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Stochastic dynamics of Izhikevich-Fitzhugh neuron model

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Abstract. This paper is concerned with stochastic stability and stochastic bifurcation of the Fitzhug-Nagumo model with multiplicative white noise. We employ largest Lyapunov exponent and singular boundary theory to investigate local and global stochastic stability at the equilibrium point. In the rest, the solution of averaging the Ito diffusion equation and extreme point of steady-state probability density function provide sufficient conditions that the stochastic system undergoes pitchfork and phenomenological bifurcations. These theoretical results of the stochastic neuroscience model are confirmed by some numerical simulations and stochastic trajectories. Finally, we compare this approach with Rulkov approach and explain how pitchfork and phenomenological bifurcations describe spiking limit cycles and stability of neuron's resting state.

Keywords: Stochastic systems, Izhikevich-FitzHugh model, stochastic stability, bifurcation.
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1 Introduction

Growing application of neuroscience in various sciences has encouraged scientists to use mathematical methods for research in this field. Mathematical modeling and dynamical systems are the most important tools for understanding neural systems. Several dynamical models arised from laboratory and clinical experiences and applications of neuroscience [7–9, 11]. The stochastic approach is an interesting and reasonable way to overcome issues caused by the existence of many unknown effective parameters in a scientific process. Recent developments in mathematical neuroscience and experimental observations indicate that reformulating neural field dynamical models as a stochastic process leads to more accurate and richer information on neural dynamics that is more consistent with the real world [2, 6, 10]. Bifurcation analysis is a performed tool to study the dynamical behavior of nonlinear mathematical models in different scientific fields, without a direct recourse to solving the equations. The main types of stochastic bifurcation are the D-bifurcation and the P-bifurcation problems, which focus on the stochastic bifurcation point in the probability and the mode of the stationary probability density function [4, 19, 21].

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Experimental evidence showed that noise is ubiquitous in neural systems and it may arise from many different sources. One source may come from synaptic noise, that is, the quasi-random release of neurotransmitters by synapses or random synaptic input from other neurons. Another major source of noise is channel noise which is the flickering of ion currents passing through the neuron membranes and cannot be suppressed experimentally [20, 22]. The HodgkinHuxley model and its simplified equations have a central role in mathematical neuroscience. FitzHugh-Nagumo model is one of these simplified equations which is a good tool to investigate the dynamical behavior of excitable systems, such as neural systems [5, 7, 11]. Berglund and Landon studied the stochastic FitzHughNagumo equations in parameter regimes characterized by mixed-mode oscillations and proved that this number has an asymptotically geometric distribution [3]. In [20] the authors considered a stochastic FitzHughNagumo neuron model in the excitable regime to estimate the firing time and proved that there exists a global random attractor for the unique stable fixed point is stable. In [10] a stochastic Izhikevich-FitzHugh dynamical model with multiplicative excitations is introduced and its stochastic stability and stochastic bifurcation are investigated.

From a dynamical systems point of view, transition from resting to sustained spiking activity is an important characteristics of neuron's dynamics. Stability and bifurcations are effective tools to study spiking and resting states of neurons [11]. On the other hand, we mentioned that many researches and experimental evidences confirm that neuron dynamics can be considered as a stochastic process. Therefore, studying stability and bifurcation neuroscience stochastic dynamical models could lead us to interesting and important results.

The main aim of this manuscript is to study the dynamical behavior of the stochastic Izhikevich-FitzHugh dynamical model, when its deterministic equation is a bistable system. The considered stochastic model has been obtained by directly adding the environmental disturbance factor to the deterministic dynamic model. In Section 2, some necessary notations and preliminary results in neuroscience and stochastic processes, that are needed in later sections, are provided. One dimensional Ito formula is the most effective tool introduced in this section to study the stability of stochastic processes. Section 3 is dedicated to introducing the stochastic Izhikevich-Fitzhugh model with multiplicative excitations. We focused on parameters that the system is bistable and rewritten so that all stability conditions are based on eigenvalues of the Jacobian matrix of the deterministic system at equilibrium points. In Section 4, largest Lyapunov exponent and singular boundary method are employed to analyze local and global stochastic stability. Moreover, we calculate conditions on steady-state probability density function that the system undergoes phenomenological and pitchfork bifurcation. Some numerical simulations including phase portrait and stochastic trajectories are given to confirm our theoretical results. Finally a comparison between our approach and Rulkov stochastic models provided and the paper ends with a summary of the research findings on spiking limit cycles and stability of neuron's resting state.

2 Preliminaries

In this section, we present some preliminary concepts and definitions of stochastic processes that will be used in the sequel.

Consider the n-dimensional stochastic differential system

$$dx(t) = f(x(t),t)dt + g(x(t),t)dW(t), \tag{1}$$

where $f \in C^3(R \times R, R)$, $g \in C^1(R \times R, R)$ and $dW(t)$ is mutually independent standard real-valued Wiener processes on the complete probability space $(\Omega, F, \mathbb{P})$. For each arbitrary initial value $x(t_0) = 0$ there exists an unique global solution for Eq. (1) that is denoted by $x(t; t_0, x_0)$.
The next theorem is an efficient tool to study stochastic stability assertions.

Theorem 1. *(The one-dimensional Ito formula [15]) Let $x(t)$ be an Ito process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(x(t),t)dt + g(x(t),t)dW(t), \tag{2}$$

where $f \in L^1(R_+; R)$, $g \in L^2(R_+; R)$ and $V \in C^{2,1}(R \times R_+; R)$. Then $V(x(t),t)$ is also an Ito process with the stochastic differential given by

$$dV(x(t),t) = [V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}V_{xx}(x(t),t)g^2(t)]dt + V_x(x(t),t)g(t)dW(t). \tag{3}$$

