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A compact discretization of the boundary value problems of the nonlinear Fredholm integro-differential equations

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Abstract. In this paper, we propose a fourth-order compact discretization method for solving a second-order boundary value problem governed by the nonlinear Fredholm integro-differential equations. Using an efficient approximate polynomial, a compact numerical integration method is first designed. Then by applying the derived numerical integration formulas, the original problem is converted into a nonlinear system of algebraic equations. It is shown that the proposed method is easy to implement and has the third order of accuracy in the infinity norm. Some numerical examples are presented to demonstrate its approximation accuracy and computational efficiency, as well as to compare the derived results with those obtained in the literature.

Keywords: Fredholm integro-differential equation, compact discretization method, boundary value problem, fourth order of accuracy, convergence order.
AMS Subject Classification 2010: 65R20, 65L10, 45B05, 45A05.

1 Introduction

In this study, we will develop a numerical method for solving a class of the second-order boundary value problems governed by the following nonlinear Fredholm integro-differential equation (FIDE)

$$\begin{cases} z''(t)+p(t)z'(t)+f_0(t)z(t)=f_1(t)+\displaystyle\int_0^1 \tilde{v}(t,s)u(z(s))ds, \quad t\in(0,1], & \text{(1a)}\\ z(0)=a_0,\ \ z(1)=b_1, & \text{(1b)}\end{cases}$$

where $p(t), f_0(t), f_1(t)$, the kernel $\tilde{v}(s,t)$, and u are known L_2 functions, while $z(t)$ is a twice differentiable function which should be determined.

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The subject of boundary value problems plays an important role in integro-differential equations in studying physical, biological and chemical phenomena [7, 23]. Moreover, the integro-differential equations serve as mathematical models in various fields of engineering and natural science such as biological models, economics, fluid dynamics, epidemic models and spatiotemporal development models [6, 19]. Unfortunately, the analytical solution to this type of problems is very complicated, and it may not be available. Therefore, numerical methods are often used to approximate their exact solutions. In the past two decades, different numerical methods are developed to solve the FIDEs including Haar wavelet method [1], wavelet method [2, 11], differential transform method [3], Bernoulli matrix method [4], multiscale Galerkin method [7], Chebyshev finite difference method [8], parametrization method [9], exponential spline method [10], Bell polynomials method [12], direct computation method [13], Walsh function method [14], Non-standard difference method [15], Legendre polynomial method [16], homotopy analysis method [17], reproducing kernel scheme [21], Sinc-collocation scheme [22] and iterative methods [23]. In [5], a hybrid method based on block pulse functions and Bernstein polynomials are used for solving the FIDEs. To the best of our knowledge, the convergence rate of the above-mentioned methods are less than or equal two. Moreover, a few numerical methods are successful in solving the nonlinear version of the problem (1).

The main purpose of this paper is to propose a straightforward and accurate numerical method for solving the second-order boundary value problem governed by the nonlinear FIDE (1). To this end, we use some fourth-order quadrature rules and a compact integration technique to discretize the presented nonlinear FIDE. By implementing the proposed method, this problem reduces to a nonlinear algebraic system of equations. The convergence analysis of the proposed method is established. We show that the convergence rate in L_∞ norm for the novel method is four.

2 A compact numerical integration method

Consider the partition $\{t_k = kh : \; k = 0, 1, \ldots, N\}$ of the interval $[0, 1]$, where $t_0 = 0$, $t_N = 1$ and $h = \frac{1}{N}$ denotes the step size. In this section, we will present a compact numerical method to approximate the following definite integrals

$$I_k(g) := \int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) g(\tau) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) g(\tau) d\tau, \quad k = 1, \ldots, N-1, \tag{2}$$

which will be applied to discretize the FIDE (1). For every function $g(t) \in \mathbb{C}^4[0, 1]$, we can propose a pair of approximate functions on $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ as follow

$$\begin{cases} \mathscr{P}_k^-(t) := g_{k-1} + \dfrac{g_k - g_{k-1}}{h}(t - t_{k-1}) + \dfrac{g_{k-1} - 2g_k + g_{k+1}}{2h^2}(t - t_k)(t - t_{k-1}), & t \in [t_{k-1}, t_k], \\ \mathscr{P}_k^+(t) := g_{k+1} + \dfrac{g_{k+1} - g_k}{h}(t - t_{k+1}) + \dfrac{g_{k-1} - 2g_k + g_{k+1}}{2h^2}(t - t_k)(t - t_{k+1}), & t \in [t_k, t_{k+1}], \end{cases} \tag{3}$$

where $g_k = g(t_k)$ and $k = 1, \ldots, N-1$. In fact, using the forward and backward interpolations of function g for support points $\{t_{k-1}, t_k, t_{k+1}\}$, the approximate functions $\mathscr{P}_k^-(t)$ and $\mathscr{P}_k^+(t)$ are respectively formulated. In the following lemma, error of the approximate functions $\mathscr{P}_k^\pm(t)$ for $g(t)$ on $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ is stated.

Lemma 1. *Let $g(t) \in \mathbb{C}^4[0,1]$, then there exist numbers $\xi_{0,k}^{\pm}$, $\xi_{1,k}^{\pm} \in (t_{k-1}, t_{k+1})$ satisfying*

$$\begin{cases} g(t) - \mathscr{P}_k^-(t) = (t-t_k)(t-t_{k-1})\left(\frac{1}{3!}(t-t_{k+1})g^{(3)}(\xi_{0,k}^-) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^-)\right), & t \in [t_{k-1}, t_k], \\ g(t) - \mathscr{P}_k^+(t) = (t-t_k)(t-t_{k+1})\left(\frac{1}{3!}(t-t_{k-1})g^{(3)}(\xi_{0,k}^+) - \frac{h^2}{4!}g^{(4)}(\xi_{1,k}^+)\right), & t \in [t_k, t_{k+1}], \end{cases} \tag{4}$$

where $k = 1, \ldots, N-1$.

