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On the blow up of solutions for hyperbolic equation involving the fractional Laplacian with source terms

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Abstract. In this paper, we study the blow-up of solutions for hyperbolic equations involving the fractional Laplacian operator with damping and source terms. We obtain the global existence results. Then, we observe the blow-up of solutions using the concavity method. Finally, we present some numerical simulation results.

Keywords: Blow up, energy function, hyperbolic equation, fractional Laplacian, source terms, fractional Sobolev spaces
AMS Subject Classification 2010: 35B44, 35B38, 35L10.

1 Introduction

In this work, we study the following boundary value problem related to the hyperbolic equation involving the fractional Laplacian with nonlinear source terms:

$$\begin{cases} u_{tt} + (-\Delta)^s u - (-\Delta)^s u_t + |u_t|^{q-1} u_t = \alpha |u|^{p-1} u, & x \in \Omega, \ \ t > 0, \\ u = 0, & (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open domain with smooth boundary $\partial\Omega$, $(-\Delta)^s$ is the fractional Laplacian such that $s \in]0,1[$, $\alpha > 0$ and $1 \le q < p \le p^*$ such that the exponent p^* satisfies

$$p^* \le \frac{2n}{n-2s} = 2_s^*, \quad n > 2s.$$

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The fractional Laplacian operator is a generalization of the differentiation operation. The concept of fractional differentiation, as a generalization of derivative to non-integer values, emerged almost simultaneously with the concept of differentiation. The first mention of this idea was made by Leibniz and Marquis in [19]. Subsequently, the concept of fractional integrodifferentiation was further developed in many works(for more details, see [10, 16]). In recent years, many mathematical models involving fractional and non-local operators have been actively studied as they arise in various applications, such as physics, image processing, population dynamics, etc.([7, 11]).
In [1, 9], the authors studied the existence of weak solutions for a fractional elliptic system.
The following equation

$$\partial_t^2 u + [u]_s^{2(\theta-1)}(-\Delta)^s u = |u|^{p-1}u, \tag{2}$$

where $\theta \in [1, 2_s^*]$ and $[u]_s$ is defined by

$$\Big(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy\Big)^{\frac{1}{2}}. \tag{3}$$

The researchers obtained the global existence and blow up of solutions for (2) by using Galerkin method combined the potential wells in [18]. Also, they showed the global existence of solutions under initial conditions. Then, in [17] the authors studied the global existence, behavior and blow up of solutions of the following damped equation

$$\partial_t^2 u + [u]_s^{2(\theta-1)}(-\Delta)^s u + |\partial_t u|^{\alpha-1}\partial_t u + u = |u|^{p-2}u, \tag{4}$$

where $2 < \alpha < 2\theta < p < 2_s^* < s$. In [14], the authors studied the blow up by a modified concavity method in finite time (see [6, 23]). And in [5] the researches proved blow-up of solutions for the following equation

$$\partial_t^2 u + (-\Delta)^s u + (-\Delta)^s \partial_t u = u|u|^{p-2}, \quad x \in \Omega,\ t > 0. \tag{5}$$

The interaction between the source terms $|u|^{p-1}u$ and the damping $|u_t|^{q-1}u_t$ makes the problem more interesting. In [12, 13], Levine was the first to study this interaction using the concavity method with $(q = 1)$. Then, Vitillaro in [21] extended Levine's results to the nonlinear case $(q > 1)$ and showed that solutions with positive initial energy blow up in finite time. In the absence of the nonlinear damping $|u_t|^{q-1}u_t$, the authors studied the blow up of solutions for the wave equation involving the fractional Laplacian using the concavity method in [5]. Messaoudi studied the decay of solutions using a combination of techniques involving the potential well method and perturbed energy in [15]. Recently, in [22] Wu and Xue proved the uniform energy decay rates of the solutions using the multiplier method. Afterwards, in [20] Piskin studied the blow up of solutions and provided lifespan estimates in three different ranges of initial energy. In [3, 4], the authors studied a quasilinear hyperbolic equation involving the weighted Laplacian operator with source terms.

In this work, first, we demonstrate the global existence of solutions for problem (1) with $\alpha = 1$. Secondly, we observe the finite-time blow up of solutions with positive initial energy, and we obtain some numerical results.