Definition 1. *[15] i− The equilibrium position of Eq. (1) is said to be stochastically stable if for every pair of $\varepsilon \in (0,1)$ and $r > 0$, there exists $\delta = \delta(\varepsilon, r, t_0) > 0$ such that*

$$\mathbb{P}\{|x(t;t_0,x_0)| < r \text{ for all } t \geq t_0\} \geq 1 - \varepsilon,$$

whenever $\|x_0\| < \delta_0$. Otherwise, it is said to be stochastically unstable.
ii− The stochastically stable equilibrium position is said to be stochastically asymptotically stable if for every $\varepsilon \in (0,1)$, there exists $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$\mathbb{P}\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} \geq 1 - \varepsilon,$$

whenever $\|x_0\| < \delta_0$. The equilibrium position is said to be global stochastically asymptotically stable if it is stochastically stable, moreover, for all $x_0 \in R^d$

$$\mathbb{P}\{\lim_{t\to\infty} x(t;t_0,x_0) = 0\} = 1.$$

In simple words, bifurcation theory is the study of dynamical behavior, equilibrium point and trajectories, of a nonlinear system in response to small changes in effective parameters. When a system is on the verge of changing from one equilibrium state to another, even small changes in parameters can have important effects. Dynamical bifurcation is concerned with a family of random dynamical systems which is differential and has invariant measure μ_α. If there exists a constant α_D satisfying the condition in any neighborhood of α_D, there exist another constant α and the corresponding invariant measure $\nu_\alpha \neq \mu_\alpha$ satisfying $\nu_\alpha \to \mu_\alpha$ as $\alpha \to \alpha_D$. Then the constant α_D is a point of dynamical bifurcation [23].

Phenomenological bifurcation is concerned with the change in the shape of the stationary probability density of a family of random dynamical systems with the change of the parameter. If there exists a constant α_0 satisfying the condition in any neighborhood of α_D, there exist two other constants α_1, α_2 and their corresponding invariant measures p_{α_1}, p_{α_1} where p_{α_1} and p_{α_1} are not equivalent. Then the constant α_0 is a point of phenomenological bifurcation [23]. From the phenomenological point of view, the multiplicative and additive noises have quite different features. In the first case, the thresholds depend on the strength of the excitation term and eventually new instabilities appear, which are completely noise-induced, while in the second case, the stability thresholds are not modified by the noise term.

3 Deterministic and stochastic Izhikevich-FitzHugh model

Here, we provide a brief description of the following deterministic Izhikevich-FitzHugh model [11]:

$$\begin{cases} \dot{v} = u(\alpha - u)(u-1) - w + I, \\ \dot{v} = \beta u - \gamma v, \end{cases} \tag{4}$$

where u mimics the membrane voltage and recovery variable v mimics the activation of an outward current. Parameter I mimics the injected current, and for the sake of simplicity we set $I = 0$ in our analysis below. Parameter α describes the shape of the cubic parabola $u(\alpha - u)(u-1)$, and parameters $\beta > 0$ and $\gamma \geq 0$ describe the kinetics of the recovery variable v.
The nullclines of this model have the simple form

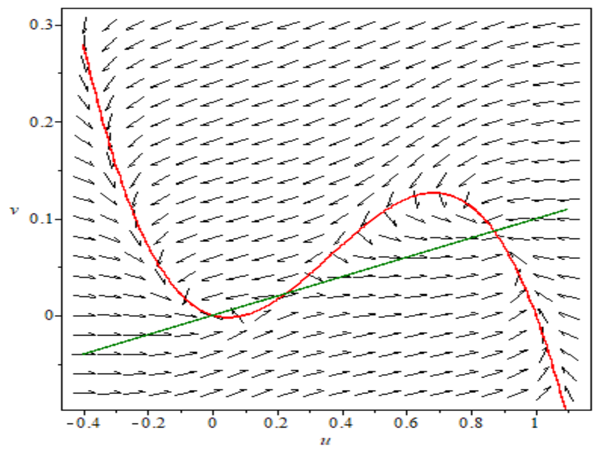

Figure 1: Nullclines in system (4) for parameters $I = 0, \beta = 0.01, \gamma = 0.1,\ \alpha = 0.1$ (Red curve is v-nullcline, green line is $w-$nullcline).

$$\begin{cases} u = u(\alpha - u)(u-1) + I, \\ v = \frac{\beta}{\gamma} u. \end{cases} \tag{5}$$

It is clear that to find the equilibrium points, we need to find the roots of the equation

$$\frac{\beta}{\gamma} u = u(\alpha - u)(u-1) + I.$$

We suppose that $I = 0$ and consequently the origin $(0,0)$ is an equilibrium point. Then $u = 0$ or $u^2 - (\alpha+1)u + \alpha + \frac{\beta}{\gamma} = 0$. When $(\frac{\alpha+1}{2})^2 - (\alpha + \frac{\beta}{\gamma}) > 0$, the roots of the second equations, are $q_1 = \frac{\alpha+1}{2} + \sqrt{(\frac{\alpha+1}{2})^2 - (\alpha + \frac{\beta}{\gamma})}$ and $q_2 = \frac{\alpha+1}{2} - \sqrt{(\frac{\alpha+1}{2})^2 - (\alpha + \frac{\beta}{\gamma})}$. So, this system has one, two or three equilibria. In this paper, we consider a state in the system tha has three equilibria $E_0 = (0,0)$, $E_1 = (q_1, \frac{\beta}{\gamma} q_1)$ and $E_2 = (q_2, \frac{\beta}{\gamma} q_2)$. Indeed, the nullclines of the model, depicted in Figure 1, always intersect at $(0,0)$ in this case. The stability of the equilibrium $(0,0)$ depends on the parameters α, β, and γ. In [10] the authors consider a stochastic scheme of this dynamical model with multiplicative excitations and proceed to study its stability and bifurcation at $(0,0)$. Now, we are going to study the stability and bifurcation of the system at equilibrium point $E_i, i = 1,2$ as a stochastic dynamical model.

