Journal of Mathematical Modeling
Vol. 12, No. 2, 2024, pp. 235–246. Research Article
JMM

Computational treatment of a convection-diffusion type nonlinear system of singularly perturbed differential equations

Manikandan Mariappan*

Department of Mathematics, School of Engineering, Presidency University, Bengaluru - 560 064, Karnataka, India
Email(s): manimaths89@yahoo.com

Abstract. In this article, a nonlinear system of singularly perturbed differential equations of convection-diffusion type with Dirichlet boundary conditions is considered on the interval $[0,1]$. Both components of the solution of the system exhibit boundary layers near $t = 0$. A new computational method involving classical finite difference operators, a piecewise-uniform Shishkin mesh and an algorithm based on the continuation method is developed to compute the numerical approximations. The computational method is proved to be first order convergent uniformly with respect to the perturbation parameters. Numerical experiments support the theoretical results.

Keywords: Nonlinear system of singularly perturbed differential equations, boundary layers, finite difference scheme, Shishkin mesh, the continuation method, parameter-uniform convergence.
AMS Subject Classification 2010: 65L11, 65L12, 65L20, 65L70.

1 Introduction

A Differential Equation (DE) in which a small positive parameter multiplying the highest derivative term in the equation and/or its lower order derivative terms with some conditions is known as a Singular Perturbation Problem (SPP). Most of the SPPs in real life follow system of nonlinear DEs. For instance, the *Navier-Stokes* equation of fluid at high *Reynolds* number follow a nonlinear system of second order DEs of Convection-Diffusion (CD) type [13]. For a broad introduction to singularly perturbed DEs of CD type one can refer to [1, 2, 13, 15].

Classical computational methods fail to resolve SPPs due to the multiscale behaviour of their solutions [1, 13, 15]. Many nonclassical computational methods are available in the literature for singularly

*Corresponding author

Received: 4 November 2023 / Revised: 7 January 2024 / Accepted: 24 January 2024
DOI: 10.22124/JMM.2024.25939.2301

perturbed linear DEs. However, only few computational methods are available in the literature for Singularly Perturbed Nonlinear Differential Equations (SPNDEs).

Articles [3] and [9] deal with the computational aspects of SPNDE of Reaction-Diffusion (RD) type whereas [5], [8] and [10] deals with the computational aspects of SPNDE of CD type. Different computational methods for nonlinear system of SPNDEs of RD type are developed in [4], [11], [12] and [16].

From the mathematical point of view fluid and gas dynamics are described by the *Navier-Stokes* equations. These comprise a system of four nonlinear partial differential equations of CD type. The singularly perturbed nature of these equations become obvious when the magnitude of the convective terms is much larger than that of the diffusion terms [13]. In 1995 Johnson et al. [6] observed that, in the particular case of incompressible *Navier-Stokes* equations, the existing analyses often contain constants that depend on e^R, where R is the *Reynolds* number, and concluded that "in the majority of cases of interest, the existing error analysis has no meaning". Still today in many works on singularly perturbed DEs either conditions are imposed on the magnitude of the perturbation parameters or artificial conditions are imposed on the problems. Such works are not helpful to obtain the parameter-uniform estimates or they weaken the nature of the original problems.

The process of obtaining robust, layer-resolving and parameter-uniform computational approximations for a nonlinear system of singularly perturbed DEs of CD type involves many challenges. To the best of the author's knowledge, no robust, layer-resolving and parameter-uniform computational method is available in the literature for a nonlinear system of singularly perturbed DEs of CD type.

In this article, a nonlinear system of singularly perturbed DEs of CD type is considered. It is worth observing that in the present study no artificial condition is imposed on the perturbation parameters and the computational method developed in this article is robust, layer-resolving and parameter-uniform.

In the present article *the intermediate value theorem* plays an important role. It has been used as a powerful tool to compute the bounds on the components $\overrightarrow{v}$ and $\overrightarrow{w}$ in Theorems 1 and 2 respectively. Also it has been used to establish a linear operator $(\overrightarrow{\mathbb{T}}^N)'$ in Section 5 which reduces the process of obtaining the error estimate to a linear case.

2 The nonlinear system

The following nonlinear system of singularly perturbed DEs is considered in this article

$$\overrightarrow{\mathbb{T}}\,\overrightarrow{u}(t) = E\,\overrightarrow{u}''(t) + A(t)\overrightarrow{u}'(t) - \overrightarrow{f}(t, \overrightarrow{u}) = \overrightarrow{0}, \text{ on } \Omega = (0,1), \tag{1}$$

$$\text{with } \overrightarrow{u}(0) = \overrightarrow{u}_0 \text{ and } \overrightarrow{u}(1) = \overrightarrow{u}_1, \tag{2}$$