Proof. Let $\mathscr{P}_k^-(t)$ and $\mathscr{P}_k^+(t)$ be functions (3) which approximate the function g on $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$, respectively. Note that it is sufficient to prove (4) in the case $t \in [t_{k-1}, t_k]$, the other case $t \in [t_k, t_{k+1}]$ can be established in the same way. For every fixed $t \in (t_{k-1}, t_k)$, we define

$$G(x) = g(x) - \bar{g}_k(x) - \frac{g(t) - \bar{g}_k(t)}{w_k(t)} w_k(x), \quad x \in [t_{k-1}, t_k],$$

where $w_k(t) = (t-t_k)(t-t_{k+1})(t-t_{k-1})$ and $\bar{g}_k(x) = g_{k-1} + \frac{g_k - g_{k-1}}{h}(x - t_{k-1}) + \frac{1}{2}(x-t_k)(x-t_{k-1})g_k''$. It is easily seen that $G''(t_k) = 0$ and $G(t) = G(t_{k-1}) = G(t_k) = 0$. Then using the well-known Rolle's theorem, $G''(x)$ has at least one zero $\varsigma_k^- \in (t_{k-1}, t_k)$, i.e., $G''(\varsigma_k^-) = 0$. Therefore, using the Rolle's theorem, $G^{(3)}(x)$ has at least one zero $\xi_{0,k}^- \in (\varsigma_k^-, t_k) \subset (t_{k-1}, t_k)$. Since $G^{(3)}(\xi_{0,k}^-) = 0$, $\bar{g}_k^{(3)}(x) \equiv 0$ and $w_k^{(3)}(x) = 3!$, we can derive

$$g(t) = \frac{1}{3!} g^{(3)}(\xi_{0,k}^-) w_k(t) + g_{k-1} + \frac{g_k - g_{k-1}}{h}(x - t_{k-1}) + \frac{1}{2}(x-t_k)(x-t_{k-1})g_k'', \quad t \in [t_{k-1}, t_k].$$

Using the Taylor series of $g(t)$, there exists $\xi_{1,k}^- \in (t_{k-1}, t_{k+1})$ satisfying

$$g_k'' = \frac{g_{k-1} - 2g_k + g_{k+1}}{h^2} - \frac{h^2}{12} g^{(4)}(\xi_{1,k}^-).$$

Consequently, the error for $\mathscr{P}_k^-(t)$ approximating $g(t)$ on $[t_{k-1}, t_k]$ is formulated by the first part of (4). This completes the proof. □

In the following lemma, we state a compact numerical integration method to approximate a pair of specified integrals over $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$.

Lemma 2. *Let $g(t) \in \mathbb{C}^4[0,1]$ and consider a pair of integrals over intervals $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ as*

$$I_k^-(g) := \int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) g(\tau) d\tau, \quad \textit{and} \quad I_k^+(g) := \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) g(\tau) d\tau, \tag{5}$$

where $t_k = kh$ and $1 \le k \le N-1$. Then for every $k = 1, 2, \ldots, N-1$, there exists a constant $C_{g_k} \in \mathbb{R}$ such that

$$I_k^+(g) + I_k^-(g) = \frac{h^2}{12}(g_{k-1} + 10g_k + g_{k+1}) + C_{g_k} h^6. \tag{6}$$

Proof. For $k = 1, \ldots, N-1$, by using the approximate functions (3), we can derive

$$\begin{aligned}
I_k^+(g) + I_k^-(g) &= \int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) g(\tau) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) g(\tau) d\tau \\
&= \int_{t_{k-1}}^{t_k} (\tau - t_{k-1}) \mathscr{P}_k^-(\tau) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) \mathscr{P}_k^+(\tau) d\tau + E_k(g) \\
&= \frac{h^2}{24}(3g_{k-1} + 10g_k - g_{k+1}) + \frac{h^2}{24}(-g_{k-1} + 10g_k + 3g_{k+1}) + E_k(g), \\
&= \frac{h^2}{12}(g_{k-1} + 10g_k + g_{k+1}) + E_k(g),
\end{aligned}$$

where $E_k(g) = \int_{t_{k-1}}^{t_k} (\tau - t_{k-1})(g(\tau) - \mathscr{P}_k^-(\tau)) d\tau + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau)(g(\tau) - \mathscr{P}_k^+(\tau)) d\tau$. If $\tau \in [t_{k-1}, t_k]$, we have

$$(\tau - t_{k-1})^2 (t_k - \tau)(t_{k+1} - \tau) \geq 0, (\tau - t_{k-1})^2 (t_k - \tau) \geq 0,$$

and when $\tau \in [t_k, t_{k+1}]$, we get

$$(\tau - t_{k-1})(\tau - t_k)(\tau - t_{k+1})^2 \geq 0, (\tau - t_{k+1})^2 (t_k - \tau) \leq 0.$$

By the Mean Value theorem for definite integrals, there exist $\eta_{0,k}^-, \eta_{1,k}^- \in (t_{k-1}, t_k)$ and $\eta_{0,k}^+, \eta_{1,k}^+ \in (t_k, t_{k+1})$, satisfying

$$\begin{aligned}
E_k(g) &= \int_{t_{k-1}}^{t_k} (\tau - t_{k-1})^2 (\tau - t_k) \left((\tau - t_{k+1}) \frac{g^{(3)}(\xi_{0,k}^-)}{3!} - h^2 \frac{g^{(4)}(\xi_{1,k}^-)}{4!} \right) d\tau \\
&\quad + \int_{t_k}^{t_{k+1}} (\tau - t_k)(\tau - t_{k+1})^2 \left((\tau - t_{k-1}) \frac{g^{(3)}(\xi_{0,k}^+)}{3!} - h^2 \frac{g^{(4)}(\xi_{1,k}^+)}{4!} \right) d\tau \\
&= \frac{g^{(3)}(\eta_{0,k}^-)}{3!} \int_{t_{k-1}}^{t_k} (\tau - t_{k-1})^2 (t_k - \tau)(t_{k+1} - \tau) d\tau + h^2 \frac{g^{(4)}(\eta_{1,k}^-)}{4!} \int_{t_{k-1}}^{t_k} (\tau - t_{k-1})^2 (t_k - \tau) dt \\
&\quad + \frac{g^{(3)}(\eta_{0,k}^+)}{3!} \int_{t_k}^{t_{k+1}} (\tau - t_{k-1})(\tau - t_k)(\tau - t_{k+1})^2 d\tau + h^2 \frac{g^{(4)}(\eta_{1,k}^+)}{4!} \int_{t_k}^{t_{k+1}} (\tau - t_{k+1})^2 (t_k - \tau) d\tau \\
&= -\frac{7h^5}{360} \left(g^{(3)}(\eta_{0,k}^+) - g^{(3)}(\eta_{0,k}^-) \right) + \frac{2h^6}{(4!)^2} \left(g^{(4)}(\eta_{1,k}^+) + g^{(4)}(\eta_{1,k}^-) \right).
\end{aligned}$$