The Jacobian matrix associated with (4) at equilibrium point E_i is given by

$$J = \begin{pmatrix} -3p^2 + 2(\alpha+1)p - \alpha & -1 \\ \beta & -\gamma \end{pmatrix}, \tag{6}$$

where $p = q_i,\ i = 1,2$. The characteristic polynomial of matrix (6) is

$$F(\lambda) = (\lambda+\gamma)(\lambda + 3p^2 - 2(\alpha+1)p + \alpha) + \beta. \tag{7}$$

The eigenvalues of this Jacobian matrix are

$$\lambda_{1,2} = \frac{-\Delta \pm \sqrt{\Delta^2 - 4(\alpha\gamma+\beta)}}{2},$$

where $\Delta = 3p^2 - 2(\alpha+1)p + \alpha + \gamma$. To shift the origin to the equilibrium E_i, we substitute $x = u - p$ and $y = v - \frac{\beta}{\gamma}p$ into system (8) and obtain its equivalent system as follows:

$$\begin{cases} \frac{dx}{dt} = (x+p)(\alpha - x - p)(x+p-1) - y - \frac{\beta}{\gamma}p, \\ \frac{dy}{dt} = \beta(x+p) - \gamma(y + \frac{\beta}{\gamma}p). \end{cases} \tag{8}$$

Assuming that the variables $x(t)$ and $y(t)$ are subject to different kinds of stochastic noise, we rewrite system (4) as follows:

$$\begin{cases} \frac{dx}{dt} = (x+p)(\alpha - x - p)(x+p-1) - y - \frac{\beta}{\gamma}p + \sigma_1 x\, dW(t), \\ \frac{dy}{dt} = \beta(x+p) - \gamma(y + \frac{\beta}{\gamma}p) + \sigma_2 y\, dW(t), \end{cases} \tag{9}$$

where σ_1 and σ_2 measure the noise intensity and when $\sigma = 0$ Eqs. (9) degenerates into Eqs. (4) and $W(t)$ denote the independent standard Wiener process.

In the sequel, we suppose that $I = 0$. Applying Taylor's expansion, we have the following equivalent system:

$$\begin{cases} dx = [-x^3 + (\alpha - 3p + 1)x^2 + (2p\alpha + 2p - 3p^2 - \alpha)x + p(\alpha - p)(p-1) - \frac{\beta}{\gamma}p - y]dt \\ \qquad + \sigma_1 x\, dW(t), \\ dy = [\beta x - \gamma y]dt + \sigma_2 y\, dW(t). \end{cases} \tag{10}$$

Also note that

$$p(\alpha - p)(p-1) - \frac{\beta}{\gamma}p = 0.$$

We rewrite system (10) as

$$\begin{cases} \frac{dx}{dt} = (2p\alpha + 2p - 3p^2 - \alpha)x - y + f_1, \\ \frac{dy}{dt} = \beta x - \gamma y + f_2, \end{cases} \tag{11}$$

where

$$f_1 = -x^3 + (\alpha - 3p + 1)x^2 + \sigma_1 x \frac{dW}{dt}, \text{ and } f_2 = \sigma_1 y \frac{dW}{dt}.$$

Let $U = \begin{bmatrix} x \\ y \end{bmatrix}$ and $f(U,\eta(t)) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ where $\eta(t)) = \frac{dW}{dt}$. Then we can rewrite system (11) as

$$\dot{U} = JU + f(U,\eta(t)). \tag{12}$$

Where, J is the Jacobian matrix associated with Eqs. (4) at equilibrium point E_2.

Assume that $\begin{bmatrix} E \\ 1 \end{bmatrix}$ and $\begin{bmatrix} F \\ 1 \end{bmatrix}$ are the eigenvectors of the Jacobian matrix J corresponding to the eigenvalues λ_1 and λ_2, respectively. Then $J\begin{bmatrix} E \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} E \\ 1 \end{bmatrix}$ and $J\begin{bmatrix} F \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} F \\ 1 \end{bmatrix}$ imply that $E = \frac{\lambda_1+\gamma}{\beta}$ and $F = \frac{\lambda_2+\gamma}{\beta}$. If we put $U = TX$, $X = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $T = \begin{bmatrix} E & F \\ 1 & 1 \end{bmatrix}$ then

$$\begin{aligned} \dot{X} =& T^{-1}JTX + T^{-1}f(TX,\eta) \\ =& \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{E-F} \begin{bmatrix} 1 & -F \\ -1 & E \end{bmatrix} \begin{bmatrix} f_1(TX,dW) \\ f_2(TX,dW) \end{bmatrix}. \end{aligned}$$

Since $U = TX = \begin{bmatrix} Ex_1 + Fy_1 \\ x_1 + y_1 \end{bmatrix}$, we have that

$$\begin{aligned} \dot{X} =& T^{-1}JTX + T^{-1}f(TX,\eta) \\ =& \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ \frac{1}{E-F} \begin{bmatrix} 1 & -F \\ -1 & E \end{bmatrix} \begin{bmatrix} -(Ex_1+Fy_1)^3 + (\alpha - 3p + 1)(Ex_1+Fy_1)^2 + \sigma_1(Ex_1+Fy_1)\frac{dW}{dt} \\ \sigma_2(x_1+y_1)\frac{dW}{dt} \end{bmatrix}. \end{aligned}$$

Therefore we reach the following stochastic dynamical model:

$$\begin{cases} \dfrac{dx_1}{dt} = \lambda_1 x_1 + a_{120}x_1^2 + a_{111}x_1y_1 + a_{102}y_1^2 + a_{130}x_1^3 + a_{121}x_1^2y_1 + a_{112}x_1y_1^2 + a_{103}x_1^3 + f_3, \\ \dfrac{dy_1}{dt} = \lambda_2 y_1 + a_{220}x_1^2 + a_{211}x_1y_1 + a_{202}y_1^2 + a_{230}x_1^3 + a_{221}x_1^2y_1 + a_{212}x_1y_1^2 + a_{203}x_1^3 + f_4, \end{cases} \tag{13}$$

where

$$\begin{aligned} f_3 &= \frac{1}{E-F}[(\sigma_1 E - \sigma_2 F)x_1 + (\sigma_1 - \sigma_2)Fy_1]\frac{dW}{dt}, \\ f_4 &= \frac{1}{E-F}[-(\sigma_1 - \sigma_2)Ex_1 + (-\sigma_1 E + \sigma_2 F)y_1]\frac{dW}{dt}. \end{aligned}$$

