\ \vec{u}(1,x) = \vec{u}_1,\ x \geq 0$ and $\vec{u}(t,0) = \vec{u}_{init}(t),\ 0 < t < 1.$

The above system of equations can be written together in vector form as

$$\begin{aligned} &-\frac{\partial \vec{u}(t,x)}{\partial x} + E\frac{\partial^2 \vec{u}(t,x)}{\partial t^2} + A(t)\frac{\partial \vec{u}(t,x)}{\partial t} - \vec{f}(t, \vec{u}(t,x)) = \vec{0}, \\ &\vec{u}(0,x) = \vec{u}_0,\ \vec{u}(1,x) = \vec{u}_1,\ x \geq 0 \text{ and } \vec{u}(t,0) = \vec{u}_{init}(t),\ 0 < t < 1, \end{aligned} \tag{35}$$

where E and A are diagonal matrices as in (7). The continuation method developed for a scalar nonlinear DE of RD type in [3] is modified appropriately for a nonlinear system of DEs of CD type as given below which is used to solve (35). For $j = 1,\ldots,N$ and $k = 1,\ldots K$,

$$-D_x^- \vec{U}(t_j,x_k) + E\,\delta_t^2 \vec{U}(t_j,x_k) + A(t_j)D_t^+ \vec{U}(t_j,x_k) - \vec{f}(t_j, \vec{U}(t_j,x_{k-1})) = \vec{0}, \tag{36}$$

$$\begin{aligned} \vec{U}(t_0,x_k) &= \vec{u}(t_0),\quad \vec{U}(t_N,x_k) = \vec{u}(t_N) \text{ for all } k \text{ and} \\ \vec{U}(t_j,x_0) &= \vec{u}_{init}(t_j) \text{ for all } t_j \in \overline{\Omega}^N, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \delta_t^2 Z(t_j,x_k) &= \frac{(D_t^+ - D_t^-)Z(t_j,x_k)}{\bar{h}_j}, \qquad D_t^+ Z(t_j,x_k) = \frac{Z(t_{j+1},x_k) - Z(t_j,x_k)}{h_{j+1}}, \\ D_t^- Z(t_j,x_k) &= \frac{Z(t_j,x_k) - Z(t_{j-1},x_k)}{h_j}, \qquad D_x^- Z(t_j,x_k) = \frac{Z(t_j,x_k) - Z(t_j,x_{k-1})}{h_x}. \end{aligned}$$

Here

$$\vec{u}_{init}(t) = \vec{u}(0) + t(\vec{u}(1) - \vec{u}(0)),$$

$h_x = x_k - x_{k-1}$ and K is determined as below. Define,

$$Err(k) = \max_{1 \leq j \leq N} \left(\frac{||\vec{U}(t_j,x_k) - \vec{U}(t_j,x_{k-1})||}{h_x} \right) \text{ for } k = 1,\ldots,K. \tag{38}$$

The h_x is chosen such that $Err(k)$ decreases with increasing k; precisely,

$$Err(k) \leq Err(k-1) \text{ for all } k, 1 < k \leq K. \tag{39}$$

and K such that

$$Err(K) \leq tol, \tag{40}$$

where tol denotes error tolerance.

Algorithm :

Step 1: Begin from x_0 with $h_x = 1$.

Step 2: Suppose (39) is not satisfied for some k, then quit the current step and begin from x_{k-1} with h_x as $h_x/2$. Continue halving h_x until finding a h_x for which (39) is satisfied.

Step 3: If (39) is satisfied at each h_x, then continue the procedure until either (40) is satisfied or $K = 100$.

Step 4: If (40) is not satisfied, then it is assumed that the stepping process is stalled due to the choice of a large h_x. In such a case, the entire process is repeated from x_0 with $h_x/2$ instead of h_x.

Step 5: If (40) is satisfied, then $\overrightarrow{U}(t_j, x_K)$ are taken as the numerical approximations to the solution of (35).