where $\overrightarrow{u}_0 = (u_{01}, u_{02})^T$ and $\overrightarrow{u}_1 = (u_{11}, u_{12})^T$ are known constant vectors. For all $t \in \overline{\Omega} = [0,1]$, $\overrightarrow{u}(t) = (u_1(t), u_2(t))^T$, $\overrightarrow{f}(t, \overrightarrow{u}) = (f_1(t, \overrightarrow{u}), f_2(t, \overrightarrow{u}))^T \in C^3(\overline{\Omega} \times \mathbb{R}^2)$. Here, E and $A(t)$ are 2×2 diagonal matrices with diagonal elements $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $a_1(t), a_2(t)$ such that $a_i(t) \in C^3(\overline{\Omega})$, for $i = 1,2$, respectively. It is assumed that for $i = 1,2$ and for all $t \in \overline{\Omega}$, $a_i(t) \geq \alpha > 0$ and for all $(t, \overrightarrow{y}) \in \overline{\Omega} \times \mathbb{R}^2$,

$$\frac{\partial f_i(t, \overrightarrow{y})}{\partial y_j} \leq 0,\ i, j = 1,2,\ i \neq j \text{ and} \tag{3}$$

$$\min_{\substack{t \in \overline{\Omega} \\ i=1,2}} \left(\frac{\partial f_i(t, \overrightarrow{y})}{\partial y_1} + \frac{\partial f_i(t, \overrightarrow{y})}{\partial y_2} \right) \geq \beta > 0. \tag{4}$$

With the above stated conditions, the existence of a unique solution $\overrightarrow{u}(t)$ to (1)-(2) such that $\overrightarrow{u}(t) \in \left(C^3(\overline{\Omega})\right)^2$ can be ensured by the implicit function theorem [14]. A reduced problem to (1)-(2) is defined to be

$$A(t)\overrightarrow{v}_0'(t) - \overrightarrow{f}(t, \overrightarrow{v}_0) = \overrightarrow{0}, \ t \in [0,1), \ \overrightarrow{v}_0(1) = \overrightarrow{u}_1. \tag{5}$$

As above, the implicit function theorem ensures a unique solution $\overrightarrow{v}_0(t)$ to (5). Moreover,

$$|v_{0i}^{(k)}(t)| \leq C \ \text{ for } \ i = 1,2, \ k = 0,1,2,3 \ \text{ and } t \in \overline{\Omega}. \tag{6}$$

In this article, C denotes a positive constant which is free from t, ε_1, ε_2 and N (the discretization parameter).

3 Some theoretical results

Decompose the solution $\overrightarrow{u}$ of (1)-(2) into $\overrightarrow{v}$ and $\overrightarrow{w}$ such that $\overrightarrow{u} = \overrightarrow{v} + \overrightarrow{w}$ where

$$E\overrightarrow{v}''(t) + A(t)\overrightarrow{v}'(t) - \overrightarrow{f}(t, \overrightarrow{v}) = \overrightarrow{0} \text{ on } \Omega, \tag{7}$$

$$\overrightarrow{v}(0) - \text{ suitably chosen}, \ \overrightarrow{v}(1) = \overrightarrow{u}_1, \tag{8}$$

and

$$E\overrightarrow{w}''(t) + A(t)\overrightarrow{w}'(t) - \overrightarrow{f}(t, \overrightarrow{v} + \overrightarrow{w}) + \overrightarrow{f}(t, \overrightarrow{v}) = \overrightarrow{0} \text{ on } \Omega, \tag{9}$$

$$\overrightarrow{w}(0) = \overrightarrow{u}_0 - \overrightarrow{v}(0), \ \overrightarrow{w}(1) = \overrightarrow{0}. \tag{10}$$

3.1 Bounds on $\overrightarrow{v}(t)$ and its derivatives

Theorem 1. *For all $t \in \overline{\Omega}$ and for $k = 0,1,2$,*

$$|v_1^{(k)}(t)| \leq C, \ \ |v_2^{(k)}(t)| \leq C, \ \ |v_1^{(3)}(t)| \leq C\varepsilon_1^{-1}, \ \ |v_2^{(3)}(t)| \leq C\varepsilon_2^{-1}.$$

Proof. From (7) and (5),

$$\varepsilon_1 v_1''(t) + a_1(t)\left(v_1' - v_{01}'\right)(t) - b_{11}(t)\left(v_1 - v_{01}\right)(t) - b_{12}(t)\left(v_2 - v_{02}\right)(t) = 0, \tag{11}$$

and

$$\varepsilon_2 v_2''(t) + a_2(t)\left(v_2' - v_{02}'\right)(t) - b_{21}(t)\left(v_1 - v_{01}\right)(t) - b_{22}(t)\left(v_2 - v_{02}\right)(t) = 0, \tag{12}$$

where $b_{ij}(t) = \dfrac{\partial f_i(t, \overrightarrow{\mu}(t))}{\partial u_j}$ are intermediate values. Equations (11) and (12) can be written together as follows

$$E\overrightarrow{v}''(t) + A(t)\overrightarrow{v}'(t) - B(t)\overrightarrow{v}(t) = A(t)\overrightarrow{v}_0'(t) - B(t)\overrightarrow{v}_0(t) = \overrightarrow{g}(t), \tag{13}$$

where

$$B(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{bmatrix}.$$

From (13) and (8), $\overrightarrow{v}$ satisfies a similar problem in [7]. Hence the bounds on $\overrightarrow{v}$ and its derivatives follow by using similar arguments as in [7]. □