And the coefficient are as the following:

$$a_{120} = \frac{1}{E-F}(\alpha - 3p + 1)E^2, \qquad a_{220} = -\frac{1}{E-F}(\alpha - 3p + 1)E^2,$$
$$a_{111} = \frac{1}{E-F}(\alpha - 3p + 1)2EF, \qquad a_{211} = -\frac{1}{E-F}(\alpha - 3p + 1)2EF,$$
$$a_{102} = \frac{1}{E-F}(\alpha - 3p + 1)F^2, \qquad a_{202} = -\frac{1}{E-F}(\alpha - 3p + 1)F^2,$$
$$a_{130} = -\frac{1}{E-F}E^3, \qquad a_{230} = \frac{1}{E-F}E^3,$$
$$a_{121} = -\frac{3}{E-F}E^2F, \qquad a_{221} = \frac{3}{E-F}E^2F,$$
$$a_{112} = -\frac{3}{E-F}EF^2, \qquad a_{212} = \frac{3}{E-F}EF^2,$$
$$a_{103} = -\frac{1}{E-F}F^3, \qquad a_{203} = \frac{1}{E-F}F^3.$$

By the polar coordinate transformation $x_1 = \rho\cos\theta$, $y_1 = \rho\sin\theta$ and following transformation:

$$\begin{cases} \frac{d\rho}{dt} = \cos\theta\frac{dx_1}{dt} + \sin\theta\frac{dy_1}{dt}, \\ \frac{d\theta}{dt} = \frac{1}{\rho}(-\sin\theta\frac{dx_1}{dt} + \cos\theta\frac{dy_1}{dt}) \end{cases} \tag{14}$$

we have

$$\begin{cases} \frac{d\rho}{dt} = \lambda_1\rho\cos^2\theta + a_{120}\rho^2\cos^3\theta + a_{111}\rho^2\cos^2\theta\sin\theta + a_{102}\rho^2\cos\theta\sin^2\theta + a_{130}\rho^3\cos^4\theta \\ \quad + a_{121}\rho^3\cos^3\theta\sin\theta + a_{112}\rho^3\cos^2\theta\sin^2\theta + a_{103}\rho^3\cos\theta\sin^3\theta \\ \quad + \lambda_2\rho\sin^2\theta + a_{220}\rho^2\sin\theta\cos^2\theta + a_{211}\rho^2\cos\theta\sin^2\theta + a_{202}\rho^2\sin^3\theta + a_{230}\rho^3\cos^3\theta\sin\theta \\ \quad + a_{221}\rho^3\cos^2\theta\sin^2\theta + a_{212}\rho^3\cos\theta\sin^3\theta + a_{203}\rho^3\sin^4\theta \\ \quad + f_3\cos\theta + f_4\sin\theta \\ \frac{d\theta}{dt} = -\lambda_1\cos\theta\sin\theta - a_{120}\rho\cos^2\theta\sin\theta - a_{111}\rho\cos\theta\sin^2\theta - a_{102}\rho\sin^3\theta - a_{130}\rho^2\cos^3\theta\sin\theta \\ \quad - a_{121}\rho^2\cos^2\theta\sin^2\theta - a_{112}\rho^2\cos\theta\sin^3\theta - a_{103}\rho^2\sin^4\theta \\ \quad + \lambda_2\cos\theta\sin\theta + a_{220}\rho\cos^3\theta + a_{211}\rho\cos^2\theta\sin\theta + a_{202}\rho\cos\theta\sin^2\theta + a_{230}\rho^2\cos^4\theta \\ \quad + a_{221}\rho^2\cos^3\theta\sin\theta + a_{212}\rho^2\cos^2\theta\sin^2\theta + a_{203}\rho^2\cos\theta\sin^3\theta \\ \quad + \frac{1}{\rho}(-f_3\sin\theta + f_4\cos\theta). \end{cases} \tag{15}$$

4 Stochastic model in polar coordinate

Let us have a stochastic system in polar coordinates as follows:

$$\begin{cases} d\rho = f_{11}(\rho,\theta)dt + g_{11}(\rho,\theta)dW(t), \\ d\theta = f_{12}(\rho,\theta)dt + g_{21}(\rho,\theta)dW(t). \end{cases} \tag{16}$$

According to the Khasminskii limiting theorem, the stochastic response process $\rho(t)$, $\theta(t)$ of system (11) weakly converges to a two-dimensional Markov diffusion process. Thus, by using the stochastic averaging method obtained in [12], we have the averaged Ito stochastic differential equation for system (11):

$$\begin{cases} d\rho = m_1(\rho)dt + \mu_1(\rho)dW_\rho(t), \\ d\theta = m_2(\rho)dt + \mu_2(\rho)dW_\theta(t), \end{cases} \tag{17}$$

where $W_\rho(t)$ and $W_\theta(t)$ are independent and standard Wiener processes, the drift coefficients $m_1(\rho), m_1(\rho)$ and the square of diffusion coefficients $\mu_1(\rho), \mu_2(\rho)$ are

$$\begin{aligned} m_1(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\{f_{11} + \frac{1}{2}[\frac{\partial g_{11}}{d\rho}g_{11} + \frac{\partial g_{11}}{d\theta}g_{21}]\}d\theta, \\ m_2(\rho) &= \frac{1}{2\pi}\int_0^{2\pi}\{f_{21} + \frac{1}{2}[\frac{\partial g_{21}}{d\rho}g_{11} + \frac{\partial g_{21}}{d\theta}g_{21}]\}d\theta, \\ (\mu_1(\rho))^2 &= \frac{1}{2\pi}\int_0^{2\pi}(g_{11}(\rho,\theta))^2 d\theta, \\ (\mu_2(\rho))^2 &= \frac{1}{2\pi}\int_0^{2\pi}(g_{21}(\rho,\theta))^2 d\theta. \end{aligned}$$