7 Illustrations

Three different problems are analyzed here. Aforesaid continuation technique is used to solve all the problems. Tolerance $'tol'$ is chosen to be 0.00001. Notations D^N, C_p^N and p^N denote the parameter-uniform maximum pointwise error, parameter-uniform error constant and parameter-uniform rate of convergence, respectively and given by

$$\begin{aligned} D^N &= \max_{\varepsilon} D_\varepsilon^N \quad \text{where } D_\varepsilon^N = \| \overrightarrow{U}^N - \overrightarrow{U}^{2N} \|, \\ p^N &= \log_2 \frac{D^N}{D^{2N}}, \; C_p^N = \frac{D^N N^{p^\star}}{1 - 2^{-p^\star}} \quad \text{where } p^\star = \min_N p^N. \end{aligned}$$

Example 1. Consider the nonlinear system

$$E \overrightarrow{u}''(t) + A(t) \overrightarrow{u}'(t) - \overrightarrow{f}(t, \overrightarrow{u}(t)) = \overrightarrow{0}, \; t \in (0,1),$$

with $\overrightarrow{u}(0) = (\sin(7), \cos(\pi/3))$, $\overrightarrow{u}(1) = \left(e^{-0.8}, \dfrac{\sqrt{3}}{5+\sqrt{2}} \right)$, where

$$E = \begin{bmatrix} \varepsilon & 0 \\ 0 & 1 \end{bmatrix}, \qquad A(t) = \begin{bmatrix} 2 - \sin(t) + \dfrac{t}{5} & 0 \\ 0 & \sqrt{2} + e^{t/2} \end{bmatrix}$$

and

$$\overrightarrow{f}(t, \overrightarrow{u}(t)) = \begin{bmatrix} (u_1(t))^7 + (3+t)u_1(t) - \cos(\frac{3}{2})u_2(t) - \dfrac{1}{\sqrt{10}} \\ (u_2(t))^5 + 2u_2(t) - u_1(t) - t^2 \end{bmatrix}.$$

For Example 1, the values of D^N, C_p^N, p^N are given in Table 1 and a graph of the numerical solution for $N = 256$ and $\varepsilon = 2^{-6}$ is depicted in Figure 1. For $N = 256$ and $\varepsilon = 2^{-1}, 2^{-4}, 2^{-8}$, changes in the components of $\overrightarrow{u}(t)$ are depicted in Figure 2. Moreover, the *Log-log plot* for the error in the suggested computational method is presented in Figure 3.

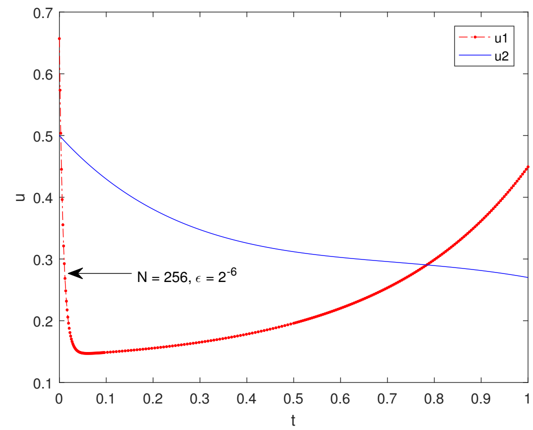

Figure 1: Solution profile of Example 1.

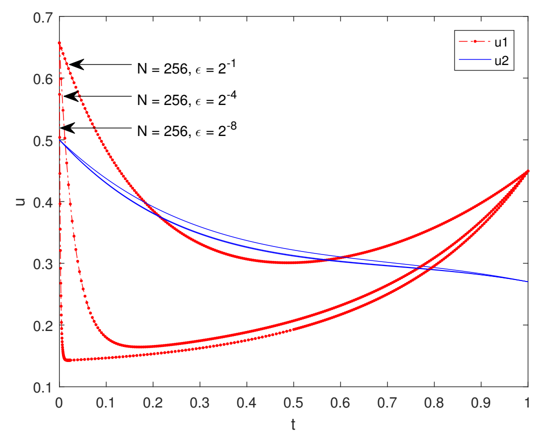

Figure 2: Changes in the components of vec$u(t)$ in Example 1.

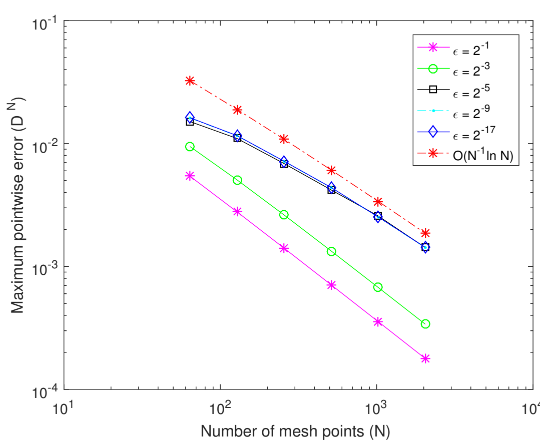

Figure 3: *Log-log plot* for the error in Example 1.

