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Tau algorithm for fractional delay differential equations utilizing seventh-kind Chebyshev polynomials

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Abstract. Herein, we present an algorithm for handling fractional delay differential equations (FDDEs). Chebyshev polynomials (CPs) class of the seventh kind is a subclass of the generalized Gegenbauer (ultraspherical) polynomials. The members of this class make up the basis functions in this paper. Our suggested numerical algorithm is derived using new theoretical findings about these polynomials and their shifted counterparts. We will use the Tau method to convert the FDDE with the governing conditions into a linear algebraic system, which can then be solved numerically using a suitable procedure. We will give a detailed discussion of the convergence and error analysis of the shifted Chebyshev expansion. Lastly, some numerical examples are provided to verify the precision and applicability of the proposed strategy.

Keywords: Chebyshev polynomials, trigonometric representation, Tau method, fractional differential equations, convergence analysis.
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1 Introduction

Fractional differential equations (FDEs) have been more significant over the past few decades due to their widespread use in several disciplines, including but not limited to physics, chemistry, and engineering. [23, 35, 37]. These equations can describe many phenomena in many applied research fields. The need for finding numerical solutions arises because exact solutions to the vast majority of these equations do not exist. We list some of the numerical algorithms employed to achieve numerical solutions: The Adomian decomposition method [8, 40], homotopy method and its variations [29, 33, 34], collocation method [38, 41, 52], wavelets methods [25, 51], and reproducing kernel method [27].

Important FDEs include fractional delay differential equations (FDDEs). Several branches of mathematics and science find a use for these equations, and various numerical methods for solving these equations have been developed. For example, in [22, 44], the authors followed an operational matrix approach for treating these equations. Taylor wavelet method is followed in [45]. The reproducing kernel method is followed in [9]. A Jacobi collocation method is applied in [36]. Another collocation procedure is followed in [39] to reduce the FDDE into a system of

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algebraic equations via Fibonacci polynomials. Furthermore, in [5], the authors presented Tau algorithm to solve the FDDEs using the first-kind shifted CPs.

Orthogonal polynomials have vital roles in numerical analysis and approximation theory, see, for example, [21, 30]. The derivatives formulas of different orthogonal polynomials are crucial in finding approximate solutions for different types of DEs. As an illustration, the author of [4] developed explicit derivative formulas for the sixth-kind CPs and used them to handle a particular kind of Burger's equation. For the purpose of addressing specific types of even-order BVPs in [19], other formulae for the third- and fourth-kind CPs were established and utilized.

It is well known that spectral methods are powerful techniques for solving DEs, particularly FDEs [6, 7, 15]. Based on these methods, solutions to DEs are expressed as linear combinations of polynomials that are often orthogonal. The Galerkin, Tau, and collocation methods are the fundamental versions of spectral methods, see [11–13]. Due to their significant, many authors used these techniques to deal with different problems; for example, the authors in [14] developed a Galerkin spectral method for the fractional Rayleigh-Stokes problem. In [16], the authors proposed a Tau approach for treating certain fractional integro-differential equations. In [48], the authors proposed a high-precision numerical algorithm for a fractional integro-differential equation using the spectral collocation method. For further similar works, one can see [17, 26, 42, 43, 46, 50].

Finding approximate solutions by spectral methods requires selecting appropriate basis functions. Among these functions, which have proven their efficiency in finding these solutions, are different kinds of CPs. Four types of CPs are known to be special ones of Jacobi polynomials; see [32]; however, both the fifth- and sixth-kind CPs are special cases of the generalized Gegenbauer polynomials; see [18, 31, 49]. It is worth noting that all six types of CPs may be written in trigonometric forms, which provides these polynomials with a great benefit. One can consult [1, 2, 47] for examples of how all six kinds of CPs were used to treat various types of DEs.

The primary purpose of this study is to present a new set of polynomials that is a subset of the more generalized Gegenbauer set. We call them CPs of the seventh kind. New theoretical formulas involving these polynomials and their shifted versions will be derived. In addition, we aim to use these polynomials in conjunction with Tau approach to address a specific class of DDEs. Here is a brief synopsis of the paper's primary objectives:

- Introducing a new kind of polynomials, namely, seventh-kind CPs.
- Establishing some new theoretical results concerned with these polynomials.
- Developing a numerical procedure for solving a type of FDDEs using shifted CPs of the seventh-kind using Tau method.
- Investigation of the error analysis of the seventh-kind Chebyshev expansion.
- Offering some examples to examine the scheme's feasibility and precision.

To the best of our knowledge, some advantages of the proposed technique can be mentioned as follows:

- Choosing the shifted CPs of the seventh-kind as basis functions and taking a few terms of the retained modes makes it possible to produce approximations with excellent precision. Less calculation is required. In addition, the resulting errors are small.
- Compared to the other CPs, the shifted CPs of the seventh-kind are not well studied or used. This motivates us to find theoretical findings concerning them. Furthermore, we found that the obtained numerical results are satisfactory if used as basis functions.

The novelties of this paper can be listed as follows:

- Some formulas of the seventh-kind CPs and their shifted ones are presented in simple forms that do not involve hypergeometric forms.
- The employment of these basis functions in the numerical treatment of the FDDEs is new.

Here is the outline for this paper: Section 2 briefly introduces the theory of fractional calculus. Moreover, a survey of the generalized Gegenbauer polynomials and CPs is provided. We also introduce the seventh-kind CPs, which are particular ones of the generalized ultraspherical polynomials. In Section 3, some crucial relations of the seventh-kind CPs and their shifted counterparts are developed. Section 4 presents a numerical method for solving a certain class of FDDE using Tau method. Convergence and error analysis is covered in Section 5. Section 6 provides some numerical examples to illustrate the theoretical outcomes. Conclusions are presented in Section 7.

2 Preliminaries and key formulas

The main aim of this section is to give a summary of some essentials of fractional calculus. Furthermore, we give an overview of the CPs. Some properties of the generalized Gegenbauer polynomials are also presented.

2.1 An outline of the fundamentals of fractional calculus

Definition 1 ([35, 37]). *The Riemann-Liouville fractional integral operator ${}_0I_t^{\alpha}$ of order α on the usual Lebesgue space $L_1[0,1]$ is defined as: for all $t \in (0,1)$*

$$({}_0I_t^{\alpha} g)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} g(\tau)\, d\tau, & \alpha > 0, \\ g(t), & \alpha = 0. \end{cases} \tag{1}$$

Definition 2 ([35, 37]). *The right side Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$(D_*^{\alpha} g)(t) = \left(\frac{d}{dt}\right)^n ({}_0I_t^{n-\alpha} g)(t), n-1 \le \alpha < n, \quad n \in \mathbb{N}. \tag{2}$$

Definition 3 ([35, 37]). *The fractional differential operator in Caputo sense is defined as*

$$(D^{\alpha} g)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} g^{(n)}(\tau)\, d\tau,\ \alpha > 0, t > 0, \tag{3}$$

where $n-1 \le \alpha < n, n \in \mathbb{N}$.

The following properties are satisfied by D^{α} for $n-1 \le \alpha < n$.

$$\begin{aligned} (D^{\alpha} I^{\alpha} g)(t) &= g(t), \\ (I^{\alpha} D^{\alpha} g)(t) &= g(t) - \sum_{k=0}^{n-1} \frac{g^{(k)}(0^+)}{k!} (t-a)^k,\ t > 0, \\ D^{\alpha}\, t^k &= \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}\, t^{k-\alpha}, \quad k \in \mathbb{N}, k \ge \lceil \alpha \rceil, \end{aligned} \tag{4}$$

where the smallest positive integer that is greater than or equal to α is denoted by $\lceil \alpha \rceil$.

2.2 A overview on Chebyshev and generalized Gegenbauer polynomials

Numerical analysis and approximation theory depend greatly on CPs of all kinds. The normalized Jacobi polynomials $R_n^{(\gamma,\delta)}(x)$ is defined as

$$R_m^{(\gamma,\delta)}(x) = \frac{m!}{(\gamma+1)_m} P_m^{(\gamma,\delta)}(x), \tag{5}$$

where $P_m^{(\gamma,\delta)}(x)$ are the classical Jacobi polynomials involving four well-known kinds of CPs [3].

