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An algorithm for multi-objective fuzzy linear programming problem with interval type-2 fuzzy numbers and ambiguity in parameters

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Abstract. Considering multiple criteria and objectives simultaneously in a single real-world problem and the fuzzy nature of this type of problem is of particular importance and application. As a result, multi-objective interval type-2 fuzzy linear programming problems have received much attention. However, there are few and a limited number of methods available for solving multi-objective interval type-2 triangular fuzzy linear programming problems with ambiguity-type imprecision (interval type-2 triangular fuzzy numbers) in almost all of the problem parameters. This research first considers a multi-objective interval type-2 fuzzy linear programming problem with ambiguity in all coefficients, in which, all problem coefficients are interval triangular fuzzy numbers. In addition, using the weighted sum method and the concept of nearest interval approximation, the problem is solved, and an example is provided.

Keywords: Interval type-2 fuzzy set, interval type-2 triangular fuzzy numbers, ambiguity in parameters, multi-objective linear programming problem.
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1 Introduction

Multi-objective problems are a group of multi-criteria decision-making problems whose solution space is continuous with an unlimited number of solutions. In fact, we are facing with multi-objective optimization problems (linear or non-linear) with more than one objective function. There are three approaches to solve multi-objective problems: weighting before solving (prior approaches), weighting after solving (posterior approaches), and interactive approaches. In the prior approach, the decision-maker's preferences are obtained before starting to solve the problem in the form of information such as the degree of importance (weights). There are several approaches in this category, including the weighted sum (WS)

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method, we refer to [27] for more details. Some recent studies in this field can be summarized as follows: Dong et al. proposed a two-stage method for a fuzzy multi-objective linear program and applied it to engineering project portfolio selection [6]; Nahar et al. proposed the statistical method to solve the multi-objective linear programming (MOLP) problem [22]; Hassanpour et al. proposed a method for solving an intuitionistic fuzzy MOLP problem [9]; Arya et al. proposed an approach for solving fully fuzzy multi-objective linear fractional optimization problems [3]. The readers can see the details and methods of solving this type of problems by referring to [5, 17, 24, 28].

Fuzzy sets (FSs) were introduced by Zadeh in 1965, [30]. Afterward, in 1975, Zadeh proposed type-2 fuzzy sets (T2FSs) as a generalized version of FSs [31–33]. One of the special cases of T2FSs is interval type-2 fuzzy sets (IT2FSs), which are less complicated and easier to understand than T2FSs [25]. Since IT2FSs include more uncertainties and information than type-1 fuzzy sets (T1FSs), they are more suitable and efficient for modeling optimization problems.

To solve the interval type-2 fuzzy linear programming (IT2FLP) problem with ambiguity in the objective function vector (OFV), resources vector (RsV), and technological coefficients (TCs), which are generally interval type-2 fuzzy numbers (IT2FNs), different methods have been introduced, such as ranking [10, 11, 13], sorting [16], and approximation methods [14]. Each method has its own advantages and drawbacks. Therefore, it is extremely complex to determine the best method. Hence, when dealing with any problem, the decision-maker must adopt an appropriate method in accordance with the existing conditions. In fact, choosing methods with more well-defined properties from the available methods is important. In the ranking methods, each type-1 fuzzy number (T1FN) is approximated by an IT2FN with a definite number. During this process, important information may be lost. Although the same concerns and doubts are evident in most defuzzification methods, using a well-defined approximate closed interval for a FS is more acceptable and logical than replacing it with a definite number. Using the concept of the nearest interval approximation (NIA) for each T1FN [14, 19] is one of the most appropriate and efficient methods used to solve IT2FLP problems with ambiguity in all coefficients. So far, rarely studies have been conducted on multi-objective interval type-2 triangular fuzzy linear programming (MOIT2TrFLP) problems with ambiguity in parameters. Therefore, in this study, we propose a new method for solving such problems. In this method, at first, by using the WS method, we convert the multi-objective problem into a single-objective problem. Then we use the concept of NIA to solve this problem.

The research sections are described below: In the second section, the basic definitions and necessary concepts, such as: the fundamental definitions of type-1 and type-2 FSs, as well as interval type-2 triangular fuzzy numbers (IT2TrFNs), the concept of NIA, the interval linear programming (ILP) problem, the MOLP problem are reviewed, and the WS method is presented as one of the best practical method to solve MOLP problems. The third section proposed a new method to solve the problem of multi-objective fuzzy linear programming (MOFLP) with IT2TrFNs by using the combination of two approaches of the WS method and concept of NIA. In the fourth section, we provide a numerical examples to demonstrate the performance of the proposed method. Finally, the conclusion section includes the obtained results and generalities of the proposed method.

2 Preliminaries

In this section, triangular fuzzy numbers (TrFNs) and their important characteristics and properties are introduced. Then, basic definitions and theorems related to IT2FSs, interval programming (IP), the MOLP problem, and a method used to solve these types of problems are introduced. Finally, the definitions and theorems related to the concept of NIA are presented. To see more definitions and proofs of theorems, we refer to [14, Page 3].

2.1 The T1FS

Definition 1 ([10, 11, 13])**.** *Let $\tilde{A}$ be a T1FS defined on the universal set X. The FS $\tilde{A}$ is made up of the ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$, in which $\mu_{\tilde{A}}(x)$ represents the membership degree of ranges between zero and one, in which zero indicates non-membership, and 1 indicates definite membership in $\tilde{A}$.*

Definition 2 ([10, 11, 13])**.** *The FS $\tilde{A}$, defined on the set of real numbers, is called a fuzzy number if:*

(i) *The FS $\tilde{A}$ is normal, that is, $supp(\tilde{A}) = 1$;*

(ii) *$\tilde{A}$ is convex, that is, for any arbitrary x_1 and x_2 belonging to the universal set X and for any $\lambda \in [0,1]$, $\mu_{\tilde{A}}(\lambda x_1 + \lambda x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$ holds;*

(iii) *$\mu_{\tilde{A}}(x)$ is a continuous function in all its intervals.*

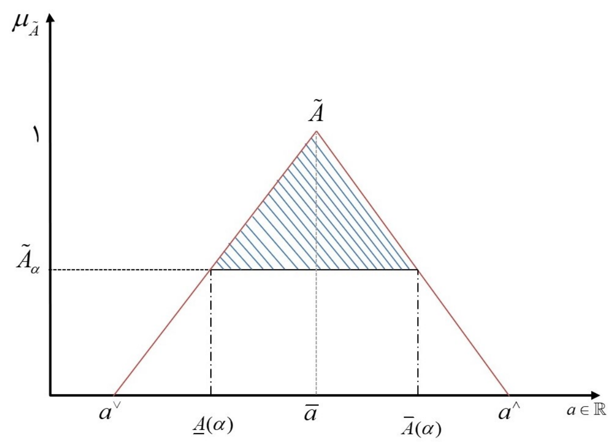

Figure 1: The T1FN and its α-cut.

Definition 3 ([8])**.** *Consider $\tilde{A}$ as T1FN (see Figure 1). The core of $\tilde{A}$ is a set described as*

$$Core(\tilde{A}) = \{x \in |\mu_{\tilde{A}}(x) = 1\}.$$

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2.2 The T2FS

Definition 4 ([20, 21, 34])**.** *The $\tilde{\tilde{A}}$ is the T2FS and is defined as:*

$$\tilde{\tilde{A}} = \int_{x\in X} \int_{u\in J_x} f_x(u)/(x,u) = \int_{x\in X} \left[\int_{u\in J_x} f_x(u)/u \right] \Big/ x,$$

where $x \in X$, $J_x \subseteq [0,1]$, $u \in [0,1]$ and $f_x(u) \in [0,1]$. The J_x and $f_x(u)$ are primary and secondary memberships, respectively. Furthermore, $\tilde{\tilde{A}}$ is called IT2FS, if $x \in X$ and $u \in J_x \subseteq [0,1]$; $\mu_{\tilde{\tilde{A}}}(x,u) = 1$.