Therefore using the Mean Value theorem, there exists $\tilde{\theta}_k \in (t_{k-1}, t_{k+1})$ such that

$$g^{(3)}(\eta_{0,k}^+) - g^{(3)}(\eta_{0,k}^-) = h\alpha_k g^{(4)}(\tilde{\theta}_k), \quad 0 < \alpha_k < 2.$$

Moreover, since $g^{(4)}(t) \in \mathbb{C}[0,1]$, there exists $\theta_k \in (t_{k-1}, t_{k+1})$ such that

$$E_k(g) = \left(-\frac{7}{360}\alpha_k + \frac{1}{(4!)^2} \right) h^6 g^{(4)}(\theta_k). \tag{7}$$

Setting $C_{g_k} = \left(-\frac{7}{360}\alpha_k + \frac{1}{(4!)^2} \right) g^{(4)}(\theta_k)$ completes the proof. □

The integration rule (6) describes a three-point difference formula which is of order four. This formula is a key for describing the compact numerical integration method in this paper.

3 A fully discrete method for the problem (1)

In this section, we formulate a fully discrete method for solving the second-order boundary value problem of the nonlinear FIDE (1). Assume that the function $p(t)$ is differentiable for all $t \in [0,1]$. Let

$$z(t) = \gamma(t)y(t), \quad \gamma(t) = \exp(-\frac{1}{2}\int_0^t p(s)ds), \quad t \in [0,1].$$

Using some simple calculations, we transform the problem (1) into

$$\begin{cases} y''(t) + q(t)y(t) = f(t) + \int_0^1 v(t,s)u\big(\gamma(s)y(s)\big)ds, \quad t \in (0,1], & \text{(8a)} \\ y(0) = a, \;\; y(1) = b, & \text{(8b)} \end{cases}$$

where $a_0 = \gamma(0)a, \;\; b_1 = \gamma(1)b$ and

$$q(t) = f_0(t) - \frac{1}{4}\big((p(t))^2 + 2\frac{d(p(t))}{dt}\big), \quad f_1(t) = \gamma(t)f(t), \quad \tilde{v}(t,s) = \gamma(t)v(t,s).$$

It is clear that $y(t)$ is a solution of (8) if and only if $z(t) = \gamma(t)y(t)$ is a solution of (1). Therefore we will implement a fully discrete method on the equivalent form (8) of the original problem (1). For mesh points $\{t_k\}_{k=0}^N$, let Y_k and $V_{k,n}$ denote the approximate values of $y_k := y(t_k)$ and $v_{k,n} := v(t_k,t_n)$, respectively. By integrating over $[t_k,t]$, and using integration by parts on the right-hand side of Eq. (8), we have

$$y'(t) - y'_k + \int_{t_k}^t q(\xi)y(\xi)d\xi = \int_{t_k}^t f(\xi)d\xi + \int_{t_k}^t \int_0^1 v(\xi,s)u(\gamma(s)y(s))dsd\xi. \tag{9}$$

Now, by integrating over $[t_k,t_{k+1}]$ and $[t_{k-1},t_k]$, we can derive

$$\int_{t_k}^{t_{k\pm1}} (y'(t) - y'_k)dt + \int_{t_k}^{t_{k\pm1}} \int_{t_k}^t q(\xi)y(\xi)d\xi dt$$
$$= \int_{t_k}^{t_{k\pm1}} \int_{t_k}^t f(\xi)d\xi dt + \int_{t_k}^{t_{k\pm1}} \int_{t_k}^t \int_0^1 v(\xi,s)u(\gamma(s)y(s))dsd\xi dt, \quad 1 \le k \le N-1.$$

Using Fubini's theorem [20], it follows

$$\begin{cases} y_{k+1} - y_k - hy'_k + \int_{t_k}^{t_{k+1}} (t_{k+1}-t)q(t)y(t)dt \\ \quad = \int_{t_k}^{t_{k+1}} (t_{k+1}-t)f(t)dt + \int_0^1 u(\gamma(s)y(s)) \left(\int_{t_k}^{t_{k+1}} (t_{k+1}-t)v(t,s)dt \right) ds, \quad k = 1,\ldots,N-1. \\ y_{k-1} - y_k + hy'_k + \int_{t_k}^{t_{k-1}} (t_{k-1}-t)q(t)y(t)dt \\ \quad = \int_{t_k}^{t_{k-1}} (t_{k-1}-t)f(t)dt + \int_0^1 u(\gamma(s)y(s)) \left(\int_{t_k}^{t_{k-1}} (t_{k-1}-t)v(t,s)dt \right) ds, \quad k = 1,\ldots,N-1. \end{cases} \tag{10}$$

