Journal of Mathematical Modeling
Vol. 12, No. 1, 2024, pp. 157–175. Research Article

JMM

A uniformly convergent computational method for singularly perturbed parabolic partial differential equation with integral boundary condition

Getu Mekonnen Wondimu[†][*], Gemechis File Duressa[§], Mesfin Mekuria Woldaregay[‡], Tekle Gemechu Dinka[‡]

[†]*Department of Mathematics, Raya University, Maichew, Ethiopia*
[§]*Department of Mathematics, Jimma University, Jimma, Ethiopia*
[‡]*Department of Applied Mathematics , Adama Science and Technology University, Adama, Ethiopia*
Email(s): getiye21@gmail.com, gammeef@gmail.com, msfnmkr02@gmail.com, tekgem@yahoo.com

Abstract. This paper presents a numerical method for a class of singularly perturbed parabolic partial differential equations with integral boundary conditions (IBC). The solution to the considered problem exhibits pronounced boundary layers on both the left and right sides of the spatial domain. To address this challenging problem, we propose the use of the implicit Euler method for time discretization and a finite difference method on a well-designed piecewise uniform Shishkin mesh for spatial discretization. The integral boundary condition is approximated using Simpson's $\frac{1}{3}$ rule. The presented method demonstrates almost second-order uniform convergence in the discretization of the spatial derivative and first-order convergence in the discretization of the time derivative. To validate the applicability and accuracy of the proposed method, two illustrative examples are employed. The computational results not only accurately reflect the theoretical estimations but also highlight the method's effectiveness in capturing the intricate features of singularly perturbed parabolic partial differential equations with integral boundary conditions.

Keywords: Singularly perturbed problems, finite difference, Shishkin mesh, uniform convergence, integral boundary condition.
AMS Subject Classification 2010: 65N06, 65N12,65N15,65L11.

1 Introduction

Singularly perturbed differential equations are usually characterized by a small parameter multiplying one or more of the highest-order terms in the differential equation, as boundary layers typically emerge in their solutions. These problems play a crucial role in contemporary scientific computations. Many

[*]Corresponding author
Received: 24 May 2023 / Revised: 10 December 2023 / Accepted: 14 December 2023
DOI: 10.22124/jmm.2023.24590.2198

mathematical models, ranging from fluid dynamics to problems in mathematical biology, are formulated as singularly perturbed problems. Examples include high Reynolds number flow in fluid dynamics, heat transport problems, plasma physics, and more. Singularly perturbed problems with integral boundary conditions have been employed to describe numerous phenomena in the applied sciences, such as heat conduction, chemical engineering, underground water flow, and so forth [2–4, 14].

The numerical analysis of singular perturbation cases has always been challenging due to the boundary layer behavior of the solution. Because of the small parameter affecting the higher derivative, solutions of the problems undergo rapid changes in the boundary layer region or elsewhere inside the problem domain [18]. The presence of a boundary layer poses difficulties for classical numerical methods (such as standard finite difference or finite element schemes) applied on uniform meshes, resulting in inaccurate numerical solutions as the singular perturbation parameter ε approaches zero. This limitation motivates the development of parameter-uniform convergent numerical methods. Among these methods, the fitted mesh method stands out as a satisfactory and popular technique that utilizes special layer-adapted meshes to overcome the numerical challenges.

Recently, researchers have focused on a category of Singularly Perturbed Differential Equations (SPDEs) with Integral Boundary Conditions (IBC), leading to the development of various numerical techniques. Notably, studies such as [1, 9, 16] have investigated the existence and uniqueness of a class of singularly perturbed problems with IBC. In [13], the authors explored an upwind finite difference scheme on a piecewise uniform mesh for singularly perturbed Ordinary Differential Equations (ODEs) containing IBC. Meanwhile, the authors in [19] proposed a finite difference scheme on a Shishkin mesh for systems of singularly perturbed ODEs of reaction-diffusion types with IBC. The work in [20] focused on a Finite Difference Method (FDM) on a Shishkin mesh for a convection-diffusion type problem with IBC, while in [7] the authors introduced a non-standard FDM for solving singularly perturbed ODEs with IBC. Additionally, the authors in [5, 6, 21–24] investigated numerical solutions for singularly perturbed ODEs with a unit delay and IBC.