Definition 5 ([14])**.** *The IT2TrFN, $\tilde{\tilde{A}}$, which is defined on interval $[\overline{a}^{\vee}, \overline{a}^{\wedge}]$, is an IT2TrFN, this is noted as:*

$$\tilde{\tilde{A}} = (\underline{A}, \overline{A}) = ((\underline{a}^{\vee}, \underline{a}, \underline{a}^{\wedge}; \underline{h}), (\overline{a}^{\vee}, \overline{a}, \overline{a}^{\wedge}; \overline{h})),$$

where $\overline{a}^{\vee} \leq \underline{a}^{\vee} \leq \underline{a} \leq \underline{a}^{\wedge} \leq \overline{a}^{\wedge}$ and $\overline{a}^{\vee} \leq \overline{a} \leq \overline{a}^{\wedge}$. Its MFs, respectively are

$$\overline{\mu}_{\tilde{\tilde{A}}}(x) = \begin{cases} \overline{h}\frac{x-\overline{a}^{\vee}}{\overline{a}-\overline{a}^{\vee}}, & \overline{a}^{\vee} \leq x \leq \overline{a}, \\ \overline{h}\frac{\overline{a}^{\wedge}-x}{\overline{a}^{\wedge}-\overline{a}}, & \overline{a} \leq x \leq \overline{a}^{\wedge}, \\ 0, & x \leq \overline{a}^{\vee}, x \geq \overline{a}^{\wedge}, \end{cases}$$

and

$$\underline{\mu}_{\tilde{\tilde{A}}}(x) = \begin{cases} \underline{h}\frac{x-\underline{a}^{\vee}}{\underline{a}-\underline{a}^{\vee}} & \underline{a}^{\vee} \leq x \leq \underline{a}, \\ \underline{h}\frac{\underline{a}^{\wedge}-x}{\underline{a}^{\wedge}-\underline{a}} & \underline{a} \leq x \leq \underline{a}^{\wedge}, \\ 0 & x \leq \underline{a}^{\vee}, x \geq \underline{a}^{\wedge}. \end{cases}$$

Figure 2 shows a complete view of an IT2TrFN of $\tilde{\tilde{A}}$ with $a \in \mathbb{R}$, its FOU, $\tilde{\tilde{A}}$-kernel, and A_e. Also, we have $\underline{a} = \bar{a}$.

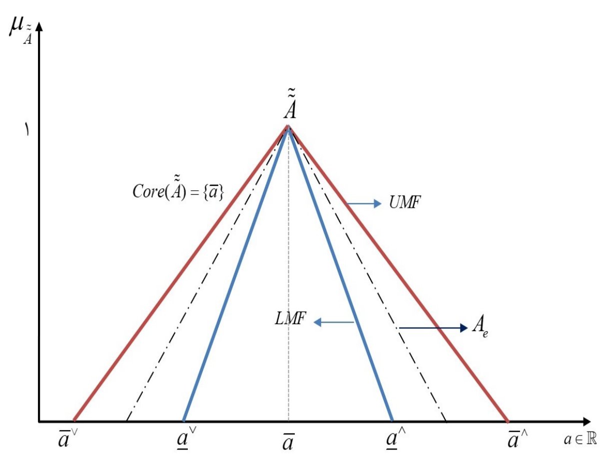

Figure 2: IT2TrFN, $\tilde{\tilde{A}}$.

2.3 The NIA

In the following, we express the definitions and theorems related to the concept of NIA.

Definition 6 ([8, 12])**.** *Assume $\tilde{A}$ is a T1FN and $[\underline{A}(\alpha), \overline{A}(\alpha)]$ is its α-cut. The closed interval $C(\tilde{A}) = [\underline{C}, \overline{C}]$ is the NIA to A, where $\underline{C} = \int_0^1 \underline{A}(\alpha) d\alpha$, $\overline{C} = \int_0^1 \overline{A}(\alpha) d\alpha$.*

Theorem 1 ([12, 14]). *The NIA for the T1FN $\tilde{A} = (a^{\vee}, \overline{a}, a^{\wedge})$ is as follows:*

$$C(\tilde{A}) = [\frac{a^{\vee} + \overline{a}}{2}, \frac{a^{\wedge} + \overline{a}}{2}].$$

2.4 The ILP

Definition 7 ([1]). *An interval number appears as $[\underline{x}, \overline{x}]$ in which the $\underline{x} \leq \overline{x}$ condition is satisfied. If $\underline{x} = \overline{x}$, x will be destroyed.*

The basic form of the ILP problem is defined as follows [18, 35]:

$$\begin{array}{lll} \max & z = \sum_{j=1}^{n} [\underline{c}_j, \overline{c}_j] x_j, & \\ s.t. & \sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \leq [\underline{b}_i, \overline{b}_i], & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n, \end{array} \tag{1}$$

where $\underline{c}_j$, $\overline{c}_j$, $\underline{a}_{ij}$, $\overline{a}_{ij}$, $\underline{b}_i$ and $\overline{b}_i$ are real numbers. The characteristic model of problem (1) is as follows:

$$\begin{array}{lll} \max & z = \sum_{j=1}^{n} c_j x_j, & \\ s.t. & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n, \end{array}$$

where $c_j \in [\underline{c}_j, \overline{c}_j]$, $a_i j \in [\overline{a}_{ij}, \overline{a}_{ij}]$ and $b_i \in [\underline{b}_i, \overline{b}_i]$. Several methods have been proposed to solve the IP problem [2, 4, 7, 29, 36]. One of the basic methods to solve this type of problem is the best worst cases (BWC) method [26]. This method divides model (1) into two best and worst sub-models.

Theorem 2 ([18]). *Consider $\sum_{j=1}^{n} [\underline{a}_{ij}, \overline{a}_{ij}] x_j \leq [\underline{b}_i, \overline{b}_i]$, for $i = 1, 2, ..., m$. The largest feasible region is $\sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i$ and the smallest feasible region is $\sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \underline{b}_i$.*

The following theorem obtains the worst and best values of the objective function.

Theorem 3 ([15, 18]). *The best value of the objective function of the ILP problem (1) is acquired by*

$$\begin{array}{lll} \max & \underline{z} = \sum_{j=1}^{n} \underline{c}_j x_j, & \\ s.t. & \sum_{j=1}^{n} \overline{a}_{ij} x_j \leq \underline{b}_i, & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n, \end{array}$$

and the worst values of the objective function of the ILP problem (1) is acquired by

$$\begin{array}{lll} \max & \overline{z} = \sum_{j=1}^{n} \overline{c}_j x_j, & \\ s.t. & \sum_{j=1}^{n} \underline{a}_{ij} x_j \leq \overline{b}_i, & i = 1, 2, ..., m, \\ & x_j \geq 0, & j = 1, 2, ..., n, \end{array}$$

The following details are the important definitions of the MOLP problem and its pareto optimal solution (POS).

2.5 The MOLP

An optimization problem with multiple linear objective functions that is simultaneously subject to linear constraints is known as the MOLP problem [23]. The basic form of the MOLP problem is as follows:

$$\begin{array}{ll} \max & \{c_1x = z_1(x)\}, \\ \max & \{c_2x = z_2(x)\}, \\ \ \vdots & \\ \max & \{c_kx = z_k(x)\}, \\ s.t. & Ax \leq b, \\ & x \geq 0, \end{array} \tag{2}$$

where k represents the number of OFVs, A is a matrix of $m \times n$, $b \in \mathbb{R}^m$ for $l = 1,...,k$, $c_l = (c_{l1},...,c_{ln})$, and $x = (x_1,...,x_n)^T$.

Definition 8 ([23])**.** *The x^* is called a POS if and only if there is no other solution like $x \in X$ (X is the reference set) such that for every i, $z_i(x) < z_i(x^*)$ and for at least one j, $z_j(x) \neq z_j(x^*)$.*

Definition 9 ([23])**.** *The x^* be a feasible solution for the problem (2):*

It is efficient, if there is no $x \in X$ such that $Cx \succ Cx^*$.

It is poorly efficient, if there is no $x \in X$ such that $Cx \succ Cx^*$.

It is perfectly optimal, if for $x \in X$, and $i = 1,...,p$, $c_ix^* \geq c_ix$.

There are three approaches to solve multi-objective problems: weighting before solving (prior approaches), weighting after solving (posterior approaches), and interactive approaches. In the prior approach, the decision-maker's preferences are obtained before starting to solve the problem in the form of information such as the degree of importance (weights). There are several approaches in this category, including the WS method. It is used to describe POSs in the MOLP problem, which is explained in detail below. For more details on other methods, refer to [23, 27].

2.6 The WS method

In this method, a weight ($\lambda_l, l = 1,2,...,k$) is assigned to each OFV of the problem (2). Then, the WS method considers all OFs of the MOLP problem. As a result, the MOLP problem (2) turns into a weighted single objective LP problem (3):

$$\begin{array}{ll} \max & \sum_{l=1}^{k} \lambda_l z_l(x), \\ s.t. & Ax \leq b, \\ & x \geq 0, \end{array} \tag{3}$$

where $\lambda_l, l = 1,...,k$, are scalars, and for $\lambda_l > 0$, $\sum_{l=1}^{k} \lambda_l = 1$.