This paper is organized as follows. In Section 2, we present some definitions, lemmas and notations. In Section 3, we show the global existence of solutions. In Section 4, we prove the blow up of solutions by using the concavity method, finally in Section 5 we obtain the some numerical simulations.

2 Preliminaries

In this section, we give some definitions, lemmas and assumptions which will be used throughout this article. Let $||\cdot||_2$ and $||\cdot||_{p+1}$ denote the usual $L^2(\Omega)$ and $L^{p+1}(\Omega)$ norm, respectively. The fractional Laplacian $(-\Delta)^s u$ of the function u is given by

$$(-\Delta)^s u(x) = C\int_\Omega \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \quad \forall x \in \Omega, \tag{6}$$

where $C = C(n,s)$ is the normalisation constant. Now, we define the fractional-order Sobolev space by

$$W^{s,2}(\Omega) = \{u \in L^2(\Omega) : \frac{|u(x)-u(y)|^2}{|x-y|^{\frac{n}{2}+s}}dy \in L^2(\Omega\times\Omega)\}, \tag{7}$$

with norm

$$||u||_{W^{s,2}(\Omega)} = \Big(\int_\Omega |u|^2 + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\Big)^{\frac{1}{2}}. \tag{8}$$

Let

$$W_0^{s,2}(\Omega) = \{u \in W^{s,2}(\Omega) : u = 0, \quad \text{a.e on} \quad \partial\Omega\}, \tag{9}$$

be a closed linear subspace of $W^{s,2}(\Omega)$ (see [5]), and its norm is given by

$$||u||_{W_0^{s,2}(\Omega)} = \Big(\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\Big)^{\frac{1}{2}}. \tag{10}$$

The space $W_0^{s,2}(\Omega)$ is Hilbert space with the inner product

$$< u,v >_{W_0^{s,2}(\Omega)} = \int_\Omega\int_\Omega \frac{|u(x)-u(y)||v(x)-v(y)|}{|x-y|^{n+2s}}dxdy. \tag{11}$$

Lemma 1 ([8]). *1. For any $s \in [1,2_s^*]$, there exists a positive constant $C_0 = C_0(n,s)$ such that for any $u \in W_0^{s,2}(\Omega)$*

$$||u||_{L^s(\Omega)} \leq C_0 \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy.$$

2. For any $s \in [1,2_s^*]$ and any bounded sequence $(u_j)_j$ in $W_0^{s,2}(\Omega)$, there exists u in $L^s(\Omega)$, with $u = 0$, such that up to a subsequence, still denoted by $(u_j)_j$

$$u_j \to u \quad \textit{strongly in } L^s(\Omega) \quad \textit{as} \quad j \to \infty.$$

3 The potential wells

In this section, we consider problem (1) in stationary case. In fact, if we replace u in this section by $u(t)$ for any $t \in [0,T)$, all the facts are still valid. We define

$$J[u(t)] = \frac{1}{2}||u||^2_{W_0^{s,2}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1},$$

and

$$I[u(t)] = ||u||^2_{W_0^{s,2}(\Omega)} - ||u||^{p+1}_{p+1}.$$

We also define the energy function

$$E[u(t)] = \frac{1}{2}||u_t||^2 + \frac{1}{2}||u||^2_{W_0^{s,2}(\Omega)} - \frac{1}{p+1}||u||^{p+1}_{p+1},$$

and we introduce the stable set as follows

$$W = \{u : u \in W_0^{s,2}(\Omega), I(u) > 0; J(u) < d\} \cup \{0\},$$

where the mountain pass level d is defined as

$$d = \inf_{u \in W_0^{s,2}(\Omega)/0} \{\sup_{\mu \geq 0} J(\mu u)\}.$$

And the Nehari manifold

$$N = \{u \in W_0^{s,2}(\Omega)/0 : I(u) = 0\},$$

with the potential depth d

$$d = \inf_{u \in N} J(u),$$

which implies that

$$dist(0,N) = \min_{u \in N} ||u||_{W_0^{s,2}(\Omega)}.$$

Definition 1. *Let us assume $s \in]0,1[, 1 \leq q < p \leq p^*, u_0 \in W_0^{s,2}(\Omega)$ and $u_1 \in L^2(\Omega)$. Then, there exists a function $u = u(t,x)$ which is a weak global solution of problem (1), if*

$$u \in L^\infty\big([0,T); W_0^{s,2}(\Omega)\big) \quad and \quad u_t \in L^\infty\big([0,T); L^2(\Omega)\big) \cap L^{q+1}\big(\Omega \times [0,T)\big),$$

$$u(0) = u_0 \quad and \quad u_t(0) = u_1 \quad a.e.\ in \quad \Omega.$$

If a weak global solution u belongs to $C((0,\infty); W_0^{s,2}(\Omega))$, we say that u is a strong global solution of problem (1).