However, fewer researchers have explored a class of Singularly Perturbed Partial Differential Equations (SPPDEs) with a large negative shift and IBC. For example, in [8] the authors studied SPPDEs of reaction-diffusion types with a unit delay and IBC using a standard FDM with a Shishkin mesh for spatial derivatives and the backward difference method for time derivatives. The authors in [10] employed the Euler method for the time direction and a non-standard FDM for the space direction to solve SPPDEs with a large negative shift and IBC. Additionally, the authors in [11] developed exponential fitted finite difference schemes on a uniform grid for solving SPDPDEs with integral boundary conditions. In [12], the authors proposed a cubic spline method on a Shishkin mesh for the spatial direction and implicit Euler method for the temporal direction to formulate a parameter uniform numerical scheme. The authors in [25] presented a finite difference scheme on a suitable piecewise uniform Shishkin mesh in the spatial direction and Crank Nicholson method in the temporal direction. In the work conducted by the authors in [26], a nonstandard finite difference method for the spatial direction and the implicit Euler method for the time derivative were developed to solve singularly perturbed partial differential equations with nonlocal boundary conditions.

To date, there has been no reported layer-resolving numerical method on a Shishkin mesh for solving singularly perturbed parabolic partial differential equations with IBC. This study aims to fill this gap by constructing a uniformly convergent numerical method for solving such equations. The proposed method involves a finite difference approach on a piecewise uniform Shishkin mesh for spatial discretization and implicit Euler method for temporal discretization. Rigorous uniform stability and convergence analyses support the method's effectiveness, demonstrating its uniform convergence regardless of the perturbation parameter.

Notation: The study employs the notations N and M to represent the number of mesh intervals in the spatial and temporal directions, respectively. Additionally, a positive constant, denoted as C, is introduced, and it remains independent of ε, N, and M throughout the analysis. The maximal norm, utilized for analyzing the convergence of numerical solutions, is defined as follows: $||z(s,t)|| := \sup|z(s,t)|$, for $(s,t) \in D$.

2 The governing equation

This paper addresses the numerical solution of a specific class of singularly perturbed parabolic partial differential equations, specifically those related to the reaction-diffusion problem with IBC. The general form of these equations is given by

$$\begin{cases} \mathscr{L}z(s,t) = \frac{\partial z(s,t)}{\partial t} - \varepsilon\frac{\partial^2 z(s,t)}{\partial s^2} + a(s,t)z(s,t) = f(s,t), \;\; (s,t) \in D = (0,1)\times(0,T], \\ z(s,0) = \phi_b(s), \;\; (s,t) \in \Gamma_b = \{(s,0), s \in [0,1]\}, \\ z(0,t) = \phi_l(s,t), \;\; (s,t) \in \Gamma_l = \{(0,t); t \in [0,T]\}, \\ \mathscr{K}z(s,t) = z(1,t) - \varepsilon\int_0^1 g(s)z(s,t)ds = \phi_r(s,t), \;\; (s,t) \in \Gamma_r = \{(1,t); t \in [0,T]\}, \end{cases} \tag{1}$$

where $\bar{D} = [0,1]\times[0,T]$, and ε is the perturbation parameter that satisfy $(0 < \varepsilon \ll 1)$. Suppose that $a(s,t) \geq \alpha > 0$, $f(s,t)$, ϕ_l, ϕ_r, ϕ_b are sufficiently smooth functions, and $g(s)$ is a non-negative monotone function that satisfy $\int_0^1 g(s)ds < 1$.

3 Properties of continuous solution

To guarantee the existence and uniqueness of the solution to the equation (1), we make the assumption that the coefficients of the problem exhibit Holder continuity and enforce appropriate compatibility criteria at the corner points, as outlined in [15].

The solution of (1) necessarily satisfy the following compatibility conditions

$$\phi_b(0,0) = \phi_l(0,0), \; \phi_b(1,0) = \phi_r(1,0),$$

and

$$\begin{aligned} -\varepsilon\frac{\partial^2\phi_b(0,0)}{\partial s^2} + a(0,0)\phi_b(0,0) + \frac{\partial\phi_l(0,0)}{\partial t} = f(0,0), \\ -\varepsilon\frac{\partial^2\phi_b(1,0)}{\partial s^2} + a(1,0)\phi_b(1,0) + \frac{\partial\phi_r(1,0)}{\partial t} = f(1,0). \end{aligned}$$

Lemma 1. *[26] [Maximum Principle] Let $\psi(s,t) \in C^{(0,0)}(\bar{D}) \cap C^{(1,0)}(D) \cap C^{(2,1)}(D)$ be a sufficiently smooth function such that $\psi(0,t) \geq 0, \psi(s,0) \geq 0, \mathscr{K}\psi(1,t) \geq 0, \mathscr{L}\psi(s,t) \geq 0, \forall (s,t) \in D$. Then $\psi(s,t) \geq 0$, $\forall (s,t) \in \bar{D}$, where $\mathscr{L}\psi(s,t) = \psi_t(s,t) - \varepsilon\psi_{ss}(s,t) + a\psi(s,t)$.*

Lemma 2 (Stability Result)**.** *Suppose $z(s,t)$ is the solution of (1). Then it satisfies the bound,*

$$z(s,t) \leq \alpha^{-1}\|f\| + \max\{\phi_b(s), \phi_l(s,t), \phi_r(s,t)\},$$

where $\|f\| = \max_{(s,t)\in D}|f(s,t)|$.