If $T_n(x), U_n(x), V_n(x)$ and $W_n(x)$ denote the four kinds of CPs in order, then we have

$$T_m(x) = R_m^{(-\frac{1}{2},-\frac{1}{2})}(x), \quad U_m(x) = (m+1)\,R_m^{(\frac{1}{2},\frac{1}{2})}(x), \quad V_m(x) = R_m^{(-\frac{1}{2},\frac{1}{2})}(x), \quad W_m(x) = (2m+1)\,R_m^{(\frac{1}{2},-\frac{1}{2})}(x).$$

An important characteristic of all four kinds of CPs is that they have trigonometric representations. This, of course, gives a great advantage to these polynomials. The following are the four trigonometric representations of CPs:

$$T_j(\cos\theta) = \cos(j\,\theta), \qquad U_j(\cos\theta) = \frac{\sin((j+1)\,\theta)}{\sin\theta}, \tag{6}$$

$$V_j(\cos\theta) = \frac{\cos\left(\left(j+\frac{1}{2}\right)\theta\right)}{\cos\left(\frac{\theta}{2}\right)}, \qquad W_j(\cos\theta) = \frac{\sin\left(\left(j+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}. \tag{7}$$

The generalized Gegenbauer polynomials $G_n^{(\lambda,\mu)}(x)$ are orthogonal polynomials on $[-1,1]$ with respect to: $w(x) = (1-x^2)^{\lambda-\frac{1}{2}}\,|x|^{2\mu}$. These polynomials can be defined as (see, [18, 49]):

$$G_k^{(\lambda,\mu)}(x) = \begin{cases} \dfrac{\left(\frac{k}{2}\right)!\,(\lambda+\mu)_{\frac{k}{2}}}{(\lambda+\mu)_k}\,P_{\frac{k}{2}}^{\left(\lambda-\frac{1}{2},\mu-\frac{1}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ even},\\[2ex] \dfrac{\left(\frac{k-1}{2}\right)!(\lambda+\mu)_{\frac{k+1}{2}}}{(\lambda+\mu)_k}\,x P_{\frac{k-1}{2}}^{\left(\lambda-\frac{1}{2},\mu+\frac{1}{2}\right)}\left(2x^2-1\right), & \text{if } k \text{ odd}, \end{cases} \tag{8}$$

where $(a)_k$ is the Pochhammer function defined as ([10]):

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

The orthogonality relation of the polynomials $G_n^{(\lambda,\mu)}(x)$ is given by ([18])

$$\int_{-1}^{1} w(x)\,G_n^{(\lambda,\mu)}(x)\,G_m^{(\lambda,\mu)}(x)\,dx = \begin{cases} h_n^{\lambda,\mu}, & n \neq m,\\ 0, & n = m, \end{cases} \tag{9}$$

where $h_n^{\lambda,\mu}$ is given by

$$h_n^{\lambda,\mu} = \frac{\left(\Gamma\left(\mu+\frac{1}{2}\right)\right)^2}{(n+\lambda+\mu)\,(\Gamma(\lambda+\mu))^2} \begin{cases} \dfrac{\Gamma\left(\lambda+\frac{n+1}{2}\right)\Gamma\left(\frac{n}{2}+\lambda+\mu\right)}{\left(\frac{n}{2}\right)!\,\Gamma\left(\frac{1+n}{2}+\mu\right)}, & n \text{ even},\\[2ex] \dfrac{\Gamma\left(\frac{n}{2}+\lambda\right)\Gamma\left(\frac{n}{2}+\lambda+\mu+\frac{1}{2}\right)}{\left(\frac{n-1}{2}\right)!\,\Gamma\left(\frac{n}{2}+\mu+1\right)}, & n \, \text{odd}. \end{cases} \tag{10}$$

Remark 1. *Note that the fifth- and sixth-kind CPs are special cases of the generalized polynomials $G_n^{(\lambda,\mu)}(x)$. We have*

$$X_n(x) = G_n^{(0,1)}(x), \quad Y_n(x) = G_n^{(1,1)}(x), \tag{11}$$

where $X_n(x)$ and $Y_n(x)$ are, respectively, the CPs of the fifth-and sixth-kinds that were introduced in [31].

Remark 2. *In this paper, we will consider a new kind of CPs denoted by $Z_k(x)$ that are orthogonal on $[-1,1]$ about $w(x) = \dfrac{x^4}{\sqrt{1-x^2}}$. This means that $Z_k(x) = G_k^{0,2}(x)$. We will call these polynomials "CPs of the seventh-kind".*

Remark 3. *The reason for calling the introduced polynomials "CPs of the seventh-kind" is due to the existence of six kinds of CPs in the literature. Furthermore, we will derive a trigonometric representation of these polynomials.*

It is clear from (8) that the polynomials $Z_k(x)$ can be defined as:

$$Z_k(x)=\begin{cases}\dfrac{(2)_{\frac{k}{2}}\,P_{\frac{k}{2}}^{\left(-\frac{1}{2},\frac{3}{2}\right)}(2x^2-1)}{\left(\frac{5}{2}\right)_{\frac{k}{2}}}, & \text{if } k \text{ even},\\[2ex] \dfrac{x\,\Gamma\left(\frac{k+5}{2}\right)P_{\frac{k-1}{2}}^{\left(-\frac{1}{2},\frac{5}{2}\right)}(2x^2-1)}{\left(\frac{5}{2}\right)_{\frac{k+1}{2}}}, & \text{if } k \text{ odd}.\end{cases} \tag{12}$$

From the orthogonality relation of $G_n^{(\lambda,\mu)}(x)$ given by (9), it is easy to conclude the orthogonality relation of $Z_n(x)$ by setting $\lambda=0$ and $\mu=2$. Explicitly, we have

$$\int_{-1}^{1}\frac{x^4}{\sqrt{1-x^2}}\,Z_n(x)\,Z_m(x)\,dx=\rho_n\,\delta_{n,m}, \tag{13}$$

where

$$\rho_n=\frac{9\,\pi}{8}\begin{cases}\dfrac{1}{(n+1)\,(n+3)}, & \text{if } \ n \text{ even},\\[2ex] \dfrac{(n+1)\,(n+3)}{n\,(n+2)^2\,(n+4)}, & \text{if } \ n \text{ odd},\end{cases} \tag{14}$$

and

$$\delta_{n,m}=\begin{cases}1, & \text{if } n=m,\\ 0, & \text{if } n\neq m.\end{cases} \tag{15}$$

The following two lemmas are the core of several formulas that will be useful in presenting our proposed algorithm in this paper.

Lemma 1. *Consider j to be any non-negative integer. The polynomials $Z_j(x)$ have the following representations:*

$$Z_{2j}(x)=\sum_{r=0}^{j}\frac{(-1)^r\,(2j-r+1)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{j-r}}\,x^{2j-2r}, \tag{16}$$

$$Z_{2j+1}(x)=\sum_{r=0}^{j}\frac{(-1)^r\,(2j-r+2)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{j-r+1}}\,x^{2j-2r+1}. \tag{17}$$

Proof. From the definition in (12), $Z_{2j}(x)$ and $Z_{2j+1}(x)$ can be respectively expressed as:

$$Z_{2j}(x)=\frac{(2)_j}{\left(\frac{5}{2}\right)_j}P_j^{\left(-\frac{1}{2},\frac{3}{2}\right)}\left(2x^2-1\right), \tag{18}$$

$$Z_{2j+1}(x)=\frac{(j+2)!}{\left(\frac{5}{2}\right)_{1+j}}\,x\,P_j^{\left(-\frac{1}{2},\frac{5}{2}\right)}\left(2x^2-1\right). \tag{19}$$

Based on the well-known hypergeometric definition of $P_n^{(\gamma,\delta)}(x)$ ([20]), we can write

$$P_n^{(\gamma,\delta)}(z)=\frac{(-1)^n\,(\delta+1)_n}{n!}\;{}_2F_1\left(\begin{matrix}-n,n+\gamma+\delta+1\\ \delta+1\end{matrix}\,\middle|\,\frac{1+z}{2}\right). \tag{20}$$

Formulas (18) and (19) along with (20) lead to the following two hypergeometric expressions:

$$Z_{2j}(x) =(-1)^j\,(j+1)\ {}_2F_1\left(\begin{array}{c} -j, j+2 \\ \frac{5}{2} \end{array}\middle| x^2\right), \tag{21}$$

$$Z_{2j+1}(x) =\frac{2}{5}(-1)^j(1+j)(2+j)x\ {}_2F_1\left(\begin{array}{c} -j, j+3 \\ \frac{7}{2} \end{array}\middle| x^2\right). \tag{22}$$

In virtue of the well-known definition of ${}_2F_1(z)$ ([10]), it is easy to see that (21) and (22) may be respectively converted into

$$Z_{2j}(x) =(-1)^j\,(j+1)\sum_{r=0}^{j}\frac{(-j)_r(j+2)_r}{\left(\frac{5}{2}\right)_r r!}x^{2r}, \tag{23}$$

$$Z_{2j+1}(x) =\frac{2}{5}(-1)^j(j+1)(j+2)\sum_{r=0}^{j}\frac{(-j)_r(j+3)_r}{\left(\frac{7}{2}\right)_r r!}x^{2r+1}, \tag{24}$$

which can be written alternatively in the following forms

$$Z_{2j}(x) =\sum_{r=0}^{j}\frac{(-1)^r\,(2j-r+1)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{j-r}}x^{2j-2r},$$

$$Z_{2j+1}(x) =\sum_{r=0}^{j}\frac{(-1)^r\,(2j-r+2)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{j-r+1}}x^{2j-2r+1}.$$

This proves formulas (16) and (17). □

The following lemma demonstrates the inversion formulae for the expressions (16) and (17).