Note: The coefficients λ_l should be chosen such that none of the objectives are excluded.

Theorem 4 ([23]). *If $\bar{x} \in S$ is the optimal solution (OS) of the WS method of LP problem* (3) *(for each $\overline{\lambda} \in \Lambda$), then, $\bar{x}$ is a POS of the MOLP problem in* (2).

According to Theorem 4, it is deduced that for any strictly positive weight vector, the WS linear programming problem has optimal solution.

Theorem 5 ([23]). *If $\bar{x} \in S$ is a POS of MOLP problem* (2)*, then for $\overline{\lambda} \in \Lambda$, $\bar{x}$ is the OS of the WS method of LP problem* (3).

In addition, Theorem 5 states that each OS is associated with at least one strictly positive weight vector, which means that there is an OS in a WS programming problem.

3 A novel approach for solving the MOFLP problems with IT2TrFNs

In this section, a new approach is proposed to solve MOIT2TrFLP problems with ambiguity in coefficients by combining the WS and NIA methods. Consider a MOFLP problem, all parameters of which are IT2TrFNs. Using the WS method, this problem is transformed into a single-objective IT2FLP problem (problem (5)). By replacing the core of these IT2TrFNs, a LP problem called the middle problem is introduced. Then, the problem (5) is rewritten as a characteristic type-1 FLP problem, all parameters of which are type-1 TrFNs. Therefore, using the concept of the NIA for each of these characteristic type-1 TrFNs, an approximate closed interval is obtained, the problem (5) is modeled as an IP problem, and finally, it is solved by applying the BWC method twice. The details of the newly proposed method are as follows:

First step: (Using the WS method) The general form of the MOIT2TrFLP problem is expressed as follows:

$$\begin{array}{lll} \max & \{\tilde{\tilde{c}}_1 x = z_1\}, & \\ \max & \{\tilde{\tilde{c}}_2 x = z_2\}, & \\ \vdots & & \\ \max & \{\tilde{\tilde{c}}_k x = z_k\}, & \\ s.t. & \sum\limits_{j=1} \tilde{\tilde{a}}_{ij} x_j \leq \tilde{\tilde{b}}_i, & i = 1, \dots, m, \\ & x_j \geq 0, & j = 1, \dots, n, \end{array} \tag{4}$$

where $\tilde{\tilde{c}}_j$, $\tilde{\tilde{a}}_{ij}$, and $\tilde{\tilde{b}}_i$ (for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $k = 1, 2, \dots l$) are IT2TrFNs with ambiguity-type imprecision. Using the WS method, the problem (4) becomes a single-objective nonlinear programming problem as follows:

$$\begin{array}{lll} \max & \tilde{\tilde{z}} = \sum\limits_{j=1}^{n} \sum\limits_{l=1}^{k} \lambda_l \tilde{\tilde{c}}_{lj} x_j, & \\ s.t. & \sum\limits_{j=1} \tilde{\tilde{a}}_{ij} x_j \leq \tilde{\tilde{b}}_i, & i = 1, \dots, m, \\ & x_j \geq 0, & j = 1, \dots, n. \end{array} \tag{5}$$

The problem (5) is a single-objective nonlinear programming problem. At the end of this step, the decision-maker can consider arbitrary values for each weight ($\lambda_l, l = 1, 2, \dots, k$, that $\lambda_1 + \lambda_2 + \dots + \lambda_l = 1$). As a result, nonlinear programming problem (5) becomes the LP problem, and the second step can be performed on the basis of it.

Second step: (Using the concept of NIA) Now, given that the cores of IT2TrFNs are singleton sets, the middle problem is introduced as follows:

$$\begin{aligned} \max \ \ & \bar{z} = \sum_{j=1}^{n} \sum_{l=1}^{k} \lambda_l \bar{c}_{lj} x_j, & \\ s.t. \ \ & \sum_{j=1} \bar{a}_{ij} x_j \le \bar{b}_i, \qquad & i = 1, ..., m, \\ & x_j \ge 0, & j = 1, ..., n, \end{aligned} \tag{6}$$

where (for $i = 1, ..., m$ and $j = 1, ..., n$), $\bar{c}_{lj}$, $\bar{a}_{ij}$ and $\bar{b}_i$ are the middle numbers in a IT2TrFNs. The problem (6) is the middle problem. By solving it, we give the middle OSs and optimal value, namely $\bar{x}_{jm}$ and $\bar{z}_m$, respectively. By representing IT2TrFNs as an infinite union of characteristic type-1 TrFNs, the problem (5) can be expressed as follows:

$$\begin{aligned} \max \ \ & z^e = \sum_{j=1}^{n} \sum_{l=1}^{k} \lambda_l c_{lj}^e x_j, & \\ s.t. \ \ & \sum_{j=1} a_{ij}^e x_j \le b_i^e, \qquad & i = 1, ..., m, \\ & x_j \ge 0, & j = 1, ..., n, \end{aligned} \tag{7}$$

where $c_{lj}^e \in FOU(\tilde{c}_{lj})$, $a_{ij}^e \in FOU(\tilde{a}_{ij})$ and $b_i^e \in FOU(\tilde{b}_i)$ are all characteristic type-1 TrFNs. Then, using the concept of the NIA for characteristic type-1 TrFNs, the IP problem equivalent to the problem (7) will be as follows:

$$\begin{aligned} \max \ \ & z^e = \sum_{j=1}^{n} \sum_{l=1}^{k} \lambda_l [c_{lj}^{\vee e}, c_{lj}^{\wedge e}] x_j, & \\ s.t. \ \ & \sum_{j=1} [a_{ij}^{\vee e}, a_{ij}^{\wedge e}] x_j \le [b_i^{\vee e}, b_i^{\wedge e}], \qquad & i = 1, ..., m, \\ & x_j \ge 0, & j = 1, ..., n, \end{aligned} \tag{8}$$

where $c_{lj}^{\vee e} \in [\underline{c}_{lj}^{\vee e}, \bar{c}_{lj}^{\vee e}]$, $c_{lj}^{\wedge e} \in [\underline{c}_{lj}^{\wedge e}, \bar{c}_{lj}^{\wedge e}]$, $a_{ij}^{\vee e} \in [\underline{a}_{ij}^{\vee e}, \bar{a}_{ij}^{\vee e}]$, $a_{ij}^{\wedge e} \in [\underline{a}_{ij}^{\wedge e}, \bar{a}_{ij}^{\wedge e}]$, $b_i^{\vee e} \in [\underline{b}_i^{\vee e}, \bar{b}_i^{\vee e}]$ and $b_i^{\wedge e} \in [\underline{b}_i^{\wedge e}, \bar{b}_i^{\wedge e}]$. The problem (8) is the IP problem. Using the BWC method, the IP problem (8) is transformed into two best (9) and worst (10) sub-models, which are presented as follows, respectively:

$$\begin{aligned} \max \ \ & z_b^{\wedge e} = \sum_{j=1}^{n} \sum_{l=1}^{k} \lambda_l c_{lj}^{\wedge e} x_j, & \\ s.t. \ \ & \sum_{j=1} a_{ij}^{\vee e} x_j \le b_i^{\wedge e}, \qquad & i = 1, ..., m, \\ & x_j \ge 0, & j = 1, ..., n, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \max \ \ & z_w^{\vee e} = \sum_{j=1}^{n} \sum_{l=1}^{k} \lambda_l c_{lj}^{\vee e} x_j, & \\ s.t. \ \ & \sum_{j=1} a_{ij}^{\wedge e} x_j \le b_i^{\vee e}, \qquad & i = 1, ..., m, \\ & x_j \ge 0, & j = 1, ..., n. \end{aligned} \tag{10}$$

As stated above, both sub-models (9) and (10) are IP problems. Therefore, the BWC method is used again, and two other sub-models are obtained for each of the two sub-models (9) and (10). In fact, we obtain the four sub-models (best-best (B-B), best-worst (B-W), worst-best (W-B), and worst-worst (W-W)), which are:

1. B-B model:

$$\begin{aligned} \max\ & \overline{z}^{\wedge}_{bb} = \sum_{j=1}^{n}\sum_{l=1}^{k} \lambda_l \overline{c}^{\wedge}_{lj} x_j, & \\ s.t.\ & \sum_{j=1} \underline{a}^{\vee}_{ij} x_j \leq \overline{b}^{\wedge}_i, & i = 1,\ldots,m, \\ & x_j \geq 0, & j = 1,\ldots,n. \end{aligned} \tag{11}$$

   The optimal values of the problem are obtained as (11) $\overline{x}^{\wedge}_{jbb}$ and $\overline{z}^{\wedge}_{bb}$.