3.2 Bounds on $\overrightarrow{w}(t)$ and its derivatives

For all $t \in \overline{\Omega}$, let $B_1(t) = e^{-\alpha t/\varepsilon_1}$ and $B_2(t) = e^{-\alpha t/\varepsilon_2}$.

Theorem 2. *For all* $t \in \overline{\Omega}$,

$$|w_1(t)| \leq CB_2(t), \quad |w_2(t)| \leq CB_2(t), \quad |w_1^{(k)}(t)| \leq C\left(\varepsilon_1^{-k}B_1(t) + \varepsilon_2^{-k}B_2(t)\right), \; k = 1,2,3,$$
$$|w_2^{(k)}(t)| \leq C\varepsilon_2^{-k}B_2(t), \; k = 1,2, \quad |w_2^{(3)}(t)| \leq C\varepsilon_2^{-1}\left(\varepsilon_1^{-1}B_1(t) + \varepsilon_2^{-2}B_2(t)\right).$$

Proof. Using (9),

$$\varepsilon_1 w_1''(t) + a_1(t)w_1'(t) - c_{11}(t)w_1(t) - c_{12}(t)w_2(t) = 0, \tag{14}$$

and

$$\varepsilon_2 w_2''(t) + a_2(t)w_2'(t) - c_{21}(t)w_1(t) - c_{22}(t)w_2(t) = 0, \tag{15}$$

where $c_{ij}(t) = \dfrac{\partial f_i(t, \overrightarrow{\gamma}(t))}{\partial u_j}$ are intermediate values. Equations (14) and (15) can be written together as follows

$$E\overrightarrow{w}''(t) + A(t)\overrightarrow{w}'(t) - C(t)\overrightarrow{w}(t) = \overrightarrow{0}, \tag{16}$$

where

$$C(t) = \begin{bmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{bmatrix}.$$

From (16) and (10), $\overrightarrow{w}$ satisfies a similar problem in [7]. Hence the bounds on $\overrightarrow{w}$ and its derivatives follow by using similar arguments as in [7]. □

4 Mesh and the discrete problem

On $\overline{\Omega}$, a piecewise-uniform Shishkin mesh with N mesh-intervals is now constructed as follows. Let $\Omega^N = \{t_j\}_{j=1}^{N-1}$ then $\overline{\Omega}^N = \{t_j\}_{j=0}^{N}$. The domain $\overline{\Omega}$ is divided into 3 sub-intervals $[0, \tau_1]$, (τ_1, τ_2) and $(\tau_2, 1]$ such that $\overline{\Omega} = [0, \tau_1] \cup (\tau_1, \tau_2) \cup (\tau_2, 1]$. The parameters τ_2 and τ_1 are defined by

$$\tau_2 = \min\left\{\frac{1}{2}, \frac{\varepsilon_2}{\alpha}\ln N\right\} \text{ and } \tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{\varepsilon_1}{\alpha}\ln N\right\}.$$

From the total N mesh points, $\frac{N}{4}$ mesh points are placed uniformly on each of the sub-domains $[0, \tau_1]$ and $[\tau_1, \tau_2]$ and the remaining $\frac{N}{2}$ mesh points are placed on the sub-domain $[\tau_2, 1]$. Let h_1, h_2 and h_3 denote the step size in $[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, 1]$, respectively. Then $h_1 = 4\tau_1/N$, $h_2 = 4(\tau_2 - \tau_1)/N$ and $h_3 = 2(1 - \tau_2)/N$.

The discrete problem corresponding to (1)-(2) is defined to be

$$\overrightarrow{\mathbb{T}}^N\overrightarrow{U}(t_j) = E\,\delta^2\overrightarrow{U}(t_j) + A(t_j)D^+\overrightarrow{U}(t_j) - \overrightarrow{f}(t_j, \overrightarrow{U}(t_j)) = \overrightarrow{0}, \text{ for } t_j \in \Omega^N, \tag{17}$$

$$\overrightarrow{U}(t_0) = \overrightarrow{u}(t_0) \text{ and } \overrightarrow{U}(t_N) = \overrightarrow{u}(t_N). \tag{18}$$

Here

$$\delta^2 Z(t_j) = \frac{(D^+ - D^-)Z(t_j)}{\overline{h}_j}, \; D^+Z(t_j) = \frac{Z(t_{j+1}) - Z(t_j)}{h_{j+1}}, \; D^-Z(t_j) = \frac{Z(t_j) - Z(t_{j-1})}{h_j},$$

with $h_j = t_j - t_{j-1}$, $\overline{h}_j = (h_{j+1} + h_j)/2$, $\overline{h}_0 = h_1/2$ and $\overline{h}_N = h_N/2$.