Table 1: .Values of D^N, C_p^N and p^N for $\alpha = 0.9$.

ε	N					
	64	128	256	512	1024	2048
2^{-1}	5.4241e-03	2.7804e-03	1.4075e-03	7.0810e-04	3.5514e-04	1.7784e-04
2^{-3}	9.4574e-03	5.0606e-03	2.6218e-03	1.3355e-03	6.7405e-04	3.3862e-04
2^{-5}	1.5099e-02	1.0997e-02	6.8717e-03	4.1941e-03	2.5723e-03	1.4230e-03
2^{-7}	1.5905e-02	1.1436e-02	7.0977e-03	4.3356e-03	2.4959e-03	1.4111e-03
2^{-9}	1.6143e-02	1.1560e-02	7.1604e-03	4.3749e-03	2.5178e-03	1.4232e-03
2^{-11}	1.6206e-02	1.1592e-02	7.1767e-03	4.3850e-03	2.5235e-03	1.4263e-03
2^{-13}	1.6222e-02	1.1600e-02	7.1809e-03	4.3876e-03	2.5249e-03	1.4271e-03
2^{-15}	1.6226e-02	1.1603e-02	7.1819e-03	4.3882e-03	2.5252e-03	1.4273e-03
2^{-17}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5253e-03	1.4274e-03
2^{-19}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
2^{-21}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
2^{-23}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
2^{-25}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
2^{-27}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
2^{-29}	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5254e-03	1.4274e-03
D^N	1.6227e-02	1.1603e-02	7.1822e-03	4.3884e-03	2.5723e-03	1.4274e-03
p^N	4.8391e-01	6.9202e-01	7.1073e-01	7.7062e-01	8.4971e-01	
C_p^N	4.2608e-01	4.2608e-01	3.6885e-01	3.1519e-01	2.5838e-01	2.0051e-01

Example 2. Let E in Example 1 be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; for this problem the values of D^N, C_p^N, p^N are given in Table 2 and a graph of the numerical solution for $N = 256$ with $\varepsilon = \varepsilon_1 = \varepsilon_2 = 1$ is depicted in Figure 4. It should be noted that in this case both the components $u_1(t)$ and $u_2(t)$ of $\overrightarrow{u}(t)$ changes smoothly throughout the domain.

Example 3. Let E in Example 1 be $\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}$; for this problem the values of D^N, C_p^N, p^N are given in Table 3 and a graph of the numerical solution for $N = 256$ with $\varepsilon = \varepsilon_1 = \varepsilon_2 = 2^{-1}, 2^{-4}, 2^{-8}$ is depicted in Figure 5. It should be noted that in this case both the components $u_1(t)$ and $u_2(t)$ of $\overrightarrow{u}(t)$ exhibits a boundary layer of same width near $t = 0$ and they are smooth elsewhere.

8 Conclusion

In this article, a robust, layer-resolving and parameter-uniform numerical method is developed for a multiscale nonlinear system of SPDEs of CD type. From the tables, it is evident that the parameter-uniform maximum pointwise error (D^N) monotonically decreases when the number of mesh points (N) increases. Further, from the tables, we also observe that the proposed method is almost first order parameter-uniform convergent. This is in agreement with Theorem 6. From Figure 1 we observe that the component $u_1(t)$ of the solution $\overrightarrow{u}(t)$ of Example 1 exhibits a boundary layer near the boundary $t = 0$ whereas the component $u_2(t)$ changes smoothly throughout the domain. Moreover, from Figure 2 we perceive that the

Table 2: Values of D^N, C_p^N and p^N for $\varepsilon_1 = \varepsilon_2 = 1$ and $\alpha = 0.9$.

	N					
	64	128	256	512	1024	2048
	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
D^N	5.1107e-03	2.6212e-03	1.3273e-03	6.6785e-04	3.3498e-04	1.6776e-04
p^N	9.6331e-01	9.8173e-01	9.9088e-01	9.9545e-01	9.9772e-01	
C_p^N	5.7644e-01	5.7644e-01	5.6913e-01	5.5835e-01	5.4605e-01	5.3318e-01

Table 3: Values of D^N, C_p^N and p^N for $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and $\alpha = 0.9$.