2. B-W model:

$$\begin{aligned} \max\ & \underline{z}^{\wedge}_{bw} = \sum_{j=1}^{n}\sum_{l=1}^{k} \lambda_l \underline{c}^{\wedge}_{lj} x_j, & \\ s.t.\ & \sum_{j=1} \overline{a}^{\vee}_{ij} x_j \leq \underline{b}^{\wedge}_i, & i = 1,\ldots,m, \\ & x_j \geq 0, & j = 1,\ldots,n. \end{aligned} \tag{12}$$

   By solving problem (12), the optimal values are obtained as $\underline{x}^{\wedge}_{jbw}$ and $\underline{z}^{\wedge}_{bw}$.

3. W-B model:

$$\begin{aligned} \max\ & \overline{z}^{\vee}_{wb} = \sum_{j=1}^{n}\sum_{l=1}^{k} \lambda_l \overline{c}^{\vee}_{lj} x_j, & \\ s.t.\ & \sum_{j=1} \underline{a}^{\wedge}_{ij} x_j \leq \overline{b}^{\vee}_i, & i = 1,\ldots,m, \\ & x_j \geq 0, & j = 1,\ldots,n. \end{aligned} \tag{13}$$

   The optimal values of the problem (13) are given as $\overline{x}^{\vee}_{jwb}$ and $\overline{z}^{\vee}_{wb}$.

4. W-W model:

$$\begin{aligned} \max\ & \underline{z}^{\vee}_{ww} = \sum_{j=1}^{n}\sum_{l=1}^{k} \lambda_l \underline{c}^{\vee}_{lj} x_j, & \\ s.t.\ & \sum_{j=1} \overline{a}^{\wedge}_{ij} x_j \leq \underline{b}^{\vee}_i, & i = 1,\ldots,m, \\ & x_j \geq 0, & j = 1,\ldots,n. \end{aligned} \tag{14}$$

   By solving problem (11), the optimal values $\underline{x}^{\vee}_{jww}$ and $\underline{z}^{\vee}_{ww}$ are obtained.

As mentioned, considering the values of the weights (λ_l), all the above problems become linear. Additionally, using the OSs obtained from the middle problem (6) and sub-models (11), (12), (13), and (14), the OSs and optimal value of the problem (5) are obtained in the form of IT2TrFNs as follows:

$$\begin{aligned} \tilde{\tilde{x}}^{*}_{opt} &= \left((\underline{x}^{\vee}_{jww}, \overline{x}_{jm}, \underline{x}^{\wedge}_{jbw}), (\overline{x}^{\vee}_{jwb}, \overline{x}_{jm}, \overline{x}^{\wedge}_{jbb}) \right), \\ \tilde{\tilde{z}}^{*}_{opt} &= \left((\underline{z}^{\vee}_{ww}, \overline{z}_{m}, \underline{z}^{\wedge}_{bw}), (\overline{z}^{\vee}_{wb}, \overline{z}_{m}, \overline{z}^{\wedge}_{bb}) \right). \end{aligned}$$

3.1 The proposed method for a bi-objective IT2FLP problem

To better understand the audience about the process of proposed method, we consider the bi-objective IT2FLP problem with ambiguity in parameters:

$$\begin{array}{ll} \max & \{\tilde{\tilde{c}}_1 x = z_1\}, \\ \max & \{\tilde{\tilde{c}}_2 x = z_2\}, \\ s.t. & \tilde{\tilde{a}}_{11}x_1 + \tilde{\tilde{a}}_{12}x_2 \leq \tilde{\tilde{b}}_1, \\ & \tilde{\tilde{a}}_{21}x_1 + \tilde{\tilde{a}}_{22}x_2 \leq \tilde{\tilde{b}}_2, \\ & x_1, x_2 \geq 0. \end{array} \tag{15}$$

In problem (15), we know $\tilde{\tilde{c}}_1 = (\tilde{\tilde{c}}_{11}, \tilde{\tilde{c}}_{12})$ and $\tilde{\tilde{c}}_2 = (\tilde{\tilde{c}}_{21}, \tilde{\tilde{c}}_{22})$. For $\tilde{\tilde{c}}_1$ and $\tilde{\tilde{c}}_2$, we consider a weight as λ_1 and λ_2 , respectively. As a result, for the objective function row we have

$$\begin{aligned} \lambda_1 \tilde{\tilde{c}}_1 x^T + \lambda_2 \tilde{\tilde{c}}_2 x^T &= \lambda_1 (\tilde{\tilde{c}}_{11}, \tilde{\tilde{c}}_{12}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \lambda_2 (\tilde{\tilde{c}}_{21}, \tilde{\tilde{c}}_{22}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \lambda_1 \tilde{\tilde{c}}_{11} x_1 + \lambda_1 \tilde{\tilde{c}}_{12} x_2 + \lambda_2 \tilde{\tilde{c}}_{21} x_1 + \lambda_2 \tilde{\tilde{c}}_{22} x_2 \\ &= (\lambda_1 \tilde{\tilde{c}}_{11} + \lambda_2 \tilde{\tilde{c}}_{21}) x_1 + (\lambda_1 \tilde{\tilde{c}}_{12} + \lambda_2 \tilde{\tilde{c}}_{22}) x_2. \end{aligned} \tag{16}$$

Therefore, by setting the (16) in the problem (15), we get

$$\begin{array}{ll} \max & (\lambda_1 \tilde{\tilde{c}}_{11} + \lambda_2 \tilde{\tilde{c}}_{21}) x_1 + (\lambda_1 \tilde{\tilde{c}}_{12} + \lambda_2 \tilde{\tilde{c}}_{22}) x_2, \\ s.t. & \tilde{\tilde{a}}_{11}x_1 + \tilde{\tilde{a}}_{12}x_2 \leq \tilde{\tilde{b}}_1, \\ & \tilde{\tilde{a}}_{21}x_1 + \tilde{\tilde{a}}_{22}x_2 \leq \tilde{\tilde{b}}_2, \\ & x_1, x_2 \geq 0. \end{array} \tag{17}$$

The problem (17) is the single-objective nonlinear programming problem in which all parameters are IT2TrFNs. All IT2TrFNs in problem (17) are treated as characteristic T1FNs. Using NIA, the problem (17) is transformed into the following IP problem:

$$\begin{array}{ll} \max & z^e = (\lambda_1 \left[\frac{c_{11}^{\vee} + \bar{c}_{11}}{2}, \frac{c_{11}^{\wedge} + \bar{c}_{11}}{2} \right] + \lambda_2 \left[\frac{c_{21}^{\vee} + \bar{c}_{21}}{2}, \frac{c_{21}^{\wedge} + \bar{c}_{21}}{2} \right]) x_1 \\ & \quad + (\lambda_1 \left[\frac{c_{12}^{\vee} + \bar{c}_{12}}{2}, \frac{c_{12}^{\wedge} + \bar{c}_{12}}{2} \right] + \lambda_2 \left[\frac{c_{22}^{\vee} + \bar{c}_{22}}{2}, \frac{c_{22}^{\wedge} + \bar{c}_{22}}{2} \right]) x_2, \\ s.t. & \left[\frac{a_{11}^{\vee} + \bar{a}_{11}}{2}, \frac{a_{11}^{\wedge} + \bar{a}_{11}}{2} \right] x_1 + \left[\frac{a_{12}^{\vee} + \bar{a}_{12}}{2}, \frac{a_{12}^{\wedge} + \bar{a}_{12}}{2} \right] x_2 \leq \left[\frac{b_1^{\vee} + \bar{b}_1}{2}, \frac{b_1^{\wedge} + \bar{b}_1}{2} \right], \\ & \left[\frac{a_{21}^{\vee} + \bar{a}_{21}}{2}, \frac{a_{21}^{\wedge} + \bar{a}_{21}}{2} \right] x_1 + \left[\frac{a_{22}^{\vee} + \bar{a}_{22}}{2}, \frac{a_{22}^{\wedge} + \bar{a}_{22}}{2} \right] x_2 \leq \left[\frac{b_2^{\vee} + \bar{b}_2}{2}, \frac{b_2^{\wedge} + \bar{b}_2}{2} \right], \\ & x_1, x_2 \geq 0, \end{array} \tag{18}$$