5 Error analysis

$\overrightarrow{\dim}$
Let $\overrightarrow{\Theta}_1$ and $\overrightarrow{\Theta}_2$ be any two vector mesh functions such that $\overrightarrow{\Theta}_1(t_0) = \overrightarrow{\Theta}_2(t_0)$ and $\overrightarrow{\Theta}_1(t_N) = \overrightarrow{\Theta}_2(t_N)$. For $t_j \in \Omega^N$,

$$\begin{aligned}
&(\overrightarrow{\mathbb{T}}^N\overrightarrow{\Theta}_1 - \overrightarrow{\mathbb{T}}^N\overrightarrow{\Theta}_2)(t_j) \\
&= E\,\delta^2(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j) + A(t_j)D^+(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j) - \overrightarrow{f}(t_j, \overrightarrow{\Theta}_1(t_j)) \\
&\quad + \overrightarrow{f}(t_j, \overrightarrow{\Theta}_2(t_j)) \\
&= E\,\delta^2(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j) + A(t_j)D^+(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j) - D(t_j)(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j) \\
&= (\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j),
\end{aligned} \tag{19}$$

where $D(t_j) = (d_{ik}(t_j))_{2\times 2},\ d_{ik}(t_j) = \partial f_i(t_j, \overrightarrow{\zeta}(t_j))/\partial u_k$ are intermediate values and $(\overrightarrow{\mathbb{T}}^N)'$ is the Frechet derivative of $\overrightarrow{\mathbb{T}}^N$. Since $(\overrightarrow{\mathbb{T}}^N)'$ is linear, it satisfies the discrete maximum principle in [7]. Thus,

$$||(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j)|| \le C||(\overrightarrow{\mathbb{T}}^N)'(\overrightarrow{\Theta}_1 - \overrightarrow{\Theta}_2)(t_j)|| = C||\overrightarrow{\mathbb{T}}^N\overrightarrow{\Theta}_1(t_j) - \overrightarrow{\mathbb{T}}^N\overrightarrow{\Theta}_2(t_j)||. \tag{20}$$

Theorem 3. *Let $\overrightarrow{u}$ be the solution of (1)-(2) and $\overrightarrow{U}$ be that of (17)-(18). Then for $t_j \in \overline{\Omega}^N$,*

$$||(\overrightarrow{U} - \overrightarrow{u})(t_j)|| \le CN^{-1}\ln N. \tag{21}$$

Proof. Let $t_j \in \Omega^N$. From (20),

$$||(\overrightarrow{U} - \overrightarrow{u})(t_j)|| \le C||(\overrightarrow{\mathbb{T}}^N\overrightarrow{U} - \overrightarrow{\mathbb{T}}^N\overrightarrow{u})(t_j)||.$$

Consider

$$||\overrightarrow{\mathbb{T}}^N\overrightarrow{u}(t_j)|| = ||(\overrightarrow{\mathbb{T}}^N\overrightarrow{u} - \overrightarrow{\mathbb{T}}^N\overrightarrow{U})(t_j)||.$$

Hence,

$$\begin{aligned}
||(\overrightarrow{\mathbb{T}}^N\overrightarrow{u} - \overrightarrow{\mathbb{T}}^N\overrightarrow{U})(t_j)|| &= ||\overrightarrow{\mathbb{T}}^N\overrightarrow{u}(t_j)|| = ||(\overrightarrow{\mathbb{T}}^N\overrightarrow{u} - \overrightarrow{\mathbb{T}}\overrightarrow{u})(t_j)|| \\
&\le E||(\delta^2\overrightarrow{u} - \overrightarrow{u}'')(t_j)|| + ||A(t_j)||\,||(D^+\overrightarrow{u} - \overrightarrow{u}')(t_j)|| \\
&\le E||(\delta^2\overrightarrow{v} - \overrightarrow{v}'')(t_j)|| + ||A(t_j)||\,||(D^+\overrightarrow{v} - \overrightarrow{v}')(t_j)|| \\
&\quad + E||(\delta^2\overrightarrow{w} - \overrightarrow{w}'')(t_j)|| + ||A(t_j)||\,||(D^+\overrightarrow{w} - \overrightarrow{w}')(t_j)||.
\end{aligned}$$

Since the bounds for $\overrightarrow{v}$ and $\overrightarrow{w}$ are same as in [7], the required result follows. □