Lemma 2. *Consider k to be any non-negative integer. The inversion formulas of the CPs of the seventh kind are given as follows:*

$$x^{2j} =2j!\sum_{r=0}^{j}\frac{(1+j-r)\left(\frac{5}{2}\right)_j}{r!(2j-r+2)!}Z_{2j-2r}(x), \tag{25}$$

$$x^{2j+1} =j!\sum_{r=0}^{j}\frac{(3+2j-2r)\left(\frac{5}{2}\right)_{1+j}}{r!(2j-r+3)!}Z_{2j-2r+1}(x). \tag{26}$$

Proof. We will prove (26). Formula (25) can be similarly proved. Consider the equation

$$x^{2j+1} =\sum_{r=0}^{j}B_{r,j}\,Z_{2j-2r+1}(x), \tag{27}$$

so to prove (26), we should compute $B_{r,j}$. Multiplying both sides of (27) by $w(x)\,Z_{2k+1}(x)$, $k \geq 0$, integrating over $[-1,1]$, yields the following formula after using the orthogonality relation (13)

$$B_{r,j} =\frac{2(1+2j-2r)(3+2j-2r)^2(5+2j-2r)}{9\pi(1+j-r)(2+j-r)}\int_{-1}^{1}\frac{x^{2j+5}}{\sqrt{1-x^2}}Z_{2j-2r+1}(x)\,dx. \tag{28}$$

Based on (17), $B_{r,j}$ may written as

$$B_{r,j} =\frac{4(-1)^{j-r}(1+2j-2r)(3+2j-2r)^2(5+2j-2r)}{45\pi}\sum_{\ell=0}^{j-r}\frac{(3+j-r)_\ell(r-j)_\ell}{\ell!\left(\frac{7}{2}\right)_\ell}\int_{-1}^{1}\frac{x^{2j+2\ell+6}}{\sqrt{1-x^2}}\,dx.$$

It is easy to see that

$$\int_{-1}^{1} \frac{x^{2j+2\ell+6}}{\sqrt{1-x^2}}\,dx = \frac{\sqrt{\pi}\,\Gamma\left(j+\ell+\frac{7}{2}\right)}{(j+\ell+3)!},$$

and accordingly, we have

$$B_{r,j} = \frac{4\sqrt{\pi}(-1)^{j-r}(1+2j-2r)(3+2j-2r)^2(5+2j-2r)}{45\pi}\sum_{\ell=0}^{j-r}\frac{\Gamma\left(\frac{7}{2}+\ell+j\right)(3+j-r)_\ell(r-j)_\ell}{\ell!(j+\ell+3)!\left(\frac{7}{2}\right)_\ell},$$

which can also be written as

$$B_{r,j} = \frac{4(-1)^{j-r}(1+2j-2r)(3+2j-2r)^2(5+2j-2r)\Gamma\left(\frac{7}{2}+j\right)}{45\sqrt{\pi}(j+3)!}\ {}_3F_2\left(\left.\begin{matrix} r-j,\frac{7}{2}+j,3+j-r \\ \frac{7}{2},j+4\end{matrix}\right|1\right). \tag{29}$$

Now, regarding the hypergeometric form that appears in the last equation, it can be summed using symbolic computation. Setting $j-r=p$, and letting

$$M_{p,j} = {}_3F_2\left(\left.\begin{matrix} -p,p+3,\frac{7}{2}+j \\ \frac{7}{2},j+4\end{matrix}\right|1\right),$$

the following recurrence relation is satisfied

$$(2p+7)(j+p+4)M_{p+1,j} + (2p+1)(j-p)M_{p,j}, \quad M_{0,j}=1,$$

and thus $M_{p,j}$ takes the closed form

$$M_{p,j} = \frac{15(-1)^p(1+j-p)_p}{(1+2p)(3+2p)(5+2p)(4+j)_p}.$$

Therefore, $B_{r,j}$ takes the simplified form

$$B_{r,j} = \frac{j!\,(3+2j-2r)\left(\frac{5}{2}\right)_{j+1}}{r!(2j-r+3)!}.$$

Now, formula (26) is obtained. □

In the upcoming section, we will derive some new formulas concerned with the CPs of the seventh-kind and their shifted polynomials that will be useful in the sequel.

3 Some crucial relations of CPs of the seventh-kind and their shifted counterparts

This section provides some basic formulas of the CPs of the seventh-kind and their shifted counterparts that will be the core to implement our proposed numerical algorithm for the FDDE.

3.1 Connection formulas with the first kind CPs

Here, we give the formulas for the connections between CPs of the first and seventh kinds. These formulas will be helpful in what follows. We provide two theorems to this end.

- The first theorem displays the expression that connects the seventh-kind CPs with the first-kind CPs.

- The second theorem displays the expression that connects the first-kind CPs with the seventh-kind CPs.

Theorem 1. *The seventh-kind CPs $Z_j(x)$ are linked with the first kind CPs $T_i(x)$ by the following two formulae:*

$$Z_{2j}(x) = \frac{6}{(2j+1)(2j+3)} \sum_{r=0}^{j} (-1)^{r+1} c_{j-r}\,(r+1)\,(-2j+r-1)\,T_{2j-2r}(x), \tag{30}$$

$$Z_{2j+1}(x) = \frac{6}{(2j+1)(2j+3)(2j+5)} \sum_{r=0}^{j} (-1)^{r}\,(r+1)\,(-2j+r-2)\,(-2j+2r-1)\,T_{2j-2r+1}(x), \tag{31}$$

where

$$c_j = \begin{cases} \frac{1}{2}, & \text{if } j=0,\\ 1, & \text{otherwise}. \end{cases} \tag{32}$$

Proof. The two formulas (30) and (31) can be proved similarly. We just prove (31). We will use the following formula:

$$Z_{2j+1}(x) = \sum_{r=0}^{j} \frac{(-1)^r (2j-r+2)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{1+j-r}} x^{2j-2r+1}. \tag{33}$$

Thanks to the inversion formula of $T_j(x)$ ([28]) given by

$$x^j = \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} c_{j-2r}\, 2^{1-j} \binom{j}{r} T_{j-2r}(x), \tag{34}$$

we get

$$x^{2j+1} = 2^{-2j} \sum_{r=0}^{j} \binom{2j+1}{r} T_{2j-2r+1}(x). \tag{35}$$

If (26) is inserted into (33), it is possible to get the following formula:

$$Z_{2j+1}(x) = \sum_{r=0}^{j} \frac{(-1)^r\,(2j-r+2)!}{(j-r)!\,r!\,\left(\frac{5}{2}\right)_{1+j-r}} \sum_{m=0}^{j-r} 2^{-2j+2r} \binom{1+2j-2r}{m} T_{2j-2r-2m+1}(x). \tag{36}$$

Another form of the final formula is as follows:

$$Z_{2j+1}(x) = \sum_{r=0}^{j} \sum_{\ell=0}^{r} \frac{(-1)^{\ell}\, 4^{\ell-j} \binom{1+2j-2\ell}{r-\ell} (2j-\ell+2)!}{(j-\ell)!\,\ell!\,\left(\frac{5}{2}\right)_{1+j-\ell}} T_{2j-2r+1}(x). \tag{37}$$

Aiming to the reduction of the sum that arises on the right-hand side of the last formula, we set

$$R_{r,j} = \sum_{\ell=0}^{r} \frac{(-1)^{\ell}\, 4^{\ell-j} \binom{1+2j-2\ell}{r-\ell} (2j-\ell+2)!}{(j-\ell)!\,\ell!\,\left(\frac{5}{2}\right)_{1+j-\ell}}.$$