where in (18), $c_{lj}^{\vee} \in [\underline{c}_{lj}^{\vee}, \bar{c}_{lj}^{\vee}]$, $c_{lj}^{\wedge} \in [\underline{c}_{lj}^{\wedge}, \bar{c}_{lj}^{\wedge}]$, $a_{ij}^{\vee} \in [\underline{a}_{ij}^{\vee}, \bar{a}_{ij}^{\vee}]$, $a_{ij}^{\wedge} \in [\underline{a}_{ij}^{\wedge}, \bar{a}_{ij}^{\wedge}]$, $b_i^{\vee} \in [\underline{b}_i^{\vee}, \bar{b}_i^{\vee}]$ and $b_i^{\wedge} \in [\underline{b}_i^{\wedge}, \bar{b}_i^{\wedge}]$. According to the theorems of the IP problem, the best sub-model is expressed as follows:

$$\begin{aligned}
\max\ \ & z_b^{\wedge} = \left(\lambda_1(\frac{c_{11}^{\wedge}+\bar{c}_{11}}{2})+\lambda_2(\frac{c_{21}^{\wedge}+\bar{c}_{21}}{2})\right)x_1+\left(\lambda_1(\frac{c_{12}^{\wedge}+\bar{c}_{12}}{2})+\lambda_2(\frac{c_{22}^{\wedge}+\bar{c}_{22}}{2})\right)x_2,\\
s.t.\ \ & (\frac{a_{11}^{\vee}+\bar{a}_{11}}{2})x_1+(\frac{a_{12}^{\vee}+\bar{a}_{12}}{2})x_2 \le (\frac{b_1^{\wedge}+\bar{b}_1}{2}),\\
& (\frac{a_{21}^{\vee}+\bar{a}_{21}}{2})x_1+(\frac{a_{22}^{\vee}+\bar{a}_{22}}{2})x_2 \le (\frac{b_2^{\wedge}+\bar{b}_2}{2}),\\
& x_1, x_2 \ge 0,
\end{aligned} \tag{19}$$

where $c_{lj}^{\wedge} \in [\underline{c}_{lj}^{\wedge}, \bar{c}_{lj}^{\wedge}]$, $a_{ij}^{\vee} \in [\underline{a}_{ij}^{\vee}, \bar{a}_{ij}^{\vee}]$ and $b_i^{\wedge} \in [\underline{b}_i^{\wedge}, \bar{b}_i^{\wedge}]$, and the worst sub-model is also expressed as:

$$\begin{aligned}
\max\ \ & z_w^{\vee} = \left(\lambda_1(\frac{c_{11}^{\vee}+\bar{c}_{11}}{2})+\lambda_2(\frac{c_{21}^{\vee}+\bar{c}_{21}}{2})\right)x_1+\left(\lambda_1(\frac{c_{12}^{\vee}+\bar{c}_{12}}{2})+\lambda_2(\frac{c_{22}^{\vee}+\bar{c}_{22}}{2})\right)x_2,\\
s.t.\ \ & (\frac{a_{11}^{\wedge}+\bar{a}_{11}}{2})x_1+(\frac{a_{12}^{\wedge}+\bar{a}_{12}}{2})x_2 \le \frac{b_1^{\vee}+\bar{b}_1}{2},\\
& (\frac{a_{21}^{\wedge}+\bar{a}_{21}}{2})x_1+(\frac{a_{22}^{\wedge}+\bar{a}_{22}}{2})x_2 \le \frac{b_2^{\vee}+\bar{b}_2}{2},\\
& x_1, x_2 \ge 0,
\end{aligned} \tag{20}$$

where $c_{lj}^{\vee} \in [\underline{c}_{lj}^{\vee}, \bar{c}_{lj}^{\vee}]$, $a_{ij}^{\wedge} \in [\underline{a}_{ij}^{\wedge}, \bar{a}_{ij}^{\wedge}]$ and $b_i^{\vee} \in [\underline{b}_i^{\vee}, \bar{b}_i^{\vee}]$. Since, the best (19) and worst (20) sub-models are IP problems, the following four sub-models are obtained:

1. B-B model:

$$\begin{aligned}
\max\ \ & \bar{z}_{bb}^{\wedge} = \left(\lambda_1(\frac{\bar{c}_{11}^{\wedge}+\bar{c}_{11}}{2})+\lambda_2(\frac{\bar{c}_{21}^{\wedge}+\bar{c}_{21}}{2})\right)x_1+\left(\lambda_1(\frac{\bar{c}_{12}^{\wedge}+\bar{c}_{12}}{2})+\lambda_2(\frac{\bar{c}_{22}^{\wedge}+\bar{c}_{22}}{2})\right)x_2,\\
s.t.\ \ & (\frac{\bar{a}_{11}^{\vee}+\bar{a}_{11}}{2})x_1+(\frac{\bar{a}_{12}^{\vee}+\bar{a}_{12}}{2})x_2 \le (\frac{\bar{b}_1^{\wedge}+\bar{b}_1}{2}),\\
& (\frac{\bar{a}_{21}^{\vee}+\bar{a}_{21}}{2})x_1+(\frac{\bar{a}_{22}^{\vee}+\bar{a}_{22}}{2})x_2 \le (\frac{\bar{b}_2^{\wedge}+\bar{b}_2}{2}),\\
& x_1, x_2 \ge 0.
\end{aligned} \tag{21}$$

2. B-W model:

$$\begin{aligned}
\max\ \ & \underline{z}_{bw}^{\wedge} = \left(\lambda_1(\frac{\underline{c}_{11}^{\wedge}+\bar{c}_{11}}{2})+\lambda_2(\frac{\underline{c}_{21}^{\wedge}+\bar{c}_{21}}{2})\right)x_1+\left(\lambda_1(\frac{\underline{c}_{12}^{\wedge}+\bar{c}_{12}}{2})+\lambda_2(\frac{\underline{c}_{22}^{\wedge}+\bar{c}_{22}}{2})\right)x_2\\
s.t.\ \ & (\frac{\underline{a}_{11}^{\vee}+\bar{a}_{11}}{2})x_1+(\frac{\underline{a}_{12}^{\vee}+\bar{a}_{12}}{2})x_2 \le (\frac{\underline{b}_1^{\wedge}+\bar{b}_1}{2}),\\
& (\frac{\underline{a}_{21}^{\vee}+\bar{a}_{21}}{2})x_1+(\frac{\underline{a}_{22}^{\vee}+\bar{a}_{22}}{2})x_2 \le (\frac{\underline{b}_2^{\wedge}+\bar{b}_2}{2}),\\
& x_1, x_2 \ge 0.
\end{aligned} \tag{22}$$

3. W-B model:

$$
\begin{aligned}
\max\ \ & \underline{z}_{wb}^{\vee} = \left(\lambda_1\left(\frac{\underline{c}_{11}^{\vee}+\bar{c}_{11}}{2}\right)+\lambda_2\left(\frac{\underline{c}_{21}^{\vee}+\bar{c}_{21}}{2}\right)\right)x_1+\left(\lambda_1\left(\frac{\underline{c}_{12}^{\vee}+\bar{c}_{12}}{2}\right)+\lambda_2\left(\frac{\underline{c}_{22}^{\vee}+\bar{c}_{22}}{2}\right)\right)x_2,\\
s.t.\ \ & \left(\frac{\underline{a}_{11}^{\wedge}+\bar{a}_{11}}{2}\right)x_1+\left(\frac{\underline{a}_{12}^{\wedge}+\bar{a}_{12}}{2}\right)x_2 \le \left(\frac{\underline{b}_{1}^{\vee}+\bar{b}_{1}}{2}\right),\\
& \left(\frac{\underline{a}_{21}^{\wedge}+\bar{a}_{21}}{2}\right)x_1+\left(\frac{\underline{a}_{22}^{\wedge}+\bar{a}_{22}}{2}\right)x_2 \le \left(\frac{\underline{b}_{2}^{\vee}+\bar{b}_{2}}{2}\right),\\
& x_1,x_2 \ge 0.
\end{aligned}
\tag{23}
$$

4. W-W model:

$$
\begin{aligned}
\max\ \ & \bar{z}_{ww}^{\vee} = \left(\lambda_1\left(\frac{\bar{c}_{11}^{\vee}+\bar{c}_{11}}{2}\right)+\lambda_2\left(\frac{\bar{c}_{21}^{\vee}+\bar{c}_{21}}{2}\right)\right)x_1+\left(\lambda_1\left(\frac{\bar{c}_{12}^{\vee}+\bar{c}_{12}}{2}\right)+\lambda_2\left(\frac{\bar{c}_{22}^{\vee}+\bar{c}_{22}}{2}\right)\right)x_2,\\
s.t.\ \ & \left(\frac{\bar{a}_{11}^{\wedge}+\bar{a}_{11}}{2}\right)x_1+\left(\frac{\bar{a}_{12}^{\wedge}+\bar{a}_{12}}{2}\right)x_2 \le \left(\frac{\bar{b}_{1}^{\vee}+\bar{b}_{1}}{2}\right),\\
& \left(\frac{\bar{a}_{21}^{\wedge}+\bar{a}_{21}}{2}\right)x_1+\left(\frac{\bar{a}_{22}^{\wedge}+\bar{a}_{22}}{2}\right)x_2 \le \left(\frac{\bar{b}_{2}^{\vee}+\bar{b}_{2}}{2}\right),\\
& x_1,x_2 \ge 0.
\end{aligned}
\tag{24}
$$

In the next section, we describe the process of solving the bi-objective FLP problem with IT2TrFNs by solving two examples. We explain the advantages and functions of the proposed method, and all the solution steps are presented in details.