6 The continuation method

The nonlinear DE in (1)-(2) is modified to an artificial nonlinear partial differential equation as given below. For $(t,x) \in (0,1) \times (0,X]$,

$$\begin{aligned}
&-\frac{\partial\overrightarrow{u}(t,x)}{\partial x} + E\,\frac{\partial^2\overrightarrow{u}(t,x)}{\partial t^2} + A(t)\frac{\partial\overrightarrow{u}(t,x)}{\partial t} - \overrightarrow{f}(t, \overrightarrow{u}(t,x)) = \overrightarrow{0}, \\
&\overrightarrow{u}(0,x) = \overrightarrow{u}(0),\ \overrightarrow{u}(1,x) = \overrightarrow{u}(1),\ x \ge 0 \ \text{and}\ \overrightarrow{u}(t,0) = \overrightarrow{u}_{init}(t),\ 0 < t < 1.
\end{aligned} \tag{22}$$

The continuation method developed for a scalar nonlinear DE of RD type in [2] is modified appropriately for a nonlinear system of DEs of CD type as given below which is used to solve (22). For $j = 1,\ldots,N$ and $k = 1,\ldots,K$,

$$-D_x^- \overrightarrow{U}(t_j,x_k) + E\,\delta_t^2 \overrightarrow{U}(t_j,x_k) + A(t_j) D_t^+ \overrightarrow{U}(t_j,x_k) - \overrightarrow{f}(t_j, \overrightarrow{U}(t_j,x_{k-1})) = \overrightarrow{0}, \tag{23}$$

$$\begin{aligned} \overrightarrow{U}(t_0,x_k) &= \overrightarrow{u}(t_0), \quad \overrightarrow{U}(t_N,x_k) = \overrightarrow{u}(t_N) \text{ for all } k \text{ and} \\ \overrightarrow{U}(t_j,x_0) &= \overrightarrow{u}_{init}(t_j) \text{ for all } t_j \in \overline{\Omega}^N, \end{aligned} \tag{24}$$

where

$$\delta_t^2 Z(t_j,x_k) = \frac{(D_t^+ - D_t^-)Z(t_j,x_k)}{\bar{h}_j}, \quad D_t^+ Z(t_j,x_k) = \frac{Z(t_{j+1},x_k) - Z(t_j,x_k)}{h_{j+1}},$$

$$D_t^- Z(t_j,x_k) = \frac{Z(t_j,x_k) - Z(t_{j-1},x_k)}{h_j}, \quad D_x^- Z(t_j,x_k) = \frac{Z(t_j,x_k) - Z(t_j,x_{k-1})}{h_x}.$$

The initial guess $\overrightarrow{u}_{init}(t)$ is taken to be $\overrightarrow{u}(0) + t(\overrightarrow{u}(1) - \overrightarrow{u}(0))$. The choices of the step size $h_x = x_k - x_{k-1}$ and the number of iterations K are determined as follows. Define

$$Err(k) \;=\max_{1\le j\le N}\left(\frac{||\overrightarrow{U}(t_j,x_k) - \overrightarrow{U}(t_j,x_{k-1})||}{h_x}\right) \text{ for } k = 1,\ldots,K. \tag{25}$$

The step size h_x is chosen sufficiently small so that $Err(k)$ decreases with the increasing k. Precisely, we choose h_x such that

$$Err(k) \le Err(k-1) \text{ for all } k, 1 < k \le K \tag{26}$$

and K such that

$$Err(K) \le tol, \tag{27}$$

where tol is a prescribed small tolerance. Algorithm given below is used to compute the numerical solution for (22).

Algorithm :

Step 1: Begin from x_0 with $h_x = 1$.

Step 2: Suppose (26) is not satisfied for some k, then quit the current step and begin from x_{k-1} with h_x as $h_x/2$. Continue halving h_x until finding a h_x for which (26) is satisfied.

Step 3: If (26) is satisfied at each h_x, then continue the procedure until either (27) is satisfied or $K = 100$.

Step 4: If (27) is not satisfied, then it is assumed that the stepping process is stalled due to the choice of a large h_x. In such a case, the entire process is repeated from x_0 with $h_x/2$ instead of h_x.

Step 5: If (27) is satisfied, then $\overrightarrow{U}(t_j,x_K)$ are taken as the numerical approximations to the solution of (22).

7 Numerical illustrations

Two examples are presented in this section. The continuation method constructed in the above section is used together with the proposed computational technique to solve the examples. The tolerance $'tol'$ is taken to be 10^{-5}. Notations D^N, p^N and C_p^N denote the parameter-uniform maximum pointwise error, parameter-uniform order of convergence and parameter-uniform error constant respectively that are given by

$$D^N = \max_{\varepsilon_1,\varepsilon_2} D_E^N \quad \text{where} \quad D_E^N = \| \overrightarrow{U}^N - \overrightarrow{U}^{2N} \|,$$
$$p^N = \log_2 \frac{D^N}{D^{2N}}, \; C_p^N = \frac{D^N N^{p^\star}}{1-2^{-p^\star}} \quad \text{where} \quad p^\star = \min_N p^N.$$