As the definition of pair of integrals $I_k^{\pm}(g)$ for $g(t)$ given in (5), the system of equations (10) can be represented as

$$\begin{cases} y_{k-1} - y_k + hy'_k + I_k^-(qy) = I_k^-(f) + \int_0^1 u(\gamma(s)y(s))I_k^-(v(.,s))ds, & \text{(11a)} \\ y_{k+1} - y_k - hy'_k + I_k^+(qy) = I_k^+(f) + \int_0^1 u(\gamma(s)y(s))I_k^+(v(.,s))ds, \; k = 1,\ldots,N-1, & \text{(11b)} \end{cases}$$

where $I_k^{\pm}(f)=\int_{t_k}^{t_{k\pm1}}(t_{k\pm1}-t)f(t)dt$ and

$$I_k^{\pm}(v(\cdot,s)) := \int_{t_k}^{t_{k\pm1}}(t_{k\pm1}-t)v(t,s)dt, \qquad I_k^{\pm}(qy)=\int_{t_k}^{t_{k\pm1}}(t_{k\pm1}-t)q(t)y(t)dt.$$

By adding the two sides of Eqs. (11), a system of integral equations is formulated

$$y_{k+1}-2y_k+y_{k-1}+I_k(qy)=I_k(f)+\int_0^1 u(\gamma(s)y(s))I_k(v(.,s))ds, \quad k=1,\dots,N-1, \tag{12}$$

where

$$\begin{cases} I_k(qy) := I_k^-(qy)+I_k^+(qy), & (13a)\\ I_k(f) := I_k^-(f)+I_k^+(f), & (13b)\\ I_k(v(.,s)) := I_k^-(v(.,s))+I_k^+(v(.,s)). & (13c)\end{cases}$$

From the boundary conditions given in (8), we have $y_0=a$ and $y_N=b$. The unknown values $y_1,\dots,y_{N-1}$ can be obtained by solving the system of (12). To find the approximate solution of the system (12), it is sufficient to utilize some numerical integration methods for integrals given in (13) and the integral term in the right-hand side of Eq. (12). To this end, we apply the compact integration formula (6) to approximate the integrals given by (13a)-(13c).

It should be noted that to approximate the integral terms in (12), we use the Simpson's rule with the nodes $\{t_j\}_{j=0}^N$ and weights $\mathbf{w}=(\omega_0=\frac{h}{3},\omega_1=4\frac{h}{3},\omega_2=2\frac{h}{3},\dots,\omega_{N-2}=2\frac{h}{3},\omega_{N-1}=4\frac{h}{3},\omega_N=\frac{h}{3})^\top$, i.e.,

$$\int_0^1 u(\gamma(s)y(s))I_k(v(.,s))ds=\frac{h^2}{12}\sum_{j=0}^N \omega_j u(\gamma_j y_j)\left(v(t_{k-1},t_j)+10v(t_k,t_j)+v(t_{k+1},t_j)\right)+C_k h^6, \tag{14}$$

where $\gamma_j=\gamma(t_j)$, $1\le k\le N-1$, and C_k depends on the forth-order derivatives of $y(t),u(\gamma(t)y(t)),v(t,s)$. Therefore, if we set $\Upsilon=[\gamma(t_1),\dots,\gamma(t_{N-1})]^\top$ and

$$\mathbf{Y}=[Y_1,\dots,Y_{N-1}]^\top,\ \ \mathbf{Q}=[q(t_1),\dots,q(t_{N-1})]^\top,\ \ \mathbf{F}=[f(t_1),\dots,f(t_{N-1})]^\top,\ \ \mathbf{V}=[v(t_k,t_n)]_{k,n=1}^{N-1},$$

then the following fully discrete system, as a compact discretization method, is developed to solve the two-point boundary value FIDE (8)

$$\left(\mathbf{J}_0+h^2\mathbf{J}\,\mathbf{diag}(\mathbf{Q})\right)\mathbf{Y}-h^3\,\mathbf{J}\,\mathbf{V}\,\mathbf{diag}(\mathbf{w})u(\Upsilon\,\mathbf{Y})=h^2\mathbf{J}\,\mathbf{F}+\mathbf{b_0}, \tag{15}$$

where $\mathbf{J}_0=\text{tridiag}(1,-2,1)$ and $\mathbf{J}=\frac{1}{12}\text{tridiag}(1,10,1)$ are two $(N-1)$-dimensional tridiagonal matrices with

$$\begin{aligned}\mathbf{b_0}=\ & h^3\omega_0u_0\mathbf{J}\mathbf{v}_0+h^3\omega_Nu_N\mathbf{J}\mathbf{v}_N+\left((-1-\frac{h^2}{12}Q_0)y_0+\frac{h^2}{12}f_0+\frac{h^3}{12}(\omega_0u_0v_{0,0}+\omega_Nu_Nv_{N,0})\right)\mathbf{I}_1\\ &+\left((-1-\frac{h^2}{12}Q_N)y_N+\frac{h^2}{12}f_N+\frac{h^3}{12}(\omega_0u_0v_{0,N}+\omega_Nu_Nv_{N,N})\right)\mathbf{I}_{N-1},\end{aligned}$$

$u_0=u(\gamma_0y_0)$, $u_N=u(\gamma_Ny_N)$, $\mathbf{v}_0=[v(t_1,t_0),\dots,v(t_{N-1},t_0)]^\top$, and $\mathbf{v}_N=[v(t_1,t_N),\dots,v(t_{N-1},t_N)]^\top$. For $i=1,N-2,N-1$, the symbol $\mathbf{I}_i$ signifies a $(N-1)$-column vector with entry 1 in position i and 0 elsewhere. For simplicity, the matrix representation of the compact discretization method (15) can be represented as

$$\mathbf{A}\,\mathbf{Y} - h^3\,\mathbf{L}\,u(\Upsilon\,\mathbf{Y}) = \mathbf{b}, \tag{16}$$

where $\mathbf{A} = \mathbf{J}_0 + h^2\mathbf{J}\,\mathbf{diag}(\mathbf{Q}),\ \mathbf{L} = \mathbf{J}\mathbf{V}\,\mathbf{diag}(\mathbf{w})$ and $\mathbf{b} = h^2\mathbf{J}\mathbf{F} + \mathbf{b_0}$. The Eq. (16) describes a nonlinear system of algebraic equations which can be solved by using a nonlinear solver such as the Newton method. It should be noted that this matrix formulation is straightforward and easy to implement. The convergence analysis of the method will be presented in the next section.