4 Numerical examples

In this section, two numerical examples of MOIT2TrFLP problem are investigated.

Example 1. Consider the following interval bi-objective FLP problem with IT2TrFNs:

$$
\begin{aligned}
\max\ \ & \{\tilde{\tilde{c}}_1 x = z_1\},\\
\max\ \ & \{\tilde{\tilde{c}}_2 x = z_2\},\\
s.t.\ \ & \tilde{\tilde{a}}_{11}x_1+\tilde{\tilde{a}}_{12}x_2 \le \tilde{\tilde{b}}_1,\\
& \tilde{\tilde{a}}_{21}x_1+\tilde{\tilde{a}}_{22}x_2 \le \tilde{\tilde{b}}_2,\\
& x_1,x_2 \ge 0.
\end{aligned}
\tag{25}
$$

Parameters which are used in the Example 1 are given in the following Tables 1, 2 and 3.

Using the WS method, the problem (25) is transformed into the following single-objective FLP problem with IT2TrFNs:

$$
\begin{aligned}
\max\ \ & (\lambda_1\tilde{\tilde{c}}_{11}+\lambda_2\tilde{\tilde{c}}_{21})x_1+(\lambda_1\tilde{\tilde{c}}_{12}+\lambda_2\tilde{\tilde{c}}_{22})x_2,\\
s.t.\ \ & \tilde{\tilde{a}}_{11}x_1+\tilde{\tilde{a}}_{12}x_2 \le \tilde{\tilde{b}}_1,\\
& \tilde{\tilde{a}}_{21}x_1+\tilde{\tilde{a}}_{22}x_2 \le \tilde{\tilde{b}}_2,\\
& x_1,x_2 \ge 0.
\end{aligned}
\tag{26}
$$

Table 1: Values of the OFVs.

$\tilde{\tilde{c}}_{11}$	$((\underline{c}_{11}^{\vee},\bar{c}_{11},\underline{c}_{11}^{\wedge}),(\bar{c}_{11}^{\vee},\bar{c}_{11},\bar{c}_{11}^{\wedge}))=((0.5,1.2,3),(0.2,1.2,3.2))$
$\tilde{\tilde{c}}_{12}$	$((\underline{c}_{12}^{\vee},\bar{c}_{12},\underline{c}_{12}^{\wedge}),(\bar{c}_{12}^{\vee},\bar{c}_{12},\bar{c}_{12}^{\wedge}))=((1,3,4.5),(0.5,3,5))$
$\tilde{\tilde{c}}_{21}$	$((\underline{c}_{21}^{\vee},\bar{c}_{21},\underline{c}_{21}^{\wedge}),(\bar{c}_{21}^{\vee},\bar{c}_{21},\bar{c}_{21}^{\wedge}))=((1,2,3),(0.8,2,3))$
$\tilde{\tilde{c}}_{22}$	$((\underline{c}_{22}^{\vee},\bar{c}_{22},\underline{c}_{22}^{\wedge}),(\bar{c}_{22}^{\vee},\bar{c}_{22},\bar{c}_{22}^{\wedge}))=((1.5,4,5),(1,4,5.5))$

Table 2: Values of the TCs.

$\tilde{\tilde{a}}_{11}$	$((\underline{a}_{11}^{\vee},\bar{a}_{11},\underline{a}_{11}^{\wedge}),(\bar{a}_{11}^{\vee},\bar{a}_{11},\bar{a}_{11}^{\wedge}))=((0.5,1,1.5),(0,1,2))$
$\tilde{\tilde{a}}_{12}$	$((\underline{a}_{12}^{\vee},\bar{a}_{12},\underline{a}_{12}^{\wedge}),(\bar{a}_{12}^{\vee},\bar{a}_{12},\bar{a}_{12}^{\wedge}))=((3,4,5),(2,4,6))$
$\tilde{\tilde{a}}_{21}$	$((\underline{a}_{21}^{\vee},\bar{a}_{21},\underline{a}_{21}^{\wedge}),(\bar{a}_{21}^{\vee},\bar{a}_{21},\bar{a}_{21}^{\wedge}))=((3,4,5),(2,4,6))$
$\tilde{\tilde{a}}_{22}$	$((\underline{a}_{22}^{\vee},\bar{a}_{22},\underline{a}_{22}^{\wedge}),(\bar{a}_{22}^{\vee},\bar{a}_{22},\bar{a}_{22}^{\wedge}))=((1.5,2.5,3.5),(0.5,2.5,4.5))$

Problem (26) is nonlinear. Considering $\lambda_1=\lambda_2=0.5$, this problem becomes an LP problem. Given that the cores of IT2TrFNs used in problem (25) are singleton sets (see Fig. 2), by placing the cores of these numbers, the following middle LP problem is obtained:

$$\begin{aligned} \max\ & \bar{z}_m = 2.1x_1+3x_2, \\ s.t.\ & x_1+4x_2 \le 11, \\ & 4x_1+2.5x_2 \le 12, \\ & x_1,x_2 \ge 0. \end{aligned} \tag{27}$$

By solving the middle problem (27), the OSs $\bar{x}_{1m}=1.519$, $\bar{x}_{2m}=2.370$, and optimal value $\bar{z}_m=10.3$ are obtained. Using the data from Example 1, the problem (21) is rewritten as follows:

$$\begin{aligned} \max\ & \bar{z}_{bb}^{\wedge} = 2.35x_1+4.375x_2, \\ s.t.\ & 0.5x_1+3x_2 \le 13, \\ & 3x_1+1.5x_2 \le 16, \\ & x_1,x_2 \ge 0. \end{aligned} \tag{28}$$

The OSs and optimal value of the problem (28) are obtained as $\bar{x}_{1bb}^{\wedge}=3.455$, $\bar{x}_{2bb}^{\wedge}=3.758$, and $\bar{z}_{bb}^{\wedge}=24.370$. The problem (22) is rewritten as:

$$\begin{aligned} \max\ & \underline{z}_{bw}^{\wedge} = 2.3x_1+4.125x_2 \\ s.t.\ & 0.75x_1+3.5x_2 \le 12, \\ & 3.5x_1+2x_2 \le 14, \\ & x_1,x_2 \ge 0. \end{aligned} \tag{29}$$

By solving the problem (29), the OSs and optimal value $\underline{x}_{1bw}^{\wedge}=2.326$, $\underline{x}_{2opt}^{\wedge}=2.930$, and $\underline{z}_{opt}^{\wedge}=17.436$

Table 3: Values of the RVs.