Example 1. Consider the BVP

$$E\,\overrightarrow{u}''(t) + A(t)\,\overrightarrow{u}'(t) - \overrightarrow{f}(t,\,\overrightarrow{u}(t)) = \overrightarrow{0}, \quad t \in (0,1)$$

with $\overrightarrow{u}(0) = (\sin(15), 0.1)$, $\overrightarrow{u}(1) = \left(e^{-0.7}, \frac{\sqrt{2}}{5}\right)$, $E = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$,

$A(t) = \begin{bmatrix} 2-\sin(t) & 0 \\ 0 & 1+e^t \end{bmatrix}$ and $\overrightarrow{f}(t, \overrightarrow{u}(t)) = \begin{bmatrix} (u_1(t))^5 + 2u_1(t) - \sin(\frac{1}{2})u_2(t) \\ (u_2(t))^3 + 4u_2(t) - u_1(t) - 2 \end{bmatrix}$.

Example 2. Consider the BVP

$$E\,\overrightarrow{u}''(t) + A(t)\,\overrightarrow{u}'(t) - \overrightarrow{f}(t,\,\overrightarrow{u}(t)) = \overrightarrow{0}, \quad t \in (0,1),$$

with $\overrightarrow{u}(0) = (\cos(1), 0.1)$, $\overrightarrow{u}(1) = \left(0.5 + e^{-0.1}, \frac{1}{\sqrt{1+\pi}}\right)$, $E = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}$,

$A(t) = \begin{bmatrix} 2-\sin(t^2) & 0 \\ 0 & \frac{3}{2}+e^{-t} \end{bmatrix}$ and $\overrightarrow{f}(t, \overrightarrow{u}(t)) = \begin{bmatrix} (u_1(t))^3 + 2u_1(t) - \cos(\frac{3}{2})u_2(t) + t \\ (u_2(t))^7 + 3u_2(t) - u_1(t) - 1 \end{bmatrix}$.

For Example 1 and Example 2, the values of D^N, C_p^N, p^N are presented in Table 1 and Table 2 respectively and graphs of the numerical solutions for $\varepsilon_1 = 2^{-6}, \varepsilon_2 = 2^{-5}$ and $N = 256$ are portrayed in Figure 1 and Figure 4, respectively. Further, for $N = 256$ and for different values of the parameters ε_1 and ε_2, the changes in the components of the solution $\overrightarrow{u}(t)$ are portrayed in Figure 2 and Figure 5, respectively. Moreover, the *Log-log plot* for the error in the suggested computational method for Example 1 and Example 2 is presented in Figure 3 and Figure 6, respectively.

8 Conclusion

In this article, a robust, layer-resolving and parameter-uniform computational method is developed for a nonlinear system of singularly perturbed DEs of CD type. From the tables, it is evident that the parameter-uniform maximum pointwise error (D^N) monotonically decreases when the number of mesh

Table 1: $\alpha = 0.9$, $\varepsilon_1 = \frac{\eta}{4}$, $\varepsilon_2 = \frac{\eta}{2}$

η	Number of mesh points N				
	128	256	512	1024	2048
2^{0}	4.7337e-03	2.4072e-03	1.2138e-03	6.0945e-04	3.0537e-04
2^{-2}	2.3629e-02	1.2741e-02	6.6174e-03	3.3724e-03	1.7024e-03
2^{-4}	2.2114e-02	1.4360e-02	9.7754e-03	5.4120e-03	2.8548e-03
2^{-6}	2.2117e-02	1.4190e-02	9.6098e-03	5.2934e-03	2.7841e-03
2^{-8}	2.2118e-02	1.4145e-02	9.5659e-03	5.2622e-03	2.7655e-03
2^{-10}	2.2118e-02	1.4133e-02	9.5548e-03	5.2543e-03	2.7608e-03
2^{-12}	2.2118e-02	1.4131e-02	9.5520e-03	5.2523e-03	2.7596e-03
2^{-14}	2.2118e-02	1.4130e-02	9.5513e-03	5.2518e-03	2.7593e-03
D^N	2.3629e-02	1.4360e-02	9.7754e-03	5.4120e-03	2.8548e-03
p^N	7.1847e-01	5.5483e-01	8.5300e-01	9.2274e-01	
C_p^N	1.0925e+00	9.7539e-01	9.7539e-01	7.9327e-01	6.1471e-01

Table 2: $\alpha = 0.9$, $\varepsilon_1 = \frac{\eta}{4}$, $\varepsilon_2 = \frac{\eta}{2}$