Based on this assumptions for system (12) we have

$$\begin{cases} d\rho = [\frac{1}{2}\rho(\lambda_1+\lambda_2) + \frac{1}{8}(3a_{130} + a_{112} + a_{221} + 3a_{203})\rho^3 + \frac{\rho}{2(E-F)^2}((\sigma_1 E - \sigma_2 F)^2 \\ \qquad + \frac{1}{8}(E-F)^2(\sigma_1-\sigma_2)^2)]dt + \mu_1(\rho)dW_\rho(t), \\ d\theta = [\frac{\rho^2}{8}(-a_{121} - 3a_{103} + 3a_{230} + a_{212}) + \frac{1}{(E-F)^2}(\frac{1}{4}(\sigma_1-\sigma_2)(E-F)(-\sigma_1 E + \sigma_2 F)]dt \\ \qquad + \mu_2(\rho)dW_\theta(t), \end{cases} \tag{18}$$

where

$$\begin{aligned} (\mu_1(\rho))^2 &= \frac{\rho^2}{2(E-F)^2}[(\sigma_1 E - \sigma_2 F)^2 + \frac{1}{4}(\sigma_1-\sigma_2)^2(E-F)^2, \\ (\mu_2(\rho))^2 &= \frac{1}{2(E-F)^2}[\frac{1}{2}(-\sigma_1 E + \sigma_2 F)^2 + \frac{1}{4}(\sigma_1-\sigma_2)^2(E+F)^2. \end{aligned}$$

Let $\sigma_1 = \sigma_2 = \sigma$, then we have the following system

$$\begin{cases} d\rho = [\frac{1}{2}\rho(\lambda_1+\lambda_2) + \frac{1}{2}\sigma^2\rho + \frac{1}{8}(3a_{130} + a_{112} + a_{221} + 3a_{203})\rho^3]dt + \frac{1}{\sqrt{2}}\sigma\rho dW_\rho(t), \\ d\theta = \frac{\rho^2}{8}(-a_{121} - 3a_{103} + 3a_{230} + a_{212})dt + \frac{1}{2}\sigma dW_\theta(t). \end{cases} \tag{19}$$

These equations show that the averaging amplitude $\rho(t)$ is a one-dimensional Markov diffusing process, then the following amplitude equation.

$$d\rho = [\frac{1}{2}\rho(\lambda_1+\lambda_2+\sigma^2) + \frac{1}{8}(3a_{130} + a_{112} + a_{221} + 3a_{203})\rho^3]dt + \frac{1}{\sqrt{2}}\sigma\rho dW_\rho(t), \tag{20}$$

could be used to study the dynamical behavior of Eqs. (9).

Then we have the following theorem.

Theorem 2. *Suppose that λ_1, λ_2 are eigenvalues of the Jacobian matrix associated with system (4) at equilibrium point $E_i, i = 1,2$. When $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} < 0$, the stochastic system (9) is stable at the equilibrium point E_i and $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} > 0$ implies that the stochastic system (9) is unstable at the equilibrium point E_i.*

Proof. The one-dimensional Ito formula implies that the solution of Eq. (20) is

$$\rho(t) = \rho(0)\exp\left(\int_0^t [\frac{1}{2}(\lambda_1 + \lambda_2 + \sigma^2) - \frac{1}{4}\sigma^2]ds + \int_0^t \frac{\sigma}{\sqrt{2}} dW_\rho(t)\right).$$

Thus the associated largest Lyapunov exponent is

$$\lambda = \lim_{t \to +\infty} \frac{\ln \|\rho(t)\|}{t} = \frac{1}{2}(\lambda_1 + \lambda_2 + \sigma^2) - \frac{1}{4}\sigma^2 = \frac{1}{2}(\lambda_1 + \lambda_2 + \frac{\sigma^2}{2}).$$

According to Oseledet's multiplicative ergodic theorem [1], negativity of the largest Lyapunov exponent is a necessary and sufficient condition for asymptotic stability with probability one of the trivial solution of system (9). This implies that the stochastic system (9) at the equilibrium point $E_i,\ i = 1,2$ is locally asymptotically stable if and only if $\lambda < 0$. Consequently local asymptotic stability stochastic system (9) at the equilibrium point $E_i,\ i = 1,2$ is equivalent to $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} < 0$. □

In the following, we consider notations on singular boundary value problems to investigate the global stability of system (4) at equilibrium points. Consider the one-dimensional Ito stochastic differential equation

$$d\rho = m(\rho)dt + \mu(\rho)dW_\rho(t). \tag{21}$$

Let x_s is the boundaries of this system, where the subscript $s = l$ or r, denoting the left or right boundary. When the diffusion term $\mu(x_s) = 0$, the singular boundary x_s is said to be the first type and in this case, we have the following definitions:
(i) if $\mu^2(x) = \lim_{x \to x_s} O \mid x - x_s \mid^{\alpha_s}$, then $\alpha_s \geq 0$ is said to be diffusion exponent of x_s.
(ii) if $\mu(x) = \lim_{x \to x_s} O \mid x - x_s \mid^{\beta_s}$, then $\alpha_s \geq 0$ is said to be drift exponent of x_s.
(iii) Characteristic value C_s is given by

$$C_l = \lim_{x \to x_l^+} \frac{2m(x)(x - x_l)^{\alpha_l - \beta_r}}{\mu^2(x)}, \quad C_r = \lim_{x \to x_r^+} \frac{2m(x)(x - x_r)^{\alpha_r - \beta_r}}{\mu^2(x)}.$$

When the drift term $m(x)$ is unbounded and $|x_s| < \infty$, then the singular boundary x_s is said to be the second type. For second type, parameters $\alpha_s,\ \beta_s$ and C_s are defined similarly to the first type which powers are multiplied by minus one.