Zeilberger's algorithm ([24]) aids in finding the recurrence relation satisfied by $R_{r,j}$ in the form

$$(1+2j-2r)(2+2j-r)(r+1)R_{r-1,j} + r(3+2j-2r)(3+2j-r)R_{r,j} = 0, \quad R_{0,j} = 1, \tag{38}$$

that can be quickly solved to provide

$$\sum_{\ell=0}^{r} \frac{(-1)^{\ell}\, 4^{-j+\ell} \binom{1+2j-2\ell}{r-\ell} (2j-\ell+2)!}{(j-\ell)!\,\ell!\,\left(\frac{5}{2}\right)_{1+j-\ell}} = \frac{6\,(-1)^r\,(r+1)\,(-2-2j+r)\,(-1-2j+2r)}{(2j+1)\,(2j+3)\,(2j+5)},$$

and consequently, formula (31) can be obtained. □

Theorem 2. *The two connection formulas between the first- and seventh-kinds CPs are given as follows:*

$$T_{2j}(x) = \frac{2j+3}{6} Z_{2j}(x) + \frac{2j}{3} Z_{2j-2}(x) + \frac{2j-3}{6} Z_{2j-4}(x), \quad j \geq 1, \tag{39}$$

$$T_{2j+1}(x) = \frac{(2j+3)(2j+5)}{12(j+1)} Z_{2j+1}(x) + \frac{(2j-1)(2j+1)(2j+3)}{12j(j+1)} Z_{2j-1}(x) + \frac{(2j-3)(2j-1)}{12j} Z_{2j-3}(x), \quad j \geq 1. \tag{40}$$

Proof. Starting from the analytic form of the first-kind CPs given by ([28])

$$T_j(x) = \sum_{r=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(-1)^r 2^{-1+j-2r} j (j-r-1)!}{r!(j-2r)!} x^{j-2r}, \quad j \geq 1, \tag{41}$$

we can write

$$T_{2j}(x) = j \sum_{r=0}^{j} \frac{(-1)^r 2^{2(j-r)}(-1+2j-r)!}{(2j-2r)!r!} x^{2j-2r}. \tag{42}$$

If we insert (25) into (42), we get

$$T_{2j}(x) = 2j \sum_{r=0}^{j} \frac{(-1)^r 2^{2(j-r)}(-1+2j-r)!}{(2j-2r)!r!} \sum_{\ell=0}^{j-r} \frac{(1+j-\ell-r)(j-r)!\left(\frac{5}{2}\right)_{j-r}}{\ell!(2+2j-\ell-2r)!} Z_{2j-2r-2\ell}(x). \tag{43}$$

The last formula can be converted into

$$T_{2j}(x) = \frac{2j}{3} \sum_{r=0}^{j} \sum_{\ell=0}^{r} \frac{(-1)^{\ell}(1+2j-2\ell)(3+2j-2\ell)(1+j-r)(2j-\ell-1)!}{\ell!(2j-\ell-r+2)!(r-\ell)!} Z_{2j-2r}(x). \tag{44}$$

In hypergeometric form, we can write

$$T_{2j}(x) = \frac{1}{3}(2j+3)(2j+1)! \sum_{r=0}^{j} \frac{1+j-r}{r!(2j-r+2)!} \, {}_3F_2\left(\left. \begin{matrix} -r, \frac{1}{2}-j, -2j+r-2 \\ 1-2j, -j-\frac{3}{2} \end{matrix} \right| 1 \right) Z_{2j-2r}(x). \tag{45}$$

It can be seen that

$${}_3F_2\left(\left. \begin{matrix} \frac{1}{2}-j, -r, -2j+r-2 \\ 1-2j, -j-\frac{3}{2} \end{matrix} \right| 1 \right) = \begin{cases} 1, & r=0, \\ \dfrac{2}{2j+3}, & r=1, \\ \dfrac{2j-3}{(j-1)(2j+1)(2j+3)}, & r=2, \\ 0, & r \geq 3, \end{cases}$$

and this leads to the following connection formula

$$T_{2j}(x) = \frac{2j+3}{6} Z_{2j}(x) + \frac{2j}{3} Z_{2j-2}(x) + \frac{2j-3}{6} Z_{2j-4}(x).$$

Formula (40) can be similarly proved. □

Remark 4. *The connection formulas between the CPs of the seventh-kind and the CPs of the first kind can be translated into trigonometric formulas that give the trigonometric representations of the seventh-kind CPs.*

Corollary 1. *The following trigonometric representations hold:*

$$Z_j(\cos(\theta)) = \begin{cases} \dfrac{3((j+3)\cos((j+1)\theta) + (j+1)\cos((j+3)\theta))\sec^3(\theta)}{4(j+1)(j+3)}, & j \text{ even}, \\ \dfrac{3((j+4)((j+3)\cos(j\theta) + 2j\cos((j+2)\theta)) + j(j+1)\cos((j+4)\theta))\sec^4(\theta)}{8j(j+2)(j+4)}, & j \text{ odd}. \end{cases} \tag{46}$$

Proof. To demonstrate formula (46), and if j is replaced respectively by $2j$ and $2j+1$, it can be shown that (46) can be decomposed into the two formulae that are listed below:

$$Z_{2j}(\cos(\theta)) = \frac{3}{4}\left(\frac{\cos((2j+1)\theta)}{2j+1} + \frac{\cos((2j+3)\theta)}{2j+3}\right)\sec^3(\theta), \tag{47}$$

$$\begin{aligned} Z_{2j+1}(\cos(\theta)) = {} & \frac{3\sec^4(\theta)}{8(2j+1)(2j+3)(2j+5)} \\ & \times ((2j+5)(2(j+2)\cos((2j+1)\theta) + 2(2j+1)\cos((2j+3)\theta)) \\ & + 2(2j+1)(j+1)\cos((2j+5)\theta)). \end{aligned} \tag{48}$$

Using symbolic computation, particularly the Mathematica software, we can verify that the two identities are consequences of the two connection formulas (30) and (31). □

3.2 The first derivative of the seventh-kind CPs

The first derivative of the seventh-kind CPs in terms of their original ones is shown in this section. The next theorem demonstrates this conclusion.

Theorem 3. *The following formulas holds for the first derivative of $Z_k(x)$*

$$\frac{dZ_{2k}(x)}{dx} = 2\sum_{p=0}^{k-1}(2k-2p+1)Z_{2k-2p-1}(x),\ k \geq 1, \tag{49}$$

$$\begin{aligned} \frac{dZ_{2k+1}(x)}{dx} = {} & \frac{4}{(1+2k)(3+2k)(5+2k)} \times \left(\sum_{p=0}^{\lfloor \frac{k}{2} \rfloor}(k-2p+1)\left((1+2k)^2(3+2k)+64(1+k)p-64p^2\right)Z_{2k-4p}(x)\right. \\ & \left. + \sum_{p=0}^{\lfloor \frac{k-1}{2} \rfloor}(k-2p)\left((1+2k)^2(11+2k)-64kp+64p^2\right)Z_{2k-4p-2}(x)\right), \quad k \geq 0. \end{aligned} \tag{50}$$

Proof. The proof of the two formulas is similar. We will prove (49). Based on (16), it is not difficult to write

$$\frac{dZ_{2k}(x)}{dx} = 2\sum_{r=0}^{k-1}\frac{(-1)^r(2k-r+1)!}{(k-r-1)!\,r!\left(\frac{5}{2}\right)_{k-r}}x^{2k-2r-1}, \tag{51}$$

which gives after the application of (26) the following formula

$$\frac{dZ_{2k}(x)}{dx} = 2\sum_{r=0}^{k-1}\frac{(-1)^r(2k-r+1)!}{r!\left(\frac{5}{2}\right)_{k-r}}\sum_{\ell=0}^{k-r-1}\frac{(1+2k-2\ell-2r)\left(\frac{5}{2}\right)_{k-r}}{\ell!(2k-\ell-2r+1)!}Z_{2k-2r-2\ell-1}(x). \tag{52}$$

The last formula can be arranged again to give the alternative one:

$$\frac{dZ_{2k}(x)}{dx} = 2\sum_{p=0}^{k-1}(2k-2p+1)\sum_{\ell=0}^{p}\frac{(-1)^\ell(2k-\ell+1)!}{\ell!(2k-\ell-p+1)!(p-\ell)!}Z_{2k-2p-1}(x). \tag{53}$$

Thanks to Zeilberger's algorithm ([24]), and letting

$$H_p = \sum_{\ell=0}^{p}\frac{(-1)^\ell(2k-\ell+1)!}{\ell!(2k-\ell-p+1)!(p-\ell)!},$$

it is easy to conclude that $H_{p+1} - H_p = 0$, $H_0 = 1$. This immediately gives

$$H_p = \sum_{\ell=0}^{p}\frac{(-1)^\ell(2k-\ell+1)!}{\ell!(2k-\ell-p+1)!(p-\ell)!} = 1,$$

and accordingly, (53) is converted into

$$\frac{d\,Z_{2k}(x)}{d\,x} = 2\sum_{p=0}^{k-1}(2k-2p+1)\,Z_{2k-2p-1}(x).$$

This ends the proof. □

3.3 Shifted CPs of the seventh-kind

This section presents some elementary properties of the shifted CPs of the seventh-kind and some of their fundamental formulas that will be required to propose our numerical algorithm.