3.1 Convergence analysis

Here, we will develop the solvability and convergence analysis of the compact discretization method (16) to solve the FIDE (8). To this end, we first recall the following lemma from [18].

Lemma 3. *[18] Let $\boldsymbol{D} = [d_{i,j}]_{i,j=1}^{N-1}$ be a tridiagonal matrix in which $d_{i,i} = d_0$, $d_{i,i\pm1} = d_1$ and $d_{i,j} = 0$ otherwise. Then, the eigenvalues of $\boldsymbol{D}$ are $\lambda_j^{\boldsymbol{D}} = d_0 + 2d_1\cos\left(\frac{j\pi}{N}\right)$ where $j = 1,2,\cdots,N-1$.*

This lemma yields that

$$\begin{cases} \lambda_j^{\mathbf{J}_0} = -4\sin^2\left(\frac{j\pi}{2N}\right), \quad j = 1,2,\ldots,N-1, \\ |\lambda_{min}(\mathbf{J}_0)| := \min\{|\lambda_j^{\mathbf{J}_0}|,\ j = 0,\ldots,N-1\} = 4\sin^2\left(\frac{\pi}{2N}\right) = \pi^2h^2 > 0, \\ \kappa(\mathbf{J}_0) \leqslant \frac{16}{\pi^2}N^2, \end{cases} \tag{17}$$

where $\kappa(\mathbf{J}_0)$ denotes the condition number of $\mathbf{J}_0$.

Corollary 1. *The compact discretization method (16) has a unique solution, and it is stable.*

Proof. We can formulate a linearization form of the system (16) as

$$\mathbf{M}\mathbf{Y} = \mathbf{b}, \tag{18}$$

where $\mathbf{M} = (\mathbf{J}_0 + h^2\mathbf{J}\,\mathbf{diag}(\mathbf{Q}) - h^3\,\mathbf{L}\Upsilon\mathbf{D}_u)$ is the coefficient matrix of the present method. The matrix $\mathbf{D}_u$ is a diagonal matrix containing the jacobian of $u(\gamma y)$, i.e., $\mathbf{D}_u = \mathrm{diag}(\left[\frac{\partial u(\gamma_k y_k)}{\partial y}\right]_{k=1}^{N-1})$. Thus this system is equivalent to the linear system

$$\mathbf{J}_0\mathbf{Y} = \mathbf{b}, \tag{19}$$

when the higher powers h^2 and h^3 are ignored, i.e., $h \to 0$. From (17), we can conclude that the matrix $\mathbf{J}_0$ is invertible and $\kappa(\mathbf{J}_0) = \mathscr{O}(N^2)$. Therefore $\kappa(\mathbf{M})$ does not grow rapidly with N, when $[0,1]$ is divided into N cells. Consequently the nonlinear system (16) has a unique solution, and it is stable. □

In the following theorem, we show that the compact discretization method (16) is of accuracy order 4 with respect to L_∞ norm.

Theorem 1. *Let the functions $f(t), v(t,s), u(z(t)), q(t) \in \mathbb{C}^4[0,1]$. If $\mathbf{y} = [y_1, y_2\ldots, y_{N-1}]^\top$ and $\mathbf{Y} = [Y_1,\ldots,Y_{N-1}]^\top$ be solutions of Eqs. (14) and (16), respectively. Then we have $\|\mathbf{y}-\mathbf{Y}\|_\infty = \mathscr{O}(h^4)$.*

Proof. If we set $\mathbf{y} = [y_1, \ldots, y_{N-1}]^\top$, then from Eqs. (6) and (14), we have

$$\mathbf{A}\,\mathbf{y} - h^3\,\mathbf{L}\,u(\Upsilon\,\mathbf{y}) = \mathbf{b} + \mathbf{R}, \tag{20}$$

where $\mathbf{R} = (\bar{\mathbf{C}} + \mathbf{C}_1)h^6$. Moreover the elements of vectors $\bar{\mathbf{C}}, \mathbf{C}_1$ depend on the forth-order derivatives of functions y, q, v, u and $\hat{\mathbf{C}} = [\hat{C}_1, \ldots, \hat{C}_{N-1}]^\top$. Hence if $\|\mathbf{v}\|_p$ denotes the norm $1 \le p \le \infty$ of the vector $\mathbf{v}$, then we have

$$\|\mathbf{R}\|_2 \leqslant \tilde{C}\,h^5\sqrt{h} \tag{21}$$

where $\tilde{C} = \max\limits_{t,s\in[0,1]} \{|f^{(4)}(t)|, |q^{(4)}(t)|, |u^{(4)}(t)|, |y^{(4)}(t)|, |\frac{\partial^4 v(t,s)}{\partial t^k \partial s^j}|, j+k=4\}$. Subtracting Eq. (16) from Eq. (20), yields

$$\mathbf{A}(\mathbf{y} - \mathbf{Y}) - h^3\,\mathbf{L}\big(u(\Upsilon\,\mathbf{y}) - u(\Upsilon\,\mathbf{Y})\big) = \mathbf{R}. \tag{22}$$

Consequently a linearization form of the error system can be achieved as

$$\mathbf{M}\,\mathbf{E}(h) = \mathbf{R}, \tag{23}$$

where $\mathbf{E}(h) = \mathbf{y} - \mathbf{Y}$, and $\mathbf{M}$ is the coefficient matrix given in (18). Let $\sigma_{min}(\mathbf{M})$ denotes the smallest singular value of $\mathbf{M}$, then from Eq. (23) we conclude that

$$\|\mathbf{E}(h)\|_2 \leqslant \frac{\|\mathbf{R}\|_2}{\sigma_{min}(\mathbf{M})} \leqslant \frac{\|\mathbf{R}\|_2}{|\lambda_{min}(\mathbf{J}_0)|}, \quad \text{as} \quad h \to 0.$$

Using inequalities (21) and (17), we have

$$\|\mathbf{E}(h)\|_2 \leqslant \frac{4\tilde{C}}{\pi^2}\,h^3\sqrt{h}, \tag{24}$$

as $h \to 0$. Since $\|\mathbf{E}(h)\|_\infty \leqslant \|\mathbf{E}(h)\|_2 \leqslant \sqrt{N}\|\mathbf{E}(h)\|_\infty$ and $Nh = 1$, we can derive

$$\|\mathbf{E}(h)\|_\infty \leqslant \frac{4\tilde{C}}{\pi^2}\,h^4. \tag{25}$$

Thus the proof is completed. □

It is known that in the finite dimensional spaces, all norms are equivalent, therefore we can derive $\|\mathbf{E}(h)\|_1 = \mathscr{O}(h^3)$ and $\|\mathbf{E}(h)\|_2 = \mathscr{O}(h^{3+\nu})$, where $\nu = \frac{1}{2}$.