η	Number of mesh points N				
	128	256	512	1024	2048
2^{0}	5.3036e-03	2.6915e-03	1.3558e-03	6.8045e-04	3.4086e-04
2^{-2}	2.6753e-02	1.4552e-02	7.5928e-03	3.8786e-03	1.9602e-03
2^{-4}	2.3239e-02	1.6007e-02	1.1128e-02	6.3042e-03	3.3657e-03
2^{-6}	2.2983e-02	1.5809e-02	1.0981e-02	6.2125e-03	3.3137e-03
2^{-8}	2.2909e-02	1.5755e-02	1.0941e-02	6.1881e-03	3.3000e-03
2^{-10}	2.2889e-02	1.5740e-02	1.0931e-02	6.1818e-03	3.2965e-03
2^{-12}	2.2884e-02	1.5737e-02	1.0929e-02	6.1802e-03	3.2956e-03
2^{-14}	2.2883e-02	1.5736e-02	1.0928e-02	6.1798e-03	3.2954e-03
D^N	2.6753e-02	1.6007e-02	1.1128e-02	6.3042e-03	3.3657e-03
p^N	7.4100e-01	5.2447e-01	8.1982e-01	9.0540e-01	
C_p^N	1.1182e+00	9.6240e-01	9.6240e-01	7.8423e-01	6.0224e-01

points (N) increases. Further, we also observe that the proposed method is almost first order parameter-uniform convergent. This is in agreement with Theorem 3.

From Figure 1 and Figure 4, we observe that both the components $u_1(t)$ and $u_2(t)$ of the solution $\overrightarrow{u}(t)$ exhibit boundary layers near the boundary $t = 0$. Moreover, from Figure 2 and Figure 5 we perceive that both the components $u_1(t)$ and $u_2(t)$ of the solution $\overrightarrow{u}(t)$ changes very rapidly near the boundary $t = 0$ when the perturbation parameters ε_1 and ε_2 tends to zero.

The $Log - log\ plot$ for the error in the suggested numerical method for Example 1 and Example 2 is presented in Figure 3 and Figure 6, respectively. From Figure 3 and Figure 6 we perceive that the maximum pointwise errors are bounded by $O(N^{-1}\ln N)$ which is proved in Theorem 3.

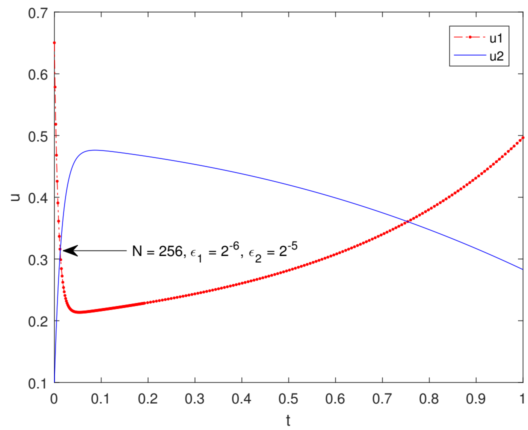

Figure 1: Solution profile of Example 1.

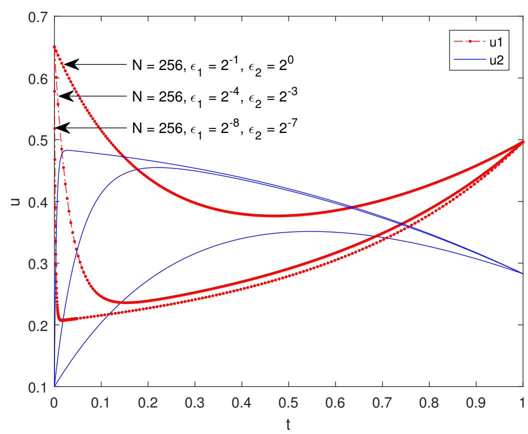

Figure 2: Changes in the components of the solution $\overrightarrow{u}(t)$ of Example 1.

References

[1] E.P. Doolan, J.J.H. Miller, W.H.A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, Ireland, 1980.

[2] P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O' Riordan, G.I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman and hall/CRC, Boca Raton, Florida, USA, 2000.

[3] P.A. Farrell, E. O'Riordan, G.I. Shishkin, *A class of singularly perturbed semilinear differential equations with interior layers*, Math. Comput. **74** (2005) 1759–1776.

[4] J.L. Gracia, F.J. Lisbona, M. Madaune-Tort, E. O'Riordan, *A system of singularly perturbed semilinear equations*, Lect. Notes Comput. Sci. **69** (2009) 163–172.

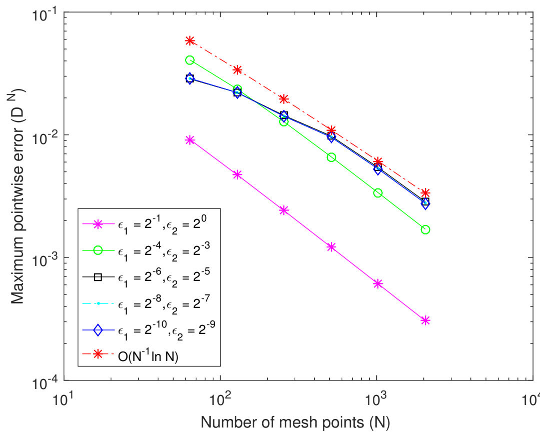

Figure 3: *Log-log plot* for the error in Example 1.