Now, we show that the energy function is a nonincreasing function along the solution of (1) in the next lemma.

Lemma 2. *Let u be a weak solution of (1). If $u_0 \in W$ and $u_1 \in L^2(\Omega)$, then $E(t)$ is a nonincreasing function and*

$$E(t) \leq E(0), \quad \forall t \in [0,T).$$

Proof. We have

$$E^{'}(t) = -\big(||u_t||^{q+1}_{q+1} + ||u_t||^2_{W_0^{s,2}(\Omega)}\big) \leq 0.$$

By integration over $[0,t]$, we obtain $E(t) - E(0) \leq 0$. □

Lemma 3. *Let $u_0 \in W$ and $u_1 \in L^2(\Omega)$. If $p^* \leq \dfrac{2n}{n-2s} = 2_s^*, n > 2s,\ s \in]0,1[$, then the solution $u \in W$, $\forall t \geq 0$.*

Proof. Since $u_0 \in W$, we have

$$I(u_0) = ||u_0||^2_{W_0^{s,2}(\Omega)} - ||u_0||^{p+1}_{p+1} > 0,$$

and by the continuity of $u(t)$, $I(t) > 0$, for some interval near $t = 0$. Let $T_m > 0$, then

$$J(t) = \frac{1}{p+1} I(t) + \frac{p-1}{2(p+1)} ||u||^2_{W_0^{s,2}(\Omega)}.$$

Since $I(t) > 0$, we have

$$J(t) \geq \frac{p-1}{2(p+1)} ||u||^2_{W_0^{s,2}(\Omega)},$$

then from $E(t)$ and $E^{'}(t)$, we obtain

$$||u||^2_{W_0^{s,2}(\Omega)} \leq \frac{2(p+1)}{p-1} J(t) \leq \frac{2(p+1)}{p-1} E(t) \leq \frac{2(p+1)}{p-1} E(0).$$

We have also

$$\begin{aligned} ||u||^{p+1}_{p+1} &\leq C_{*s} ||u||^{p+1}_{W_0^{s,2}(\Omega)} = C_{*s} ||u||^{p-1}_{W_0^{s,2}(\Omega)} ||u||^2_{W_0^{s,2}(\Omega)} \\ &\leq C_{*s} \Big(\frac{2(p+1)}{p-1} E(0) \Big)^{\frac{p-1}{2}} ||u||^2_{W_0^{s,2}(\Omega)} \\ &= \beta ||u||^2_{W_0^{s,2}(\Omega)} \leq ||u||^2_{W_0^{s,2}(\Omega)} \quad \forall t \in [0, T_m], \end{aligned} \tag{12}$$

with $C_{*s} = C^{p+1}_{2_s^*}$. When we repeat the procedure, T_m is extended to T, so the proof is completed. □

4 Blow up results

In this section, we prove the blow up result for problem (1), when $q = 1$.

Definition 2. *A solution u of problem (1) is called blow up if there exists a finite time T^* such that*

$$\lim_{t \to T^{*-}} ||u(t,x)||_2 = \infty.$$

We put

$$A(t) = ||u(t,x)||_2 \quad \textit{for} \quad t \geq 0.$$

Lemma 4. *Let $u_0 \in W$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) > 0$ and $\int_\Omega u_0 u_1 dx > 0$, then any solution u blows up in a finite time.*

Proof. We have

$$A^{'}(t) = 2 < u_t, u >_{L^2(\Omega)}, \quad t \in [0, T],$$

and

$$A^{''}(t) = 2 < u_{tt}, u >_{L^2(\Omega)} + 2||u_t||_2, \quad t \in [0, T].$$

Now, we multiply problem (1) by u, then we integrate over Ω. We obtain

$$< u_{tt}, u >_{L^2(\Omega)} = -||u||_{W_0^{s,2}(\Omega)} - < u_t, u >_{W_0^{s,2}(\Omega)} - \int_\Omega u_t u dx + \int_\Omega |u|^{p-1} u dx.$$