4 Numerical examples

The performance of the compact discretization method (15) to solve the FIDE (8) is demonstrated in this section. In the following numerical simulations, the step size is selected as $h = 2^{-k}, k = 2, 3, \ldots$. The error $\mathbf{E}(h)$ with respect to L_1, L_2, L_∞ norms and condition number of the coefficient matrix $\mathbf{M}$ are computed. Moreover, the accuracy order of the present method with respect to L_p norm is calculated by $\log_2\left(\frac{\|\mathbf{E}(h)\|_p}{\|\mathbf{E}(\frac{h}{2})\|_p}\right)$ with $p = 1, 2, \infty$.

Table 1: Errors and the accuracy order of the proposed method (16) for Example 1.

	$\alpha=4,\beta=1$							$\alpha=4.5,\beta=2$						
N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$
4	2.69e-04	–	1.87e-04	–	1.49e-04	–	5.74e+00	1.10e-02	–	6.38e-03	–	4.32e-03	–	5.67e+00
8	3.73e-05	2.85	1.65e-05	3.50	9.40e-06	3.99	2.47e+01	1.45e-03	2.92	5.69e-04	3.49	2.72e-04	3.99	2.41e+01
16	4.77e-06	2.97	1.46e-06	3.50	6.06e-07	3.95	9.99e+01	1.83e-04	2.99	5.02e-05	3.50	1.70e-05	4.00	9.69e+01
32	5.99e-07	2.99	1.29e-07	3.50	3.79e-08	4.00	3.99e+02	2.29e-05	3.00	4.42e-06	3.50	1.06e-06	4.00	3.85e+02
64	7.49e-08	3.00	1.14e-08	3.50	2.37e-09	4.00	1.59e+03	2.85e-06	3.00	3.90e-07	3.50	6.62e-08	4.00	1.53e+03
128	9.37e-09	3.00	1.01e-09	3.50	1.48e-10	4.00	6.38e+03	3.56e-07	3.00	3.44e-08	3.50	4.13e-09	4.00	6.13e+03
256	1.17e-09	3.00	8.92e-11	3.50	9.24e-12	4.00	2.55e+04	4.44e-08	3.00	3.04e-09	3.50	2.58e-10	4.00	2.45e+04
512	1.41e-10	3.06	7.43e-12	3.59	5.47e-13	4.08	1.02e+05	5.52e-09	3.01	2.67e-10	3.51	1.61e-11	4.01	9.80e+04

Table 2: Errors and the accuracy order of the proposed method (16) for Example 1.

	$\alpha=2.5,\beta=2$							$\alpha=4,\beta=50$						
N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$
4	1.38e-01	–	8.07e-02	–	5.39e-02	–	5.65e+00	2.48e+00	–	1.46e+00	–	9.59e-01	–	8.66e+00
8	1.88e-02	2.87	7.41e-03	3.45	3.55e-03	3.92	2.41e+01	3.68e-01	2.75	1.44e-01	3.34	6.69e-02	3.84	8.76e+00
16	2.41e-03	2.97	6.61e-04	3.49	2.25e-04	3.98	9.66e+01	4.81e-02	2.94	1.31e-02	3.46	4.32e-03	3.95	2.00e+01
32	3.03e-04	2.99	5.85e-05	3.50	1.41e-05	4.00	3.84e+02	6.08e-03	2.98	1.16e-03	3.49	2.72e-04	3.99	6.35e+01
64	3.79e-05	3.00	5.18e-06	3.50	8.84e-07	4.00	1.53e+03	7.62e-04	3.00	1.03e-04	3.50	1.70e-05	4.00	2.26e+02
128	4.74e-06	3.00	4.58e-07	3.50	5.52e-08	4.00	6.11e+03	9.53e-05	3.00	9.11e-06	3.50	1.06e-06	4.00	8.53e+02
256	5.92e-07	3.00	4.05e-08	3.50	3.45e-09	4.00	2.44e+04	1.19e-05	3.00	8.05e-07	3.50	6.65e-08	4.00	3.31e+03
512	7.40e-08	3.00	3.58e-09	3.50	2.16e-10	4.00	9.76e+04	1.49e-06	3.00	7.11e-08	3.50	4.16e-09	4.00	1.30e+04

Example 1. Consider the following second-order boundary value problem of the FIDE

$$\begin{cases} y''(t)+\beta y'(t)-e^{-\beta t}y(t)=f(t)+\int_0^1(\alpha+2)st^{\beta}\cos(\pi\beta t)y(s)ds, & t\in[0,1], \\ y(0)=0,\ \ y(1)=\beta, \end{cases} \tag{26}$$