The shifted CPs of seventh-kind $\bar{Z}_n(x)$ can be defined on $[0,1]$ by

$$\bar{Z}_n(x) = Z_n(2x-1). \tag{54}$$

It is evident from (13) that the polynomials $\bar{Z}_n(x), n \geq 0$, are orthogonal on the interval $[0,1]$. We have

$$\int_0^1 \frac{(2x-1)^4}{\sqrt{x-x^2}}\,\bar{Z}_n(x)\,\bar{Z}_m(x)\,dx = \rho_n\,\delta_{n,m}, \tag{55}$$

where ρ_n are as given in (14). The formulas concerned with the seventh-kind CPs can be changed to yield their counterparts of shifted polynomials. Some of these results are direct, and others need some procedures to be deduced.

Corollary 2. *The shifted CPs of the seventh-kind have the following connections with the shifted polynomials of the first kind $T_j^*(x)$:*

$$\bar{Z}_{2j}(x) = \frac{6}{(2j+1)(2j+3)}\sum_{r=0}^{j}(-1)^{r+1}\,c_{j-r}\,(r+1)\,(-2j+r-1)\,T^*_{2j-2r}(x), \tag{56}$$

$$\bar{Z}_{2j+1}(x) = \frac{6}{(2j+1)(2j+3)(2j+5)}\sum_{r=0}^{j}(-1)^{r}\,(r+1)\,(-2j+r-2)\,(-2j+2r-1)\,T^*_{2j-2r+1}(x). \tag{57}$$

Proof. The proof of this corollary can be easily obtained after replacing x with $(2x-1)$ in the two connection formulas (30) and (31). □

Theorem 4. *The analytic form of the polynomial $\bar{Z}_i(x)$ is given as follows:*

$$\bar{Z}_i(x) = \sum_{p=0}^{i} g_{p,i}\,x^p, \tag{58}$$

where

$$g_{p,i} = \begin{cases} \dfrac{3(i+2)}{2(i+1)(i+3)}, & \text{if } p=0, \quad i \text{ even},\\[2ex] \dfrac{3(-1)^{i/2}4^p}{(i+1)(i+3)(2p)!}\displaystyle\sum_{j=\left\lfloor\frac{p+1}{2}\right\rfloor}^{\frac{i}{2}} \dfrac{(-1)^{j+p}(2+i-2j)j(2+i+2j)(2j+p-1)!}{(2j-p)!}, & \text{if } p\neq 0, \quad i \text{ even},\\[2ex] \dfrac{3(-1)^{\frac{i+1}{2}}2^{-1+2p}}{i(i+2)(i+4)(2p)!}\displaystyle\sum_{j=\left\lfloor\frac{p}{2}\right\rfloor}^{\frac{i-1}{2}} \dfrac{(-1)^{j+p}(1+i-2j)(1+2j)^2(3+i+2j)(2j+p)!}{(2j-p+1)!}, & \text{if } p\neq 0, \quad i \text{ odd}. \end{cases} \tag{59}$$

Proof. The proof can be done by starting with the inversion formula of the shifted CPs of the first kind given by

$$x^{\ell} = (2\ell)!\sum_{r=0}^{\ell}\frac{c_{\ell-r}\,2^{1-2\ell}}{(2\ell-r)!\,r!}\,T^{*}_{\ell-r}(x), \quad \ell \geq 0, \tag{60}$$

along with the two connection formulas (39) and (40) (but replacing x with $(2x-1)$), after some lengthy algebraic computations □

Theorem 5. *The inversion formula to the analytic forms of $\bar{Z}_i(x)$ is given as follows*

$$x^m = \sum_{r=0}^{m} H_{r,m}\,\bar{Z}_r(x), \tag{61}$$

where

$$H_{r,m} = \frac{2^{-2m}}{3}\times \begin{cases} (2m)!\left(\dfrac{r+1}{(m-r-4)!\,(m+r+4)!}+\dfrac{r+3}{(m-r)!\,(m+r)!}+\dfrac{2\,(r+2)}{(m-r-2)!\,(m+r+2)!}\right), & \textit{if } r \textit{ even},\\ \dfrac{2^{2-2m}(2m)!(2+r)}{3(m-r)!(m+r+4)!}\times & \\ \left(24+10m^3+m^4+r(4+r)(8+r(4+r))+m^2(35+2r(4+r))+m(50+6r(4+r))\right), & \textit{if } r \textit{ odd}. \end{cases} \tag{62}$$

Proof. The proof of (61) can be done starting with the inversion formula of CPs of the first kind in (34) along with the two connection formulas (39) and (40). □

4 Tau method for treating a type of FDDEs

Here, we offer a numerical technique for solving certain FDDEs. Consider the FDDE below:

$$D^{\gamma}y(t)+\xi\,y'(t)+\eta\,y(t)+\mu\,y(\tau t) = f(t), \qquad t\in(0,1), \tag{63}$$

governed by the boundary conditions

$$y(0)=\rho_0, \quad y(1)=\rho_1, \tag{64}$$

or by the initial conditions

$$y(0)=\tilde{\rho}_0, \quad y'(0)=\tilde{\rho}_1, \tag{65}$$

where $\gamma\in(1,2)$, $\tau\in(0,1)$, $\xi,\eta,\mu\,\rho_0,\rho_1,\tilde{\rho}_0$ and $\tilde{\rho}_1$ are real constants and $f(t)$ is a known continuous function. Let $y(t)\in L^2_{\omega(t)}(0,1)$, where

$$L^2_{\omega(t)}[0,1] = \operatorname{span}\{\bar{Z}_i(t): i=0,1,2,\ldots\}, \tag{66}$$

and $\omega(t)=\frac{(2t-1)^4}{\sqrt{t-t^2}}$. Assume that $y(t)$ can be expanded as

$$y(t) = \sum_{i=0}^{\infty} b_i\,\bar{Z}_i(t), \tag{67}$$

where

$$b_i = \frac{1}{\rho_i}\int_0^1 y(t)\,\bar{Z}_i(t)\,\omega(t)\,dt. \tag{68}$$

Hence, it can be approximated as

$$y(t) \approx y_N(t) = \sum_{i=0}^{N} b_i \bar{Z}_i(t) = \mathbf{B}\,\mathbf{C}(t), \tag{69}$$

where

$$\mathbf{B} = (b_0, b_1, \dots, b_N), \qquad \mathbf{C}(t) = \begin{pmatrix} \bar{Z}_0(t) \\ \bar{Z}_1(t) \\ \vdots \\ \bar{Z}_N(t) \end{pmatrix}.$$

So, to move further with our proposed numerical technique for solving (63) governed by (64) or (65), the following results are required.

Corollary 3. *The following formula holds for the first derivative of $Z_i(x)$*

$$\frac{d\bar{Z}_{2k}(x)}{dx} = 2\sum_{p=0}^{k-1} (2k-2p+1)\,\bar{Z}_{2k-2p-1}(x), \quad k \geq 1, \tag{70}$$

$$\begin{aligned} \frac{d\bar{Z}_{2k+1}(x)}{dx} = \frac{4}{(1+2k)(3+2k)(5+2k)} \Bigg(& \sum_{p=0}^{\lfloor \frac{k}{2} \rfloor} (k-2p+1)\left((1+2k)^2(3+2k)+64(1+k)p-64p^2\right) Z_{2k-4p}(x) \\ & + \sum_{p=0}^{\lfloor \frac{k-1}{2} \rfloor} (k-2p)\left((1+2k)^2(11+2k)-64kp+64p^2\right) \bar{Z}_{2k-4p-2}(x) \Bigg), \quad k \geq 0. \end{aligned} \tag{71}$$

Proof. You may get the formulas (70) and (71) by simply substituting $(2x-1)$ for x in (49) and (50). $\square$