$\tilde{\tilde{b}}_1$	$\left((\underline{b}_1^{\vee},\overline{b}_1,\underline{b}_{11}^{\wedge}),(\overline{b}_{11}^{\vee},\overline{b}_{11},\overline{b}_1^{\wedge})\right)=((9,11,13),(7,11,15))$
$\tilde{\tilde{b}}_2$	$\left((\underline{b}_2^{\vee},\overline{b}_2,\underline{b}_2^{\wedge}),(\overline{b}_2^{\vee},\overline{b}_2,\overline{b}_2^{\wedge})\right)=((8,12,16),(4,12,20))$

are obtained. The problem (23) is rewritten in the following problem:

$$\begin{aligned} \max\ & \underline{z}_{wb}^{\vee}=1.175x_1+2.375x_2, \\ s.t.\ & 1.25x_1+4.5x_2\leq 10, \\ & 4.5x_1+3x_2\leq 10, \\ & x_1,x_2\geq 0. \end{aligned} \tag{30}$$

The OSs and optimal value of the problem (30) are given as $\underline{x}_{1wb}^{\vee}=0.910$, $\underline{x}_{2wb}^{\vee}=1.970$, and $\underline{z}_{wb}^{\vee}=5.746$. The problem (24) is rewritten as:

$$\begin{aligned} \max\ & \overline{z}_{ww}^{\vee}=1.05x_1+2.125x_2, \\ s.t.\ & 1.5x_1+5x_2\leq 9, \\ & 5x_1+3.5x_2\leq 8, \\ & x_1,x_2\geq 0. \end{aligned} \tag{31}$$

By solving the problem (31), the OSs and optimal value $\overline{x}_{1ww}^{\vee}=0.430$, $\overline{x}_{2ww}^{\vee}=1.671$, and $\overline{z}_{ww}^{\vee}=4.003$ are obtained. Therefore, the OSs and optimal value are calculated up to three decimal places and presented as the following IT2TrFNs:

$$\begin{aligned} \tilde{\tilde{x}}_{1opt}&=((0.910,1.519,2.326),(0.430,1.519,3.455)), \\ \tilde{\tilde{x}}_{2opt}&=((1.970,2.370,2.930),(1.671,2.370,3.758)), \\ \tilde{\tilde{z}}_{opt}&=((5.746,10.300,17.436),(4.003,10.3,24.370)), \end{aligned}$$

As mentioned, the optimal solutions and the optimal value of the objective function are IT2TrFNs. In fact, with this solution method, the numbers retain their fuzzy nature. See the OSs in Figure 3, and the optimal value in Figure 4.

As you can see, the OSs and optimal value are in the form of IT2TrFNs.

Example 2. Consider the three-objective FLP problem with the IT2TrFNs:

$$\begin{aligned} \max\ & \{\tilde{\tilde{c}}_1x=z_1\}, \\ \max\ & \{\tilde{\tilde{c}}_2x=z_2\}, \\ \max\ & \{\tilde{\tilde{c}}_3x=z_3\}, \\ s.t.\ & \tilde{\tilde{a}}_{11}x_1+\tilde{\tilde{a}}_{12}x_2\leq\tilde{\tilde{b}}_1, \\ & \tilde{\tilde{a}}_{21}x_1+\tilde{\tilde{a}}_{22}x_2\leq\tilde{\tilde{b}}_2, \\ & x_1,x_2\geq 0. \end{aligned} \tag{32}$$

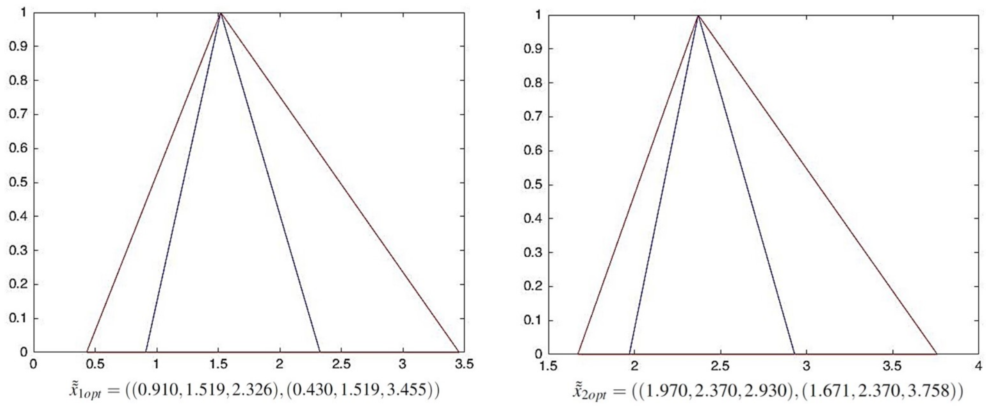

Figure 3: The OSs of Example 1.

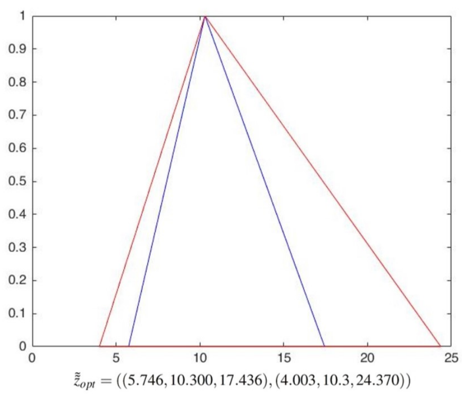

Figure 4: The optimal value of Example 1.

Table 4: Values of the OFVs.

$\tilde{\tilde{c}}_{11}$	$((\underline{c}_{11}^{\vee}, \underline{c}_{11}, \underline{c}_{11}^{\wedge}), (\overline{c}_{11}^{\vee}, \overline{c}_{11}, \overline{c}_{11}^{\wedge})) = ((1.5, 2, 2.5), (1, 2, 3))$
$\tilde{\tilde{c}}_{12}$	$((\underline{c}_{12}^{\vee}, \underline{c}_{12}, \underline{c}_{12}^{\wedge}), (\overline{c}_{12}^{\vee}, \overline{c}_{12}, \overline{c}_{12}^{\wedge})) = ((2.5, 3, 3.5), (2, 3, 4))$
$\tilde{\tilde{c}}_{21}$	$((\underline{c}_{21}^{\vee}, \underline{c}_{21}, \underline{c}_{21}^{\wedge}), (\overline{c}_{21}^{\vee}, \overline{c}_{21}, \overline{c}_{21}^{\wedge})) = ((1, 1.5, 2), (0, 1.5, 3))$
$\tilde{\tilde{c}}_{22}$	$((\underline{c}_{22}^{\vee}, \underline{c}_{22}, \underline{c}_{22}^{\wedge}), (\overline{c}_{22}^{\vee}, \overline{c}_{22}, \overline{c}_{22}^{\wedge})) = ((2, 3, 4), (1, 3, 5))$
$\tilde{\tilde{c}}_{31}$	$((\underline{c}_{31}^{\vee}, \underline{c}_{31}, \underline{c}_{31}^{\wedge}), (\overline{c}_{31}^{\vee}, \overline{c}_{31}, \overline{c}_{31}^{\wedge})) = ((2, 2.5, 3), (1.5, 2.5, 3))$
$\tilde{\tilde{c}}_{32}$	$((\underline{c}_{32}^{\vee}, \underline{c}_{32}, \underline{c}_{32}^{\wedge}), (\overline{c}_{32}^{\vee}, \overline{c}_{32}, \overline{c}_{32}^{\wedge})) = ((2.5, 4, 4.5), (2, 4, 5))$

Table 5: Values of the TCs.

$\tilde{\tilde{a}}_{11}$	$((\underline{a}_{11}^{\vee}, \underline{a}_{11}, \underline{a}_{11}^{\wedge}), (\overline{a}_{11}^{\vee}, \overline{a}_{11}, \overline{a}_{11}^{\wedge})) = ((0.5, 1, 1.5), (0, 1, 2))$
$\tilde{\tilde{a}}_{12}$	$((\underline{a}_{12}^{\vee}, \underline{a}_{12}, \underline{a}_{12}^{\wedge}), (\overline{a}_{12}^{\vee}, \overline{a}_{12}, \overline{a}_{12}^{\wedge})) = ((3, 4, 5), (2, 4, 6))$
$\tilde{\tilde{a}}_{21}$	$((\underline{a}_{21}^{\vee}, \underline{a}_{21}, \underline{a}_{21}^{\wedge}), (\overline{a}_{21}^{\vee}, \overline{a}_{21}, \overline{a}_{21}^{\wedge})) = ((3, 4, 5), (2, 4, 6))$
$\tilde{\tilde{a}}_{22}$	$((\underline{a}_{22}^{\vee}, \underline{a}_{22}, \underline{a}_{22}^{\wedge}), (\overline{a}_{22}^{\vee}, \overline{a}_{22}, \overline{a}_{22}^{\wedge})) = ((1.5, 2.5, 3.5), (0.5, 2.5, 4.5))$

Table 6: Values of the RVs.

$\tilde{\tilde{b}}_1$	$\left((\underline{b}_1^{\vee},\underline{b}_1,\underline{b}_{11}^{\wedge}),(\overline{b}_{11}^{\vee},\overline{b}_{11},\overline{b}_1^{\wedge})\right)=((9,11,13),(7,11,15))$
$\tilde{\tilde{b}}_2$	$\left((\underline{b}_2^{\vee},\underline{b}_2,\underline{b}_2^{\wedge}),(\overline{b}_2^{\vee},\overline{b}_2,\overline{b}_2^{\wedge})\right)=((8,12,16),(4,12,20))$

Parameters which are used in the problem (32) are given in the Tables 4, 5 and 6.