Theorem 3. *Suppose that λ_1 and λ_2 are eigenvalues of the Jacobian matrix associated with system (4) at equilibrium point $E_i, i = 1,2$. When $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} < 0$ and $3a_{130} + a_{112} + a_{221} + 3a_{203} < 0$, the stochastic system (9) is globally asymptotic stable at the equilibrium point E_i.*

Proof. Consider Eq. (20). Since $\rho \to 0^+$, the asymptotic expressions for the drift term $m(\rho)$ and the square of the diffusion term $\mu^2(\rho)$ are of the forms

$$m(\rho) = \frac{1}{2}\rho(\lambda_1 + \lambda_2 + \sigma^2) + O(\rho), \quad \mu^2(\rho) = \frac{1}{2}\sigma^2\rho^2.$$

So, the diffusion exponent $\alpha_l = 2$, drift exponent $\beta_l = 1$ and $C_l = \frac{2(\lambda_1+\lambda_2+\sigma^2)}{\sigma^2}$, which, the left boundary $\rho = 0$ of the averaged Ito belongs to the singular boundary of the first type.
Similarly, for the right boundary $\rho \to +\infty$ the asymptotic expressions for the drift term $m(\rho)$ and the square of the diffusion term $\mu^2(\rho)$ are of the forms

$$m(\rho) = \frac{1}{8}(3a_{130} + a_{112} + a_{221} + 3a_{203})\rho^3, \quad \mu^2(\rho) = \frac{1}{2}\sigma^2\rho^2,$$

which is a second type singular boundary such that the diffusion exponent $\alpha_r = 2$, drift exponent $\beta_r = 3$ and $C_r = -\frac{3a_{130}+a_{112}+a_{221}+3a_{203}}{2\sigma^2}$. Therefore $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} < 0$ and $3a_{130} + a_{112} + a_{221} + 3a_{203} < 0$ imply that $C_l < 1$ and $C_r > -1$, respectively. Then similar argument to proof of Theorem 3.2 of [12] shows that the stochastic system (9) is globally asymptotic stable at the equilibrium point E_i. □

5 Stochastic bifurcation

In this section, we consider conditions in the stochastic system that undergo pitchfork and phenomenological bifurcations.

Theorem 4. *Suppose that λ_1, λ_2 are eigenvalues of the Jacobian matrix associated with system (4) at equilibrium point $E_i, i = 1,2$. When $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} < 0$, the solution of stochastic system (9) possesses exactly one invariant measure that is stable. If $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} > 0$, the solution of stochastic system (9) possesses two unstable and one stable random Dirac measures. Then the stochastic system undergoes a stochastic pitchfork bifurcation at $\lambda_1 + \lambda_2 = -\frac{\sigma^2}{2}$.*

Proof. Suppose that $\mu_3 = 3a_{130} + a_{112} + a_{221} + 3a_{203}$ and $z_t = (-\frac{\mu_3}{8})^{\frac{1}{2}}\rho$. Then we can rewrite stochastic system (20) as

$$dz_t = [\frac{1}{2}(\lambda_1 + \lambda_2 + \sigma^2)z_t - z_t^3]dt + (\frac{\sigma^2}{2})^{\frac{1}{2}} z_t dW_\rho(t)$$

The Stratonovich form of this stochastic system is as follows:

$$dz_t = (\varphi z_t - z_t^3)dt + \eta z_t o dW_\rho(t), \tag{22}$$

where $\varphi = \frac{1}{2}(\lambda_1 + \lambda_2 + \sigma^2)$ and $\eta = (\frac{\sigma^2}{2})^{\frac{1}{2}}$. This is well known that system (22) generates a local random dynamical system of the form:

$$\psi_\varphi(t, \omega, z) = \frac{z\exp(\varphi t + \eta W_t(\omega))}{(1 + 2z^2\int_0^t \exp[2(\varphi s + \eta W_s(\omega))]ds)^{\frac{1}{2}}}.$$

Therefore condition of Section 4.1 in [13] established for system (20). Consequently system (20) undergoes pitchfork bifurcation at $\eta_D = 0$, i.e. $\lambda_1 + \lambda_2 = -\frac{\sigma^2}{2}$ and the proof is completed. □

Now, we would like to use a phenomenological approach to study the stochastic bifurcation of system (20). The stochastic P-bifurcation is a type of stochastic bifurcation that occurs in a stochastic system. This bifurcation describes the mode of the stationary probability density function or the invariant measure of the stochastic process. Let steady-state probability density function be denoted by $P_{st}(\rho)$. The extreme

value of $P_{st}(\rho)$ illustrates the stationary behavior of the Fokker-Planck equation arising from nonlinear stochastic systems. Stability at a point ρ^* in probability viewpoint means that the sample trajectory will stay for a longer time in the neighborhood of ρ^*. A sufficient condition for stability in the meaning of probability is that $P_{st}(\rho)$ has a maximum value at ρ^*. Minimum value of $P_{st}(\rho)$ at ρ^* gives an opposite property [12, 21]. Stochastic systems undergo stochastic P-bifurcation when the mode of the stationary probability density function changes in nature. It indicates the jump in the distribution of the random variable in the probability sense. To investigate the P-bifurcation of a stochastic system (9) and its polar coordinate transformation (20), we use probability density functions.
We obtain the corresponding FPK equation of amplitude Eq. (20) as follows:

$$\frac{\partial P(\rho)}{\partial t} = -\frac{\partial}{\partial \rho}\{[\frac{1}{2}(\rho(\lambda_1+\lambda_2+\sigma^2)+\frac{\mu_3}{8}\rho^3]P(\rho)\}+\frac{1}{2}\frac{\partial^2}{\partial \rho^2}(\frac{\sigma^2\rho^2}{2})P(\rho)],$$

where $\mu_3 = 3a_{130}+a_{112}+a_{221}+3a_{203}$. By calculating the solution of the degenerate system when $\frac{\partial P(\rho)}{\partial t} = 0$, the steady-state probability density function $P_{st}(\rho)$ can be obtained as follows:

$$P_{st}(\rho) = \begin{cases} \delta(\rho), & \lambda_1+\lambda_2+\frac{\sigma^2}{2} \le 0, \\ \dfrac{\rho^{\frac{2(\lambda_1+\lambda_2)}{\sigma^2}}\exp(\frac{\mu_3}{4\sigma^2}r^2)}{\Gamma(\frac{2(\lambda_1+\lambda_2)+\sigma^2}{2\sigma^2})(\frac{\mu_3}{4\sigma^2})^{\frac{2(\lambda_1+\lambda_2)+\sigma^2}{2\sigma^2}}}, & \lambda_1+\lambda_2+\frac{\sigma^2}{2} > 0, \end{cases} \tag{23}$$