Theorem 6. *The following formula holds for $1 < \gamma < 2$.*

$$D^{\gamma}\bar{Z}_i(x) \approx \sum_{k=0}^{N} \varepsilon_{k,i}\,\bar{Z}_k(x), \tag{72}$$

where

$$\varepsilon_{k,i} = \sum_{p=2}^{i} \frac{(p)!\,g_{p,i}\,a_k}{(p-\gamma)!}, \tag{73}$$

and

$$\begin{aligned} a_k = \frac{1}{h_k}\sum_{m=0}^{k} g_{m,k} \Bigg(& 16\,\beta\left(m+p-\gamma+\frac{9}{2},\frac{1}{2}\right) - 32\,\beta\left(m+p-\gamma+\frac{7}{2},\frac{1}{2}\right) + 24\,\beta\left(m+p-\gamma+\frac{5}{2},\frac{1}{2}\right) \\ & - 8\,\beta\left(m+p-\gamma+\frac{3}{2},\frac{1}{2}\right) + \beta\left(m+p-\gamma+\frac{1}{2},\frac{1}{2}\right)\Bigg). \end{aligned} \tag{74}$$

Proof. First, apply D^{γ} to (55) to get

$$D^{\gamma}\bar{Z}_i(x) = \sum_{p=0}^{i} g_{p,i}\,(D^{\gamma}x^p) = \sum_{p=2}^{i} \frac{(p)!\,g_{p,i}}{(p-\gamma)!}\,x^{p-\gamma}. \tag{75}$$

Now, $x^{p-\gamma}$ is approximated in terms of $\bar{Z}_i(x)$ as

$$x^{p-\gamma} \approx \sum_{k=0}^{N} a_k\,\bar{Z}_k(x). \tag{76}$$

To find a_k, based on the orthogonality relation (55) and the analytic form of $\bar{Z}_i(t)$, in (58), we get

$$\begin{aligned} a_k &= \frac{1}{h_k} \int_0^1 x^{p-\gamma} \bar{Z}_k(x) \frac{(2x-1)^4}{\sqrt{x-x^2}}\, dx \\ &= \frac{1}{h_k} \sum_{m=0}^{k} g_{m,k} \int_0^1 x^{m+p-\gamma-\frac{1}{2}} \frac{16x^4 - 32x^3 + 24x^2 - 8x + 1}{\sqrt{1-x}}\, dx. \end{aligned} \tag{77}$$

The last equation can be simplified to give the following result:

$$\begin{aligned} a_k = \frac{1}{h_k} \sum_{m=0}^{k} g_{m,k} \Bigg(& 16\beta\left(m+p-\gamma+\frac{9}{2}, \frac{1}{2}\right) - 32\beta\left(m+p-\gamma+\frac{7}{2}, \frac{1}{2}\right) + 24\beta\left(m+p-\gamma+\frac{5}{2}, \frac{1}{2}\right) \\ & - 8\beta\left(m+p-\gamma+\frac{3}{2}, \frac{1}{2}\right) + \beta\left(m+p-\gamma+\frac{1}{2}, \frac{1}{2}\right)\Bigg), \end{aligned} \tag{78}$$

where $\beta(\cdot,\cdot)$ denotes the celebrated beta function .
Now, inserting Eq. (76) into Eq. (75), the outcome of Theorem 6 is obtained. □

Lemma 3. *The duplication formula for $\bar{Z}_i(x)$ is given by*

$$\bar{Z}_i(\tau x) = \sum_{p=0}^{i} \lambda_{p,i} \bar{Z}_p(x), \tag{79}$$

where

$$\lambda_{p,i} = \sum_{r=p}^{i} \tau^r g_{r,i} H_{p,r}. \tag{80}$$

Proof. Formula (58) and its inversion formula in (61) are the core of the proof of Lemma 3. □

4.1 The proposed Tau algorithm

We can now propose our Tau approach to solve (63) constrained by (64) or (65). Thanks to Lemma 3 and Theorems 3, 6, the residual $\mathbf{R}(t)$ of (63) may be expressed as

$$\mathbf{R}(t) = D^{\gamma} y_N(t) + \xi\, y_N'(t) + \eta\, y_N(t) + \mu\, y_N(\tau t) - f(t). \tag{81}$$

As a result of Tau method [13], one has

$$(\mathbf{R}(t), \bar{Z}_j(t))_{\omega(t)} = 0, \quad j: 0, 1, \ldots, N-2, \tag{82}$$

where $(y(t), v(t))_{\omega(t)} = \int_0^1 y(t)\, v(t)\, \omega(t)\, dt$.
Eq. (82) can be rewritten in matrix form as

$$\mathbf{B}\bar{\mathbf{H}}(t) + \xi\, \mathbf{B}\bar{\mathbf{D}}(t) + \eta\, \mathbf{B}\bar{\mathbf{C}}(t) + \mu\, \mathbf{B}\bar{\mathbf{G}}(t) - \mathbf{F} = 0, \tag{83}$$

where

$$\begin{aligned} &\bar{\mathbf{H}}(t) = (h_{i,j}), \quad h_{i,j} = (D^{\gamma}\bar{Z}_i(t), \bar{Z}_j(t))_{\omega(t)}, \\ &\bar{\mathbf{D}}(t) = (d_{i,j}), \quad d_{i,j} = (\frac{d\bar{Z}_i(t)}{dt}, \bar{Z}_j(t))_{\omega(t)}, \\ &\bar{\mathbf{C}}(t) = (\bar{c}_{i,j}), \quad \bar{c}_{i,j} = (\bar{Z}_i(t), \bar{Z}_j(t))_{\omega(t)}, \\ &\bar{\mathbf{G}}(t) = (g_{i,j}), \quad g_{i,j} = (\bar{Z}_i(\tau t), \bar{Z}_j(t))_{\omega(t)}, \\ &\mathbf{F}(t) = (f_j), \quad f_j = (f(t), \bar{Z}_j(t))_{\omega(t)}. \end{aligned} \tag{84}$$

The boundary conditions (64) yield

$$\mathbf{B}\,\mathbf{C}(0) = \rho_0, \quad \mathbf{B}\,\mathbf{C}(1) = \rho_1, \tag{85}$$

and the he initial conditions (65) yield

$$\mathbf{B}\,\mathbf{C}(0) = \tilde{\rho}_0, \quad \mathbf{B}\,\mathbf{C}'(0) = \tilde{\rho}_1. \tag{86}$$

Eq. (83) along with the boundary conditions (85) or the initial conditions (86) produce a linear system of algebraic equations of dimension $(N+1)$ in the unknown expansion coefficients b_i , that can be solved using the Gaussian elimination method.

Theorem 7. *The elements of the matrices $\bar{\mathbf{H}}(t)$, $\bar{\mathbf{C}}(t)$, $\bar{\mathbf{G}}(t)$ and $\bar{\mathbf{D}}(t)$, can be computed explicitly as follows:*

$$h_{i,j} = \sum_{k=0}^{N} \varepsilon_{k,i}\, \rho_k\, \delta_{k,j}, \tag{87}$$

$$\bar{c}_{i,j} = \rho_i\, \delta_{i,j}, \tag{88}$$

$$g_{i,j} = \sum_{p=0}^{i} \lambda_{p,i}\, \rho_p\, \delta_{p,j}, \tag{89}$$

$$d_{2i,j} = \sum_{p=0}^{i-1} 2\,(2i-2p+1)\, \rho_{2i-2p-1}\, \delta_{2i-2p-1,j}, \tag{90}$$

$$\begin{aligned} d_{2i+1,j} = & \sum_{p=0}^{\lfloor \frac{i-1}{2} \rfloor} \frac{\left(4\,(i-2p)\,\left(-64\,i\,p+(2i+11)\,(2i+1)^2+64\,p^2\right)\right)}{(2i+1)\,(2i+3)\,(2i+5)}\, \rho_{2i-4p}\, \delta_{2i-4p,j} \\ & + \sum_{p=0}^{\lfloor \frac{i}{2} \rfloor} \frac{\left(4\,(i-2p+1)\,\left(64\,(i+1)\,p+(2i+3)\,(2i+1)^2-64\,p^2\right)\right)}{(2i+1)\,(2i+3)\,(2i+5)}\, \rho_{2i-4p-2}\, \delta_{2i-4p-2,j}. \end{aligned} \tag{91}$$

Proof. The proof is easily obtained by applying the orthogonality relation (55) to the inner products in (84) to compute the elements of the matrices $\bar{\mathbf{H}}(t)$, $\bar{\mathbf{C}}(t)$, $\bar{\mathbf{G}}(t)$ and $\bar{\mathbf{D}}(t)$. □

5 Error analysis of the proposed Chebyshev expansion

In this section, we are interested in investigating the convergence of the shifted CPs of the seventh-kind expansion. We first state and prove a lemma in which an upper bound for the polynomials $\bar{Z}_k(t)$ is found.