By Cauchy-Schwartz inequality, we get

$$A^{'2}(t) = 4 < u, u_t >_{L^2}^2 \leq 4||u||_2^2||u_t||_2^2, \quad t \in [0,T],$$

which implies that

$$\begin{aligned} A^{''}(t)A(t) - (1+\delta)(A^{'}(t))^2 & \\ &\geq \big(2||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u\big)A(t) - 4(1+u) < u_t, u >^2_{L^2} \\ &\geq \big(2||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u\big)A(t) - 4(1+u)||u_t||_2^2||u||_2^2 \\ &= \big(-2||u||^2_{W_0^{s,2}(\Omega)} - 2 < u_t, u >_{L^2(\Omega)} + 2\int_\Omega u|u|^{p-1}u - 4(1+2\delta)||u_t||_2^2\big)A(t), \end{aligned}$$

where $\delta > 0$. Then, we put

$$M(t) = -2||u||^2_{W_0^{s,2}(\Omega)} + 2\int_\Omega u|u|^{p-1}u - 4(1+2\delta)||u_t||_2^2.$$

After that

$$\begin{aligned} M(t) &= -2||u||^2_{W_0^{s,2}(\Omega)} + 2||u||_{p+1}^{p+1} - 4(1+2\delta)||u_t||_2^2 \\ &\geq -4(1+2\delta)||u_t||_2^2 - 2||u||^2_{W_0^{s,2}(\Omega)} + (p+1)||u_t||_2^2 + (p+1)||u||^2_{W_0^{s,2}(\Omega)} - 2(p+1)E(0) \\ &= -(8\delta - p + 1)||u_t||_2^2 + (p-1)||u||^2_{W_0^{s,2}(\Omega)} - 2(p+1)E(0) \\ &\geq -(8\delta - p + 1)||u_t||_2^2 + (p-1)||u||^2 + (p-1)||u||_2^2 - 2(p+1)E(0), \end{aligned}$$

for $t \in [0,T)$. Set $\delta = \dfrac{p-1}{8} > 0$, then we get

$$M(t) \geq (p-1)||u||_2^2 - 2(p+1)E(0) \geq 0.$$

So, we obtain

$$A^{''}(t)A(t) - (1+\delta)(A^{'}(t))^2 > 0, \quad t \in [0,T).$$

This implies that

$$\begin{aligned} (A^{-\delta})^{'} &= -\delta A^{-\delta-1}A^{'}(t) < 0, \\ (A^{-\delta})^{''} &= -\delta A^{-\delta-2}\big(A^{''}(t)A(t) - (1+\delta)(A^{'}(t))^2\big) < 0, \end{aligned}$$

for all $t \in [0,T)$, which means that the function $A^{-\delta}$ is concave. Obviously, $A(0) > 0$, then there must exist T^* such that

$$\lim_{t \longrightarrow T^*} A^{-\delta}(t) = 0.$$

So, that

$$\lim_{t \longrightarrow T^{*-}} A(t) = \infty.$$

Thus, the proof is completed. □

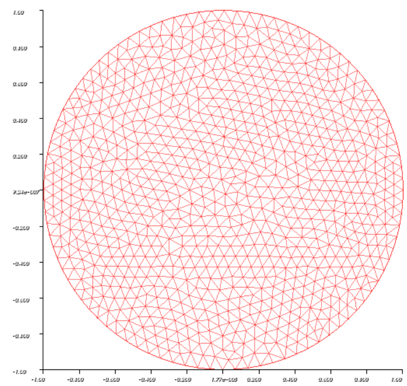

Figure 1: Uniform mesh grid.

5 Numerical simulation

In this section, we give some numerical simulation to illustrate the theoretical results for problem (1). We solve problem (1) under specific initial data and Dirichlet boundary conditions. We use a numerical schema based on the finite element method [2, 24]. The error between the exact solution and the approximate solution in the example is draw, by using FreeFEM++. Then, we draw the curse of the error. The numerical approximation error L is $L = u_h - u_e$ (u_h: the approximate solution, and u_e: the exact solution) at different time iterations $t = 0$, $t = 0.5$ and $t = 0.9$.