To obtain the extreme value point of the probability density $P_{st}(\rho)$, we need to solve $\frac{dP_{st}(\rho)}{d\rho} = 0$, that is

$$(\frac{2(\lambda_1+\lambda_2)}{\sigma^2}+2\frac{\mu_3}{4\sigma^2}\rho^2)\frac{\rho^{\frac{2(\lambda_1+\lambda_2)-\sigma^2}{\sigma^2}}\exp(\frac{\mu_3}{4\sigma^2}r^2)}{\Gamma(\frac{2(\lambda_1+\lambda_2)+\sigma^2}{2\sigma^2})(\frac{\mu_3}{4\sigma^2})^{\frac{2(\lambda_1+\lambda_2)+\sigma^2}{2\sigma^2}}} = 0.$$

It is clear that $\rho = 0$ or $\rho^* = \sqrt{-\frac{4(\lambda_1+\lambda_2)}{\mu_3}}$. Therefore $P_{st}(\rho)$ has extreme point if $\frac{4(\lambda_1+\lambda_2)}{\mu_3} < 0$ and $\frac{\partial^2 P_{st}(\rho)}{\partial \rho^2} < 0$ holds at ρ^* for $\mu_3 < 0$. These conditions imply that $P_{st}(\rho)$ possesses a maximum value at the point $\rho = \rho^*$. Consequently, we have the following theorem to obtain phenomenological bifurcation.

Theorem 5. *Suppose that λ_1, λ_2 are eigenvalues of the Jacobian matrix associated with system (4) at equilibrium point $E_i, i = 1,2.$ and $\mu_3 = 3a_{130}+a_{112}+a_{221}+3a_{203}$. If $\mu_3 < 0$ and $\frac{(\lambda_1+\lambda_2)}{\mu_3} < 0$, then Eq. (20) undergoes phenomenological bifurcation at $\rho^* = \sqrt{-\frac{4(\lambda_1+\lambda_2)}{\mu_3}}$.*

6 Numerical simulation

In this section, we present some numerical examples including phase portrait and trajectories evolution in time to illustrate theoretical results on stochastic stability and stochastic bifurcation. We choose the value of parameters as $\gamma = 0.01$, $\beta = 0.01$ and $\alpha = 0.1$. The eigenvalues of system (4) are $\lambda_1 = -0.0386$ and $\lambda_2 = -0.518$, then Theorem 2 implies that, for every $\sigma > 0$ that $\lambda_1+\lambda_2+\frac{\sigma^2}{2} < 0$, the stochastic system (9) is stable at the equilibrium point q_1. Then for every $\sigma < \sqrt{1.13}$, the equilibrium point q_1 is

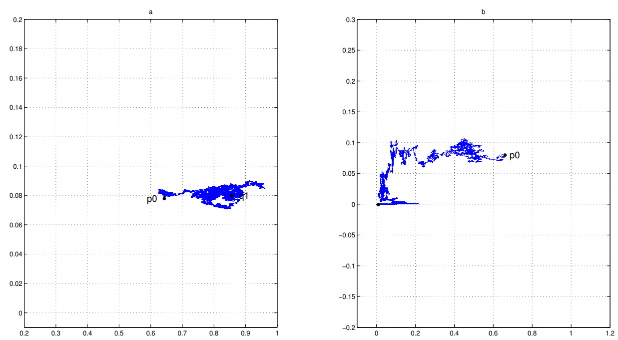

Figure 2: Phase portrait for system (10) for $\gamma = 0.1$, $\beta = 0.01$, $\alpha = 0.1$, initial conditions $(u_0, v_0) = (0.65, 0.08)$ and $\sigma = 0.045$ (a), $\sigma = 1.2$(b) for equilibrium point $p = q_1$.

asymptotically stable in probability one and for $\sigma > \sqrt{1.13}$ it is unstable. Figure 2 represents the phase portrait of the stochastic system for two different values of the noise σ. In case (a) $\sigma = 0.045 < \sqrt{1.13}$ and trajectory of $p = (0.65, 0.08)$ convergence to equilibrium point q_1. In case (b) $\sigma = 1.12 > \sqrt{1.13}$ and trajectory of $p = (0.65, 0.08)$ convergence to equilibrium point $(0, 0)$.

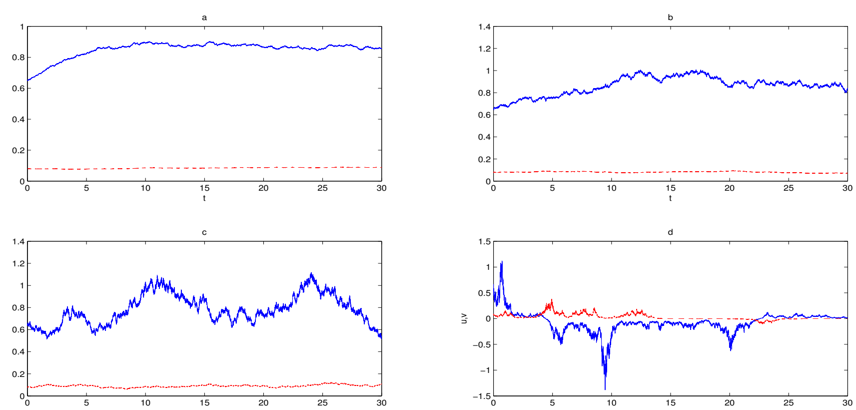

Figure 3: Time series of system (10) for $\gamma = 0.1$, $\beta = 0.01$, $\alpha = 0.1$, initial conditions $(u_0, v_0) = (0.65, 0.08)$ and $\sigma = 0.015$ (a), $\sigma = 0.045$ (b), $\sigma = 0.13$(c), $\sigma = 1.1$(d) for equilibrium point $p = q_1$.