ε	N					
	64	128	256	512	1024	2048
2^{-1}	8.9385e-03	4.6732e-03	2.3892e-03	1.2079e-03	6.0732e-04	3.0451e-04
2^{-4}	1.8745e-02	1.5090e-02	1.0311e-02	7.1346e-03	3.4133e-03	2.3220e-03
2^{-6}	1.9000e-02	1.5238e-02	1.0365e-02	7.1664e-03	4.0345e-03	2.1461e-03
2^{-8}	1.9086e-02	1.5288e-02	1.0385e-02	7.1785e-03	4.0399e-03	2.1487e-03
2^{-10}	1.9109e-02	1.5303e-02	1.0391e-02	7.1820e-03	4.0415e-03	2.1495e-03
2^{-12}	1.9116e-02	1.5306e-02	1.0393e-02	7.1828e-03	4.0419e-03	2.1497e-03
2^{-14}	1.9117e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1497e-03
2^{-16}	1.9117e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1497e-03
2^{-18}	1.9118e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1498e-03
2^{-20}	1.9118e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1498e-03
2^{-22}	1.9118e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1498e-03
2^{-24}	1.9118e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.1498e-03
D^N	1.9118e-02	1.5307e-02	1.0393e-02	7.1831e-03	4.0420e-03	2.3220e-03
p^N	3.2067e-01	5.5859e-01	5.3294e-01	8.2953e-01	7.9969e-01	
C_p^N	3.6401e-01	3.6401e-01	3.0867e-01	2.6644e-01	1.8724e-01	1.3434e-01

component $u_1(t)$ of the solution $\overrightarrow{u}(t)$ of Example 1 changes very rapidly near the boundary $t = 0$ when the perturbation parameter ε tends to zero. From Figure 4 we notice that both the components $u_1(t)$ and $u_2(t)$ of the solution $\overrightarrow{u}(t)$ of Example 2 changes smoothly throughout the domain. From Figure 5 we notice that both the components $u_1(t)$ and $u_2(t)$ of the solution $\overrightarrow{u}(t)$ of Example 3 exhibits a boundary layer of same width near $t = 0$ and they are smooth elsewhere. The *Log* $-$ *log plot* for the error in the suggested numerical method for Example 1 is presented in Figure 3. From this figure we perceive that the maximum pointwise errors are bounded by $O(N^{-1}\ln N)$ which is proved in Theorem 6.

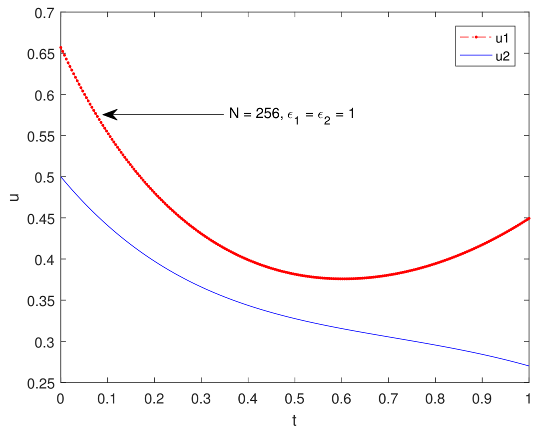

Figure 4: Solution profile of Example 2.

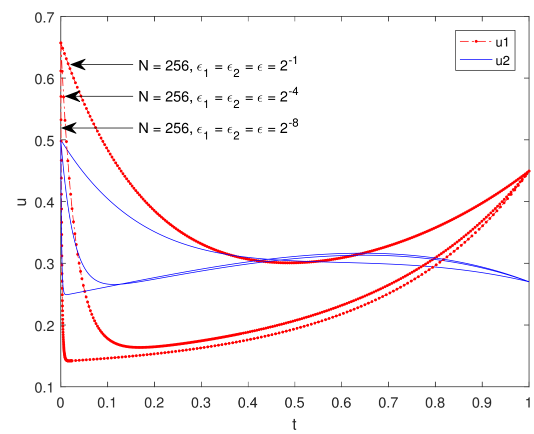

Figure 5: Solution profile of Example 3.

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