Test: We consider the domain $\Omega = \{(x,y) : x^2 + y^2 < 1\}$ with a triangulation discretization which consists 1682 triangle and 888 vertices. And we use the initial conditions

$$u_0(x,y) = \sinh(x)\sinh(y), \quad u_1 = -2\sinh(x)\sinh(y).$$

We consider problem (1) in two space-dimension and take $q = p = 1$ and $\alpha = 7$. We show the numerical approximation of solutions u at different time iterations $t = 0$, $t = 0.5$ and $t = 0.9$ respectively.

It is noticed that the value of the error (L) decreases if t increases, so we have the stability and convergence of solutions.

6 Conclusion

In this work, we obtained the global existence results, the blow up of solutions and some numerical simulations for a hyperbolic equation involving the fractional Laplacian with source terms in a bounded domain. This improves and extends many results in the literature.

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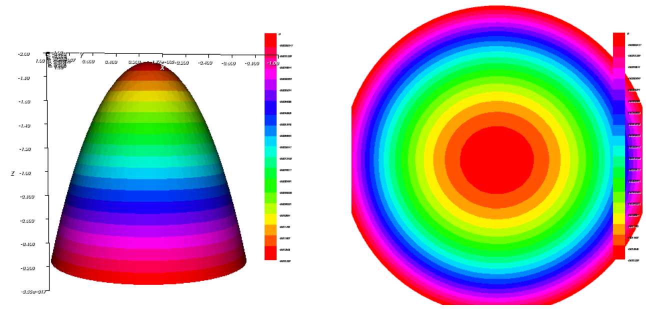


Figure 2: t=0.

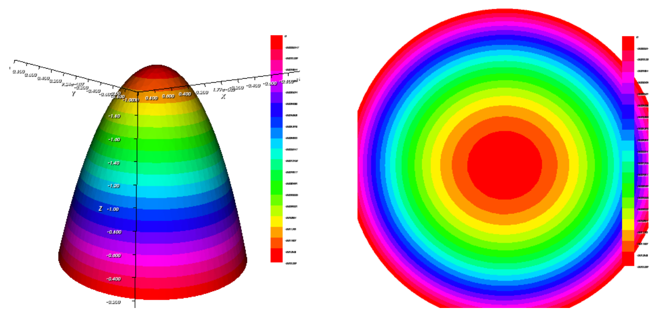


Figure 3: t=0.5.

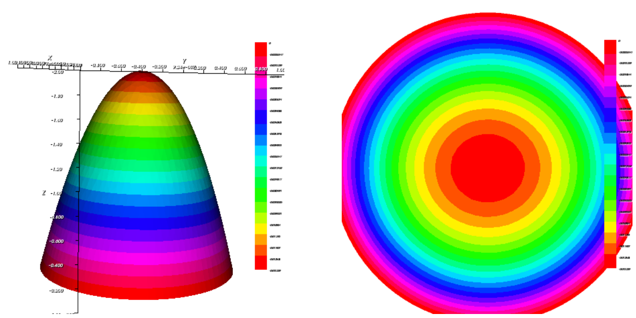
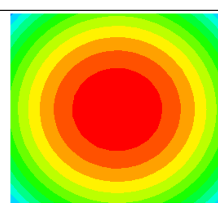
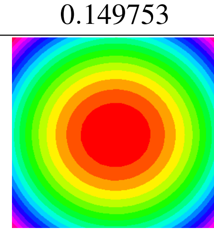
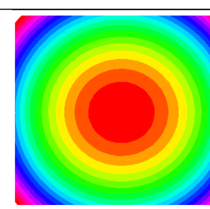


Figure 4: t=0.9.

t	$t=0$	$t=0.5$	$t=0.9$
Error	0.409438	0.149753	0.0668976
Zoom of the Error			

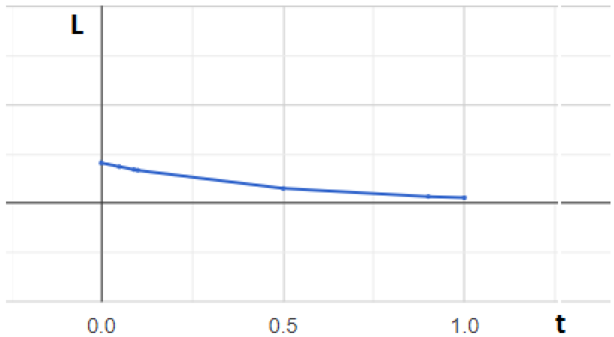


Figure 5: Curve of L.

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