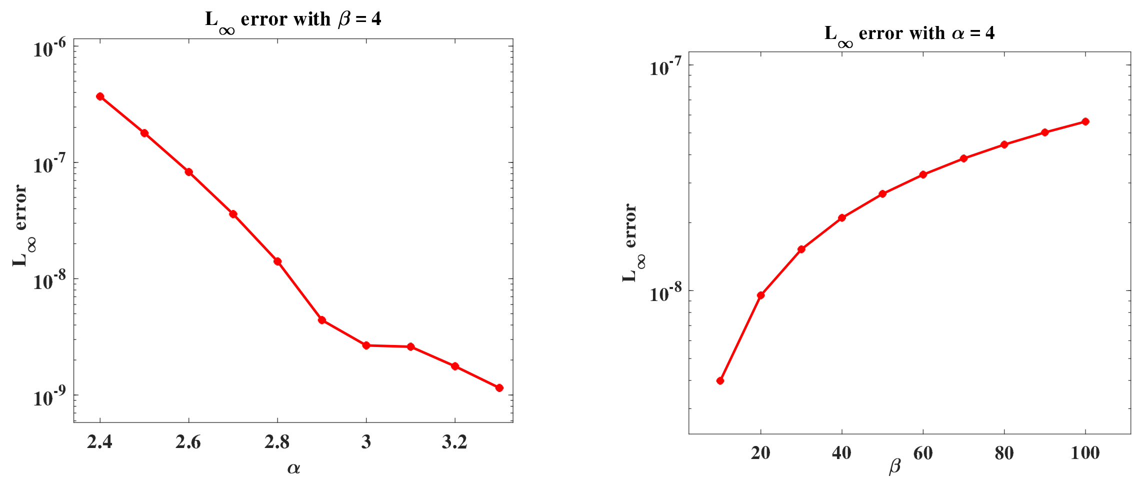
where the function $f(t)$ is chosen such that the exact solution is $y(t)=\beta t^{\alpha}$.

The numerical solution of this boundary value problem is computed by the proposed method. Numerical results containing the L_1, L_2, L_∞ errors, condition number of the coefficient matrix $\mathbf{M}$ and the rate of convergence of the compact discretization method are displayed in Tables 1-2 and Figure 1 when the computational domain $[0,1]$ is divided into N cells. From Table 1 we can see that the desired order of convergence of the presented method is achieved for $\alpha=4,\beta=1$ and $\alpha=4.5,\beta=2$, which the theoretical results presented by Theorem 1. Note that the third-order derivative of the exact solution with $\alpha=2.5$ is nonsingular at $t=0$, and the kernel of the integral term of Eq. (26) is oscillatory when $\beta\gg 1$. To demonstrate the efficiency of the method, the numerical results with $\alpha=2.5,\beta=2$ and $\alpha=4,\beta=50$ are provided in Table 2. Moreover, the log plots of L_∞ error versus the values of α and β are depicted by Figure 1(a) and Figure 1(b), respectively. Tables 1-2 verify that $\kappa(\mathbf{M})$ does not grow rapidly with N.

Example 2. Consider the following FIDE

$$\begin{cases} y''(t)+\sin(\pi t)y'(t)+(t^3+1)y(t)=f(t)+\displaystyle\int_0^1 s\cos(\pi t)\exp(y(s))ds, & t\in[0,1], \\ y(0)=1,\ \ y(1)=2, \end{cases}$$

where $f(t)=3+t^2+t^3+t^5-\frac{1}{2}(\exp(1)-1)\exp(1)\cos(\pi t)+2t\sin(\pi t)$, and the exact solution is $y(t)=t^2+1$.



(a) The log plots of L_∞ error versus the values of α. (b) The log plots of L_∞ error versus the values of β.

Figure 1: For $N = 2^8$ cells, the absolute error of the compact discretization method (15) to solve the nonlinear FIDE (26) with the exact solution $y(t) = \beta t^\alpha$.

Table 3: Errors, the accuracy order and the CPU time (in second) of the proposed method (16) for Example 2.

N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	CPU time (s)
4	2.25e-03	–	1.35e-03	–	9.27e-04	–	6.47e+00	0.01
8	2.76e-04	3.02	1.13e-04	3.58	6.02e-05	3.94	6.59e+00	0.01
16	3.44e-05	3.01	9.84e-06	3.52	3.70e-06	4.02	2.90e+01	0.01
32	4.29e-06	3.00	8.67e-07	3.50	2.31e-07	4.00	1.18e+02	0.02
64	5.36e-07	3.00	7.66e-08	3.50	1.44e-08	4.00	4.73e+02	0.02
128	6.71e-08	3.00	6.77e-09	3.50	9.02e-10	4.00	1.89e+03	0.03
256	8.37e-09	3.00	5.98e-10	3.50	5.63e-11	4.00	7.55e+03	0.03

For $N = 2, 2^2, \ldots, 2^8$, the numerical results of the compact discretization method are illustrated in Table 3. As this table shows that the present method is successful for this problem containing the high-nonlinear term $\exp(y)$, and the numerical results follow the theoretical results given by Corollary 1 and Theorem 1. Moreover Table 3 shows that the present method is very fast in solving this high-nonlinear FIDE.

Example 3. Consider the following linear two-point boundary value problem of the FIDE [7]

$$\begin{cases} y''(t) + ty'(t) + \pi^2 y(t) = \pi t \cos(\pi t) - \frac{2t+1}{\pi} + \int_0^1 (s+t)y(s)ds, & t \in [0,1], \\ y(0) = y(1) = 0. \end{cases}$$

The exact solution is $y(t) = \sin(\pi t)$.

For $N = 2^2, \ldots, 2^9$ cells, the errors with respect to L_1, L_2, L_∞ norms, $\kappa(\mathbf{M})$ and the accuracy order of the present method (15) and the multiscale Galerkin method [7] are displayed by Table 4. It can be

Table 4: Computational results of the multiscale Galerkin method [7] and present method for Example 3.

	Present method							Method of [7]			
N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	L_2 error	Order	L_1 error	Order
4	6.59e-03	–	3.87e-03	–	2.75e-03	–	3.80e+01	–	–	–	–
8	8.71e-04	2.92	3.48e-04	3.48	1.75e-04	3.98	1.78e+02	–	–	–	–
16	1.11e-04	2.98	3.09e-05	3.49	1.10e-05	3.99	7.39e+02	1.60e-2	–	1.33e-1	–
32	1.39e-05	2.99	2.73e-06	3.50	6.88e-07	4.00	2.98e+03	4.06e-3	1.98	6.39e-2	1.06
64	1.74e-06	3.00	2.42e-07	3.50	4.31e-08	4.00	1.19e+04	1.02e-3	1.99	3.16e-2	1.02
128	2.17e-07	3.00	2.14e-08	3.50	2.69e-09	4.00	4.77e+04	2.55e-4	2.00	1.57e-2	1.00
256	2.79e-08	2.96	1.94e-09	3.46	1.73e-10	3.96	1.98e+05	6.38e-5	2.00	7.87e-3	1.00
512	3.34e-09	3.06	1.69e-10	3.52	9.99e-12	4.11	7.63e+05	1.59e-5	2.00	3.93e-3	1.00

seen that the present method has the third- and fourth-order of accuracy with respect to L_1 and L_∞ norms, respectively. While the multiscale Galerkin method [7] is of order 1 with respect to L_1 norm. Moreover the L_2 errors of the present method and the multiscale Galerkin method are 1.69×10^{-10} and 1.59×10^{-5}, respectively. Consequently the present method is accurate than the method given by [7].