Lemma 4. *The following inequality is satisfied for all $k \geq 0$*

$$|\bar{Z}_k(t)| < k+3, \qquad \forall\, t \in [0,1]. \tag{92}$$

Proof. Consider the following two cases to prove the inequality (92):

Case 1: $k = 2j$: Using the connection formula in (56) together with the simple inequality: $|T_m^*(t)| \leq 1$, we get

$$|\bar{Z}_k(t)| \leq \frac{6}{(2j+1)\,(2j+3)} \sum_{r=0}^{j} (r+1)\,(-2j+r-1) = \frac{(j+1)\,(j+2)\,(4j+3)}{(2j+1)\,(2j+3)} < 2j+3 = k+3.$$

Case 2: $k = 2j+1$: With the aid of (57) and the inequality: $|T_m^*(t)| \leq 1$ yield

$$\begin{aligned} |\bar{Z}_k(t)| &\leq \frac{6}{(2j+1)\,(2j+3)\,(2j+5)} \sum_{r=0}^{j} (r+1)\,(-2j+r-2)\,(-2j+2r-1) \\ &= \frac{3\,(j+1)^2\,(j+2)^2}{(2j+1)\,(2j+3)\,(2j+5)} < 2j+4 = k+3. \end{aligned}$$

For every $k \geq 0$, the following estimate follows from the previous two cases:

$$|\bar{Z}_k(t)| < k+3, \qquad \forall\, t \in [0,1]. \qquad \square$$

Theorem 8. *Assume that a function $y(t) \in L^2_{\omega(t)}[0,1]$ with $|u^{(4)}(t)| \leqslant M$, can be expanded as an infinite series of the shifted seventh kind of CPs as*

$$y(t) = \sum_{i=0}^{\infty} b_i \bar{Z}_i(t). \tag{93}$$

The series in (93) converges uniformly to $y(t)$. Moreover, the following bound on the expansion coefficients in (93) holds

$$|b_i| < \frac{M}{i^3}, \quad \forall\, i > 3. \tag{94}$$

Proof. The substitution: $t = \frac{1}{2}(1+\cos\theta)$, allows us to express the expansion coefficients b_i in (68) as

$$b_i = \frac{1}{\rho_i} \int_0^{\pi} u\left(\frac{1}{2}(1+\cos\theta)\right) \bar{Z}_i(\cos\theta) \cos^4\theta \, d\theta. \tag{95}$$

Now, to check the validity of the inequality in (94), we consider the following two cases:

Case 1: i is even: Based on Eq. (47), Eq. (95) turns into

$$\begin{aligned} b_i =& \frac{1}{3\pi} \int_0^{\pi} u\left(\frac{1}{2}(1+\cos\theta)\right) \\ & \times \left(2\,(i+2)\cos((i+2)\theta) + (i+3)\cos((i+3)\theta) + (i+1)\cos((i+4)\theta)\right) d\theta. \end{aligned} \tag{96}$$

The integration by parts three times on the right-hand side of Eq.(96) leads to the following formula:

$$b_i = \frac{1}{192\,\pi} \int_0^{\pi} u'''\left(\frac{1}{2}(1+\cos\theta)\right) \Omega_i(\theta)\, d\theta, \tag{97}$$

where

$$\begin{aligned} \Omega_i(\theta) =& \frac{(i+3)}{i\,(i-1)\,(i-2)} \left(\cos((i-3)\theta) - \cos((i-1)\theta)\right) \\ & - \frac{8}{i\,(i-1)\,(i+1)} \left(\cos((i-1)\theta) - \cos((i+1)\theta)\right) \\ & - \frac{2}{(i+1)\,(i+2)} \left(\cos(i\theta) - \cos((i+2)\theta)\right) \\ & + \frac{(i+1)}{(i+4)\,(i+5)\,(i+6)} \left(\cos((i+5)\theta) - \cos((i+7)\theta)\right) \\ & + \frac{2}{(i+2)\,(i+3)} \left(\cos((i+2)\theta) - \cos((i+4)\theta)\right) \\ & - \frac{2\,(i+1)}{(i+3)\,(i+4)\,(i+5)} \left(\cos((i+3)\theta) - \cos((i+5)\theta)\right) \\ & - \frac{2\left(i^2 - 5\,i - 18\right)}{i\,(i+1)\,(i+2)\,(i+3)\,(i+4)} \left(\cos((i+1)\theta) - \cos((i+3)\theta)\right). \end{aligned} \tag{98}$$

If we integrate the right-hand side of (97) again by parts and using the hypothesis of the theorem, then the following inequality is obtained:

$$|b_i| < \frac{M}{i^3}, \quad \forall\, i > 3. \tag{99}$$

Case 2: i is odd: With the aid of Eq. (48), Eq. (95) can be turned into

$$\begin{aligned} b_i =& \frac{i\,(i+2)^2\,(i+4)}{3\,\pi\,(i+1)^2\,(i+3)^2\,(i+5)} \\ & \times \int_0^\pi u\left(\frac{1}{2}(1+\cos\theta)\right)\,[(i+4)\,(i+5)\,\cos((i+1)\theta)+2\,(i+1)\,(i+5)\,\cos((i+3)\theta) \\ & +(i+1)\,(i+2)\,\cos((i+5)\theta)]\,d\theta. \end{aligned} \tag{100}$$

By integrating the right-hand side of the previous equation by parts three times, one has

$$b_i = \frac{i\,(i+2)^2\,(i+4)}{192\,\pi\,(i+1)^2\,(i+3)^2\,(i+5)} \int_0^\pi u'''\left(\frac{1}{2}(1+\cos\theta)\right)\bar{\Omega}_i(\theta)\,d\theta, \tag{101}$$

where

$$\begin{aligned} \bar{\Omega}_i(\theta) =& \frac{(i+4)\,(i+5)}{i\,(i+1)\,(i-1)}\,(\cos((i-2)\theta)-\cos(i\theta)) \\ & + \frac{12\,(i+1)}{(i+3)\,(i+5)\,(i+6)}\,(\cos((i+4)\theta)-\cos((i+6)\theta)) \\ & - \frac{2\,\left(i^2+6i-19\right)}{(i+1)\,(i+3)\,(i+5)}\,(\cos((i+2)\theta)-\cos((i+4)\theta)) \\ & - \frac{12\,(i+5)}{i\,(i+1)\,(i+3)}\,(\cos(i\theta)-\cos((i+2)\theta)) \\ & + \frac{(i+1)\,(i+2)}{(i+5)\,(i+6)\,(i+7)}\,(\cos((i+6)\theta)-\cos((i+8)\theta)). \end{aligned} \tag{102}$$

Now, with the aid of $|u^{(4)}(t)| \leqslant M$, after integrating the right hand side of (101) by parts, we get

$$|b_i| < \frac{M}{i^3}, \quad \forall\, i > 2. \tag{103}$$

Finally, the two inequalities (99) and (103), enable us to write

$$|b_i| < \frac{M}{i^3}, \quad \forall\, i > 3.$$

This finalizes the proof of Theorem 8. □

Theorem 9. *Let $y(t)$ satisfy the assumptions of Theorem 8, and let $y_N(t) = \sum_{i=0}^{N} b_i\,\bar{Z}_i(t)$ be the approximate solution. Then, the following estimation is satisfied:*

$$|y(t) - y_N(t)| < \frac{2M}{N}.$$

Proof. We have

$$y(t) - y_N(t) = \sum_{i=N+1}^{\infty} b_i\,\bar{Z}_i(t). \tag{104}$$

With the aid of Theorem 8 along with Lemma 4, the previous equation can be rewritten as

$$|y(t) - y_N(t)| < M \sum_{i=N+1}^{\infty} \frac{i+3}{i^3}. \tag{105}$$

Table 1: Comparison of ME of Example 1.