Using the WS method, the problem (32) is transformed into the following problem:

$$\begin{aligned}
\max\ & (\lambda_1\tilde{\tilde{c}}_{11}+\lambda_2\tilde{\tilde{c}}_{21}+\lambda_3\tilde{\tilde{c}}_{31})x_1+(\lambda_1\tilde{\tilde{c}}_{12}+\lambda_2\tilde{\tilde{c}}_{22}+\lambda_3\tilde{\tilde{c}}_{32})x_2,\\
s.t.\ & \tilde{\tilde{a}}_{11}x_1+\tilde{\tilde{a}}_{12}x_2\leq\tilde{\tilde{b}}_1,\\
& \tilde{\tilde{a}}_{21}x_1+\tilde{\tilde{a}}_{22}x_2\leq\tilde{\tilde{b}}_2,\\
& x_1,x_2\geq 0.
\end{aligned}\tag{33}$$

The problem (33) is a single-objective nonlinear programming problem. Now, we consider $\lambda_1=0.4,\lambda_2=0.3$ and $\lambda_3=0.3$, note that $\lambda_1+\lambda_2+\lambda_3=1$. The middle LP problem is as follows:

$$\begin{aligned}
\max\ & \underline{z}_m=1.75x_1+3x_2,\\
s.t.\ & x_1+4x_2\leq 11,\\
& 4x_1+2.5x_2\leq 12,\\
& x_1,x_2\geq 0.
\end{aligned}\tag{34}$$

By solving the problem (34), the OSs, $\underline{x}_{1m}=1.518$, $\underline{x}_{2m}=2.370$, and optimal value, $\underline{z}_m=9.768$ are obtained. Problem (21) is rewritten as follows:

$$\begin{aligned}
\max\ & \overline{z}_{bb}^{\wedge}=2.375x_1+3.75x_2,\\
s.t.\ & 0.9x_1+3x_2\leq 13,\\
& 3x_1+1.5x_2\leq 16,\\
& x_1,x_2\geq 0.
\end{aligned}\tag{35}$$

The OSs and optimal value of the problem (35) are obtained as $\overline{x}_{1bb}^{\wedge}=3.454$, $\overline{x}_{2bb}^{\wedge}=3.758$, and $\overline{z}_{bb}^{\wedge}=22.296$. Then, the problem (22) is rewritten as:

$$\begin{aligned}
\max\ & \underline{z}_{bw}^{\wedge}=2x_1+3.375x_2,\\
s.t.\ & 0.75x_1+3.5x_2\leq 12,\\
& 3.5x_1+2x_2\leq 14,\\
& x_1,x_2\geq 0.
\end{aligned}\tag{36}$$

By solving the problem (36), the OSs and optimal value are obtained as $\underline{x}_{1bw}^{\wedge}=2.326$, $\underline{x}_{2bw}^{\wedge}=2.930$, and

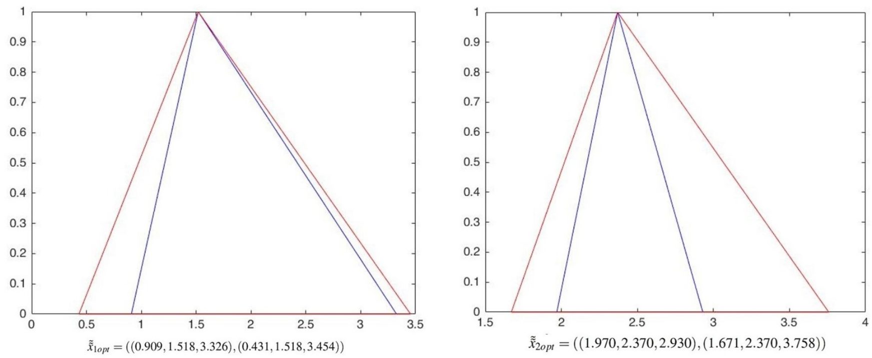

Figure 5: The OSs of Example 2.

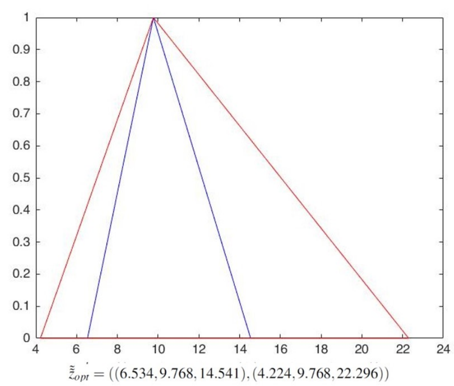

Figure 6: The optimal value of Example 2.

$\underline{z}^{\wedge}_{bw} = 14.541$. And, the problem (23) is rewritten in the following problem:

$$\begin{aligned} \max \quad & \underline{z}^{\vee}_{wb} = 1.5x_1 + 2.625x_2, \\ s.t. \quad & 1.25x_1 + 4.5x_2 \leq 10 \\ & 4.5x_1 + 3x_2 \leq 10, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{37}$$

The OSs and optimal value of the problem (37) are given as $\underline{x}^{\vee}_{1wb} = 0.909$, $\underline{x}^{\vee}_{2wb} = 1.970$, and $\underline{z}^{\vee}_{wb} = 6.534$. The problem (24) is rewritten as:

$$\begin{aligned} \max \quad & \bar{z}^{\vee}_{ww} = 1.125x_1 + 2.25x_2, \\ s.t. \quad & 1.5x_1 + 5x_2 \leq 9 \\ & 5x_1 + 3.5x_2 \leq 8, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{38}$$

By solving the problem (38), the OSs and optimal value are given as $\bar{x}^{\vee}_{1ww} = 0.431$, $\bar{x}^{\vee}_{2ww} = 1.671$, and

$\bar{z}^{\vee}_{ww} = 4.244$. Therefore, the OSs and optimal value are presented as the following IT2TrFNs:

$$\tilde{\tilde{x}}_{1opt} = ((0.909, 1.518, 3.326), (0.431, 1.518, 3.454)),$$
$$\tilde{\tilde{x}}_{2opt} = ((1.970, 2.370, 2.930), (1.671, 2.370, 3.758)),$$
$$\tilde{\tilde{z}}_{opt} = ((6.534, 9.768, 14.541), (4.224, 9.768, 22.296)),$$

Figures 5 and 6 show the OSs and the optimal value of Example 2, respectively.

As you can see, the OSs and optimal value are in the form of IT2TrFNs, and their fuzzy nature is preserved.

5 Conclusions

In this study, an MOFLP problem with the coefficients of IT2TrFNs with ambiguity-type imprecision was investigated. First, using the WS method, the MOLP problem was transformed into a single-objective programming problem. In addition, the concept of NIA for type-1 TrFN was used to solve this new problem. Every IT2TrFN is created by the infinite union of type-1 TrFNs. Therefore, using the concept of the NIA, an interval is assigned to each IT2TrFN in the problem, and the single-objective LP problem with IT2TrFNs is transformed into two IP problems where all coefficients are of the interval type. Then, using the BWC method, each of the above two problems is transformed into two LP sub-models. Therefore, four sub-models are obtained, and by solving them, the optimal value and OSs are obtained in the form of IT2TrFNs. We used this method to solve two multi-objective problems (bi-objective and three-objective). One of the important reasons for conducting this study is the lack of sufficient study on MOLP problems using parameters such as INT2TrFNs. The advantages of the proposed method can be summarized as follows:

A combination of two simple and practical methods: the weighted sum method and the nearest interval approximation approach, to solve the multi-objective problem.

Using the weighted sum method to convert a multi-objective problem into a single-objective problem, the decision-maker can allocate a proportional weight vector to each objective function based on the conditions of the real problem. Consequently, the obtained results are much more desirable;

The approximation used to convert a triangular fuzzy number into an interval is easily computable;

Applying the decision-makers opinion before starting the problem-solving process.

The obtained optimal solutions are in the form of IT2TrFNs. In fact, the fuzzy nature of the numbers is preserved.

Computational simplicity and desirable results;

Since the obtained results for OSs and optimal objective function values are IT2TrFNs, the obtained results have less error.

Since no specific study has been conducted on this type of problem (MOFLP problems) with such parameters (IT2TrFNs), this study can be considered as one of the first related studies.

For future studies, the proposed method can be used for multi-objective problems with some IT2FNs, such as IT2 trapezoidal FNs and interval-typed TrFNs.

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