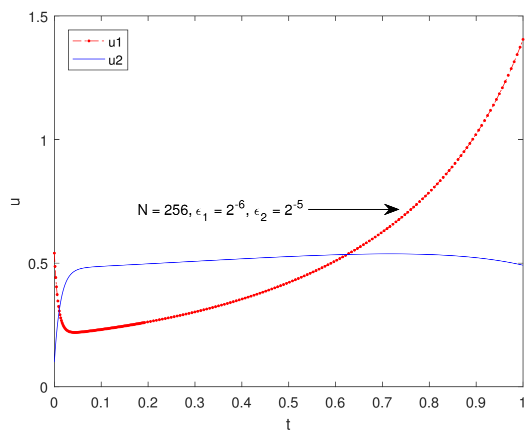

Figure 4: Solution profile of Example 2.

[5] A.F. Hegarty, E. O'Riordan, *A numerical method for singularly perturbed convection-diffusion problems posed on smooth domains*, J. Sci. Comput. **92** (2022) 84.

[6] C. Johnson, R. Rannacher, M. Boman, *Numerics and hydrodynamic stability: toward error control in computational fluid dynamics*, SIAM J. Numer. Anal. **32** (1995) 1058–1079.

[7] S.S. Kalaiselvan, J.J.H. Miller, V. Sigamani, *A parameter uniform fitted mesh method for a weakly coupled system of two singularly perturbed convection-diffusion equations*, Math. Commun. **24** (2019) 193–210.

[8] N. Kopteva, M. Stynes, *A robust adaptive method for a quasi-linear one-dimensional convection-diffusion problem*, SIAM J. Numer. Anal. **39** (2002) 1446–1467.

[9] N. Kopteva, M. Stynes, *Numerical analysis of a singularly perturbed nonlinear reaction-diffusion problem with multiple solutions*, Appl. Numer. Math. **51** (2004) 273–288.

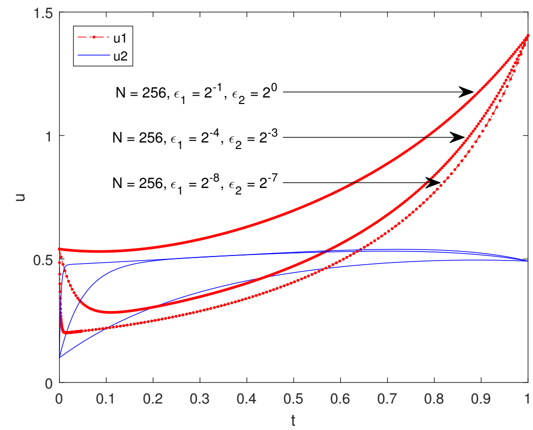

Figure 5: Changes in the components of the solution $\overrightarrow{u}(t)$ of Example 2.

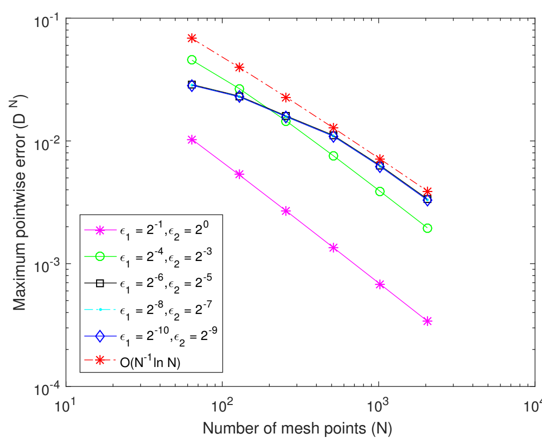

Figure 6: *Log-log plot* for the error in Example 2.

[10] M. Mariappan, *Computational analysis on a class of singularly perturbed nonlinear differential equations of convection-diffusion type*, Submitted to Mediterr. J. Math..

[11] M. Mariappan, A. Tamilselvan, *Higher order numerical method for a semilinear system of singularly perturbed differential equations*, Math. Commun. **26** (2021) 41–52.

[12] M. Mariappan, A. Tamilselvan, *Higher order computational method for a singularly perturbed nonlinear system of differential equations*, J. Appl. Math. Comput. **68** (2022) 1351–1363.

[13] J.J.H. Miller, E. O'Riordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong, 1996.

[14] J.M.J. Ortega, W.S. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[15] H.G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations, Convection-Diffusion and Flow Problems*, Springer-Verlag, New York, 1996.

[16] L. Shishkina, G.I. Shishkin, *Conservative numerical method for a system of semilinear singularly perturbed parabolic reaction-diffusion equations*, Math. Modell. Anal. **14** (2009) 211-228.