In Figure 3, we investigate the effect of the noise in system (10) with the fixed parameters $\gamma = 0.01$, $\beta = 0.01$ and $\alpha = 0.1$ and initial condition $(u_0, v_0) = (0.65, 0.08)$. This figure shows that if the intensity of the noise is increased, then the equilibrium point q_1 tend to be unstable. In case (d) $\lambda_1 + \lambda_2 + \frac{\sigma^2}{2} > 0$ and time series after a fluctuation moves away from equilibrium point q_1 and convergence to origin which confirm that q_1 is unstable and the origin is stochastic asymptotically stable, as claimed in [10]. In Figure 4, we plot the time series evaluation of system (10) for 10 different initial condition (u_0, v_0). This figure confirms stochastic asymptotical stability conditions provided in Theorem

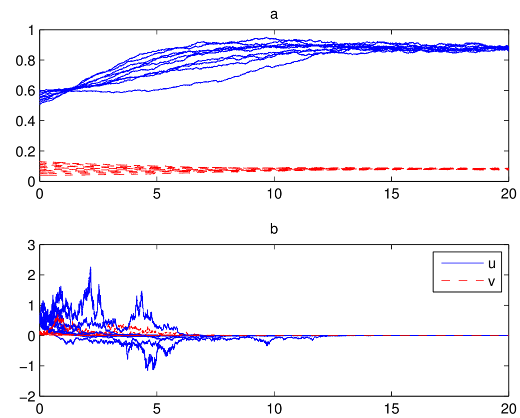

Figure 4: Time series of system (10) for $\gamma = 0.1,\ \beta = 0.01,\ \alpha = 0.1$, 10 different initial conditions (u_0, v_0) and $\sigma = 0.015$ (a), $\sigma = 1.12$ (b) for equilibrium point $p = q_1$.

(2) and the above simulations. Consider the parameters as $\gamma = 0.02,\ \beta = 0.002,\ \alpha = 0.3$, in this case

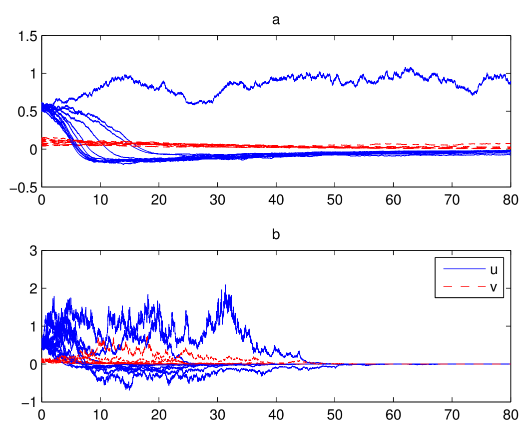

Figure 5: Time series of system (10) for $\gamma = 0.02,\ \beta = 0.002,\ \alpha = 0.3$, 12 different initial conditions (u_0, v_0) and $\sigma = 0.06$ (a), $\sigma = 0.45$ (b) for equilibrium point $p = q_2 = (0.5, 0.05)$.

$q_2 = (0.5, 0.05)$ is an unstable equilibrium point with eigenvalues $\lambda_1 = 0.1873$ and $\lambda_1 = 0.0427$. Then the stochastic system is unstable for every $\sigma > 0$. Consider the time series evaluation of system (10) for 12 different initial condition (u_0, v_0) in Figure 5. This figure confirms stochastic asymptotical stability conditions provided in Theorem 2. In case (a) $\sigma = 0.06$ and some trajectories converge to equilibrium point q_1 and the other convergence to origin as a stable equilibrium point. In case (b) $\sigma = 0.45$ and all trajectories converge to the origin as a stable equilibrium point. In any case, all trajectories move away from equilibrium point q_2.

7 Discussion and conclusion

There exist two main approaches to develop a stochastic model corresponding to a deterministic model to investigate the effect of environmental disturbance on its dynamic behavior. The first one is to add environmental disturbance factor directly to the deterministic dynamical model such that expressive the effect of a randomly fluctuating environment without affecting any particular parameter. One of the strengths of this approach is that the influence of any parameter in the deterministic model does not change separately. This paper was based on this approach which does not depend on specific data.
The second one is to replace the time-independent parameters involved with the deterministic model system by some random parameters or the values of the parameters are randomly selected. In [14] the authors used this approach to study the stochastic Brusselator system and proved that stochastic Hopf bifurcation could arise from the variation of intensity of the random parameter alone.
In general, neurons are excitable because they are near bifurcations from resting to spiking activity, so the type of the bifurcation determines the excitable properties of the neuron. Indeed different classes of excitability occur because neurons have different bifurcations of resting and spiking states [11]. Based on neurobiological experiments and numerical simulations, individual neurons have irregular bursts, while ensembles of such irregularly bursting neurons can synchronize and produce regular, rhythmical bursting. In this direction, Rulkov used the second approach and showed that synchronization among chaotically bursting cells can lead to the onset of regular bursting [17]. The proposed model combines two one-dimensional fast and slow subsystems and is known as the Rulkov map. Recursive nonlinear and mean-reverting properties of Rulkov maps caused this model to be highly suitable for the modeling of financial time series, such as the occurrence of data clusters, mutual synchronization, chaos and regularization of bursts of activity across the markets [16, 18].

First of all, the Rulkov map and its results confirm the need to consider neuron dynamics as a stochastic process. Also, Izhikevich explains the noise-induced bursting in a two-dimensional system (4) coexistence of an equilibrium and a limit cycle attractor [11]. Rulkov in [17] concluded that this mechanism of chaos regularization is due to the dynamical features of each cell at the beginning and at the end of the chaotic burst which are similar to saddle-node bifurcation and the appearance of a homoclinic orbit. All of these results are obtained based on similarity of figures and numerical simulations. But, in the present paper, we consider the first approach mentioned above to determine the necessary and sufficient conditions on noise and deterministic parameters for stability and bifurcations of the FitzHugh-Nagumo model as a stochastic model. The sign of largest Lyapunov exponent is a simple and efficient tool to study stochastic stability and stochastic bifurcation. Theorem 2 determines the largest Lyapunov exponent depending on the eigenvalues and noise of the stochastic systems. We provided conditions on eigenvalues of the Jacobian matrix at non-origin equilibrium points and additive noise that our stochastic system undergoes pitchfork and phenomenological bifurcations. Phenomenological and pitchfork bifurcations describe spiking limit cycles and stability of neuron's resting state of neuron dynamics, respectively, as a stochastic process. Researching bursting in the stochastic FitzHugh-Nagumo model with the first approach mentioned above can be an attractive topic for further studies.

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