		$N=8$		$N=10$		$N=12$	
γ	τ	Method in [5]	Our method	Method in [5]	Our Method	Method in [5]	Our method
1.25	$\frac{1}{4}$	5.38×10^{-9}	1.90859×10^{-10}	2.27×10^{-11}	9.23151×10^{-14}	4.68×10^{-14}	4.99612×10^{-16}
	$\frac{1}{2}$	6.39×10^{-9}	1.86591×10^{-10}	7.95×10^{-11}	9.39804×10^{-14}	6.23×10^{-14}	3.33067×10^{-16}
	$\frac{3}{4}$	2.39×10^{-9}	2.10726×10^{-10}	5.95×10^{-11}	1.08581×10^{-13}	7.21×10^{-14}	2.77556×10^{-16}
1.5	$\frac{1}{4}$	5.92×10^{-9}	4.84044×10^{-10}	5.61×10^{-11}	1.98008×10^{-12}	6.34×10^{-14}	3.88578×10^{-16}
	$\frac{1}{2}$	6.38×10^{-9}	4.88578×10^{-10}	9.34×10^{-11}	1.79545×10^{-12}	2.15×10^{-14}	1.66533×10^{-16}
	$\frac{3}{4}$	2.36×10^{-9}	4.79537×10^{-10}	5.27×10^{-11}	1.81299×10^{-12}	2.27×10^{-14}	3.33067×10^{-16}
1.75	$\frac{1}{4}$	2.96×10^{-9}	1.92485×10^{-10}	3.65×10^{-11}	2.93111×10^{-12}	2.65×10^{-14}	9.99201×10^{-16}
	$\frac{1}{2}$	2.37×10^{-9}	1.99649×10^{-10}	5.92×10^{-11}	3.11839×10^{-12}	2.84×10^{-14}	1.22125×10^{-16}
	$\frac{3}{4}$	3.98×10^{-9}	2.00782×10^{-10}	9.34×10^{-11}	3.16003×10^{-12}	2.38×10^{-14}	1.55431×10^{-15}

Moreover, the inequality

$$\frac{i+3}{i^3} < \frac{i+3}{i(i^2-1)}, \quad \forall\, i > 1,$$

implies that

$$|y(t) - y_N(t)| < M \sum_{i=N+1}^{\infty} \frac{i+3}{i(i^2-1)} = \frac{(N+2)M}{N\,(N+1)}. \tag{106}$$

At the end, the following identity: $\frac{(N+2)}{N\,(N+1)} < \frac{2}{N}$, $\forall\, N > 0$, enable us to reach the next estimation:

$$|y(t) - y_N(t)| < \frac{2M}{N}. \qquad \square$$

6 Illustrative examples

Example 1. Consider the following FDDE [5]

$$\begin{aligned} &D^{\gamma} y(t) + \xi\, y'(t) + \eta\, y(t) + \mu\, y(\tau t) = f(t), \qquad t \in (0,1), \\ &y(0) = 1, \quad y(1) = \frac{1}{e}, \end{aligned} \tag{107}$$

where $\xi = \eta = \mu = 1$ and $f(t)$ is such that the exact solution of this problem is $y(t) = e^{-t}$. Table 1 presents a comparison of maximum point-wise error (ME) for some values of γ, τ and N, between our method with the method developed in [5].

Example 2. Consider the following FDDE:

$$\begin{aligned} &D^{\gamma} y(t) + \xi\, y'(t) + \eta\, y(t) + \mu\, y(\tau t) = f(t), \qquad t \in (0,1), \\ &y(0) = 0, \quad y'(0) = \gamma, \end{aligned} \tag{108}$$

where $\xi = \eta = 1$, $\mu = \frac{1}{2}$ and $f(t)$ is chosen such that the exact solution of this problem is $y(t) = \sin(\gamma t)$. Table 2 lists the ME at $\gamma = 1.5$ and different values of τ and N. Figure 1 sketches the ME for some values of N at $\gamma = 1.1$, $\tau = 0.9$ and $\gamma = 1.9$, $\tau = 0.5$.

Table 2: ME of Example 2 at $\gamma = 1.5$.

N	2	4	6	8	10	12	14
$\tau = \frac{1}{4}$	4.555×10^{-1}	5.201×10^{-3}	3.571×10^{-5}	1.022×10^{-7}	2.189×10^{-10}	2.660×10^{-13}	3.330×10^{-16}
$\tau = \frac{3}{4}$	4.555×10^{-1}	5.108×10^{-3}	3.472×10^{-5}	1.004×10^{-7}	2.126×10^{-10}	2.606×10^{-13}	3.330×10^{-16}

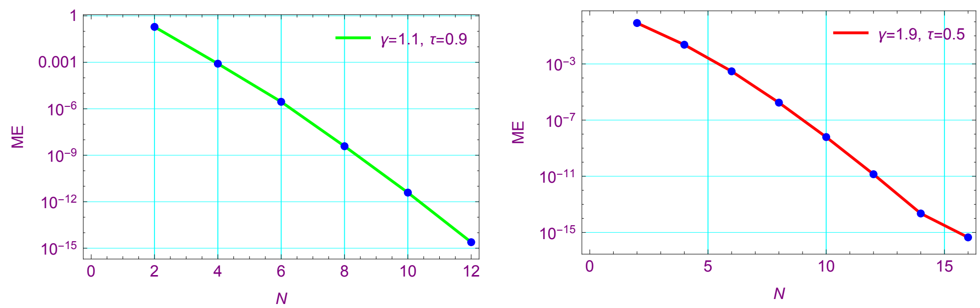

Figure 1: The ME for Example 2.

Example 3. Consider the following FDDE:

$$\begin{aligned} &D^{\gamma} y(t) + \xi\, y'(t) + \eta\, y(t) + \mu\, y(\tau t) = f(t), \qquad t \in (0,1), \\ &y(0) = 1, \quad y(1) = E_{\alpha,\beta}(1), \end{aligned} \tag{109}$$

where $\xi = 0.25$, $\eta = \mu = 1$ and $f(t)$ is chosen such that the exact solution of Eq. (109) is $y(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(ak+b)} = E_{a,b}(t)$ and $E_{a,b}(t)$ is the Mittag-Leffler function of the two parameters $a, b > 0$. Eq. (109) is solved in two cases corresponding to $a = 2, b = 1$ and $a = \frac{1}{2}, b = 1$.

Case 1: For $a = 2, b = 1$, this exact solution reduces to $y(t) = E_{2,1}(t) = \cosh(t)$. In Table 3, the ME for the case corresponding to $\gamma = 1.5$, $\tau = 0.3$ are displayed. In addition, Fig. 2 illustrates the AE (left) and approximate solution (right) at $\gamma = 1.9$, $\tau = 0.7$ and $N = 9$.

Case 2: For $a = \frac{1}{2}, b = 1$, this exact solution reduces to $y(t) = E_{\frac{1}{2},1}(t) = e^{t^2}$erfc(-t), where erfc(-t) is the complementary error function and defined as:

$$\text{erfc(t)} = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-x^2} dx = 1 - \text{erf(t)}. \tag{110}$$

Figure 3 illustrates the ME for different values of N at $\gamma = 1.2$ and $\tau = 0.8$.

Table 3: ME of Example 3.

N	1	3	6	9
ME	1.0948×10^{-2}	4.8354×10^{-6}	1.35891×10^{-13}	4.44089×10^{-16}

7 Concluding remarks

This paper considers a kind of orthogonal polynomials related to CPs. Due to their trigonometric expression, we call these polynomials "seventh-kind CPs". Several new theoretical results concerned with these polynomials are

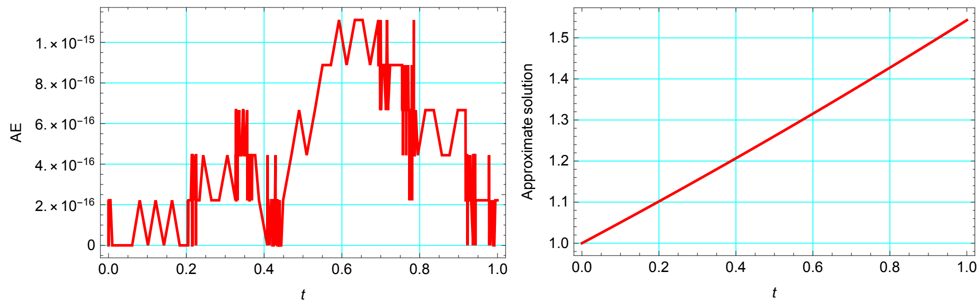

Figure 2: The AE (left) and approximate solution (right) for Example 3.

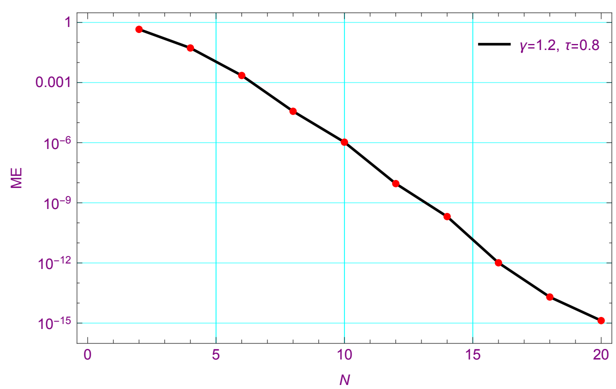

Figure 3: The ME for Example 3.

established and used to solve DDEs with the help of Tau method. The new basis functions used here are the shifted CPs of the seventh-kind. Convergence and error analysis were discussed in depth. Finally, numerical examples in Section 6 illustrated the technique's applicability, efficiency, and accuracy. Many problems in the applied sciences may benefit from this method if successfully implemented.

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