Example 4. Consider the following linear Fredholm integro-differential boundary value problem with $p(t)\equiv 0$ and $q(t)\equiv 0$ [6, 10]

$$\begin{cases} y''(t)-\int_0^t (s+t)y(s)ds = -\pi^2\sin(\pi t)-\dfrac{2t+1}{\pi}, & t\in[0,1],\\ y(0)=y(1)=0. \end{cases} \tag{27}$$

This problem has the exact solution in form $y(t)=\sin(\pi t)$.

Here, numerical results of three numerical techniques the presented compact discretization method (15), the fast multiscale Galerkin method [6] and exponential spline method [10] for solving the linear two-point boundary value FIDE (27) are compared in Table 5. As it is shown that the accuracy order of the present method is 3, 3.5 and 4 with respect to L_1, L_2 and L_∞ norms, respectively. While the fast multiscale Galerkin method [6] and exponential spline method [10] have the accuracy-order one and two with respect to L_1 and L_∞ norms, respectively. Moreover for $N=256$ cells, the errors of present method and method given in [6] and [10] are of order $\mathcal{O}(10^{-11}), \mathcal{O}(10^{-3})$ and $\mathcal{O}(10^{-7})$, respectively. Therefore the present method is more accurate than those techniques given in the literature.

Example 5. Consider the following nonlinear two-point boundary value FIDE [1]

$$\begin{cases} y''(t)=\exp(t)+\frac{1}{4}(\exp(2)-2)t+\frac{1}{2}\int_0^1 t(s-y^2(s))ds, & t\in[0,1],\\ y(0)=1, y(1)=\exp(1), \end{cases} \tag{28}$$

where its exact solution is $y(t)=\exp(t)$.

The numerical results containing the L_1, L_2, L_∞ errors, $\kappa(\mathbf{M})$ and the convergence order of the compact discretization method (15) and the Haar wavelet method [1] with collocation/Guass points for solving the nonlinear two-point boundary value FIDE (28) are compared in Table 6 when $[0,1]$ is divided into $N=2,2^2,\ldots,2^7$ cells. Table 6 indicates that the presented method is more accurate than the method given by [1]. Moreover, the numerical results given in this table follow the theoretical results presented by Corollary 1 and Theorem 1.

Table 5: Numerical results of the fast multiscale Galerkin method [6], exponential spline method [10] and the present method (15) for Example 4.

	Present method							Method of [6]		Method of [10]	
N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	L_1 error	Order	L_∞ error	Order
4	3.21e-03	–	1.88e-03	–	1.34e-03	–		–	–	–	–
8	4.11e-04	2.96	1.6394e-04	3.52	8.25e-05	4.02	5.39e+00	2.55e-1	–	–	–
16	5.17e-05	3.00	1.44e-05	3.50	5.12e-06	4.00	2.34e+01	1.28e-1	1.00	2.42e-4	–
32	6.47e-06	3.00	1.28e-06	3.50	3.21e-07	4.00	9.53e+01	6.40e-2	1.00	5.96e-5	2.02
64	8.09e-07	3.00	1.13e-07	3.50	2.00e-08	4.00	3.83e+02	3.20e-2	1.00	1.48e-5	2.01
128	1.01e-07	3.00	9.96e-09	3.50	1.25e-09	4.00	1.53e+03	1.60e-2	1.00	3.71e-6	2.00
256	1.26e-08	3.00	8.80e-10	3.50	7.83e-11	4.00	6.14e+03	8.00e-3	1.00	9.23e-7	2.01

Table 6: Numerical results of the present method (15) and the Haar wavelet method [1] with collocation/Guass points for Example 5.

								Haar wavelet method [1] with			
	Present method							collocation points		Guass points	
N	L_1 error	Order	L_2 error	Order	L_∞ error	Order	$\kappa(\mathbf{M})$	L_2 error	Order	L_2 error	Order
4	8.15e-05	–	4.79e-05	–	3.26e-05	–	6.26e+00	1.95e-3	–	7.71e-4	–
8	1.01e-05	2.90	4.35e-06	3.46	2.13e-06	3.94	2.71e+01	5.12e-4	1.93	1.96e-4	1.98
16	1.39e-06	2.97	3.87e-07	3.49	1.35e-07	3.98	1.11e+02	1.31e-4	1.96	4.93e-5	1.99
32	1.75e-07	2.99	3.42e-08	3.50	8.45e-09	4.00	4.45e+02	3.33e-5	1.98	1.24e-5	1.99
64	2.18e-08	3.00	3.02e-09	3.50	5.27e-10	4.00	1.78e+03	8.37e-6	1.99	3.10e-6	2.00
128	2.55e-09	3.09	2.50e-10	3.59	3.08e-11	4.09	7.14e+03	2.10e-6	2.00	7.77e-7	2.00

5 Conclusions

In this paper, the compact discretization method was used to approximate the solution of second-order boundary value problem governed by the nonlinear FIDEs. In the compact discretization approach, the desired solution is obtained through some fourth-order numerical integrations. The accuracy of the proposed method is displayed using illustrative test examples which were recently considered using other techniques. The compact discretization method is very fast and easy to implement for the nonlinear Fredholm integro-differential boundary value problems. Moreover, the numerical results show the excellent performance of this novel method for solving the considered boundary value problems.

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