Proof. The required bound is obtained by constructing a comparison functions:

$$\Theta^{\pm}(s,t) = \alpha^{-1}\|f\| + \max\{\phi_b(s), \phi_l(s,t), \phi_r(s,t)\} \pm z(s,t), (s,t) \in \bar{D}$$

and using the maximum principle in Lemma 1. □

The subsequent classical lemma furnishes sufficient criteria for the existence of a unique solution to the problem (1).

Lemma 3. *If the coefficient satisfies $f(s,t), a(s,t) \in C^{(2+\alpha_1, 1+\alpha_1/2)}(\bar{D})$ and the boundary conditions satisfies $\phi_l \in C^{2+\alpha_1/2}([0,T]), \phi_b \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\Gamma_b), \phi_r \in C^{2+\alpha_1/2}([0,T])$, where $\alpha_1 \in (0,1)$, then the problem in (1) has exactly one solution z which satisfies $z \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{D})$. And the derivatives of z are bounded as*

$$\left\|\frac{\partial^{i+j}z}{\partial s^i \partial t^j}\right\| \leq C\varepsilon^{\frac{-i}{2}}, \quad \textit{for} \ \ 0 \leq i+2j \leq 4.$$

Proof. For the proof, interested reader can refer [8]. □

The nonclassical bounds in singular and regular components and their derivatives are established in the following lemma since the bounds in the above lemma are not adequate for the proof of parameter-uniform error estimate.

Lemma 4. *If $a(s,t), f(s,t) \in C^{(4+\alpha_1, 2+\alpha_1/2)}(\bar{D})$, and the boundary condition satisfies $\phi_l \in C^{(3+\alpha_1/2)}([0,T])$, $\phi_b \in C^{(6+\alpha_1, 3+\alpha_1/2)}(\Gamma_b)$, $\phi_r \in C^{(3+\alpha_1/2)}([0,T])$, where $\alpha_1 \in (0,1)$, then*

$$\left\|\frac{\partial^{i+j}v}{\partial s^i \partial t^j}\right\|_{\bar{D}} \leq C\left(1+\varepsilon^{1-i/2}\right), \qquad (s,t) \in D, \tag{2}$$

$$\left|\frac{\partial^{i+j}w_l}{\partial s^i \partial t^j}\right| \leq C\varepsilon^{\frac{-i}{2}} e^{\frac{s}{\sqrt{\varepsilon}}}, \qquad (s,t) \in D, \tag{3}$$

$$\left|\frac{\partial^{i+j}w_r}{\partial s^i \partial t^j}\right| \leq C\varepsilon^{\frac{-i}{2}} e^{\frac{-(1-s)}{\sqrt{\varepsilon}}}, \qquad (s,t) \in D, \tag{4}$$

where C is a constant independent of ε, $0 \leq i+2j \leq 4$.

Proof. For the proof one can refer [26]. □

4 The numerical method

In this context, the discretization of the time derivative is achieved through the application of the implicit Euler technique. Additionally, we employ the conventional finite difference method to discretize the space derivative, introducing a Shishkin-type layer-adapted mesh to enhance the numerical treatment.

4.1 Time semi-discretization

Let Ω^M represents a uniform mesh employed in temporal semi-discretization that is defined as $\Omega^M = \{t_j = j\Delta t, j = 0,1,\ldots,M, \ \ \Delta t = T/M\}$, where M is a positive integer. The problem described in (1) is semi-discretized utilizing the implicit Euler method as follows:

$$\mathscr{L}^{\Delta t} Z^{j+1}(s) = g(s,t_{j+1}), \ \ j = 0,1,\ldots,M-1, \tag{5}$$

with discretized boundary condition

$$\begin{cases} Z^{j+1}(s) = \phi_l^{j+1}(s), & \phi_l^{j+1}(s) \in \Gamma_l, \\ \mathscr{K} Z^{j+1}(s) = Z^{j+1}(1) - \varepsilon \int_0^1 g(s)z(s)ds = \phi_r^{j+1}(s), & \phi_r^{j+1}(s) \in \Gamma_r, \\ Z^{j+1}(s) = \phi_b^{j+1}(s), & \phi_b^{j+1}(s) \in \Gamma_b, \end{cases} \tag{6}$$

where $\mathscr{L}^{\Delta t} Z^{j+1}(s) = -\varepsilon Z_{ss}^{j+1} + r(s)Z^{j+1}(s), g(s,t_{j+1}) = \frac{Z^j(s)}{\Delta t} + f(s,t_{j+1})$, and $r(s) = \frac{1}{\Delta t} + a(s)$. Here, $Z^{j+1}(s)$ is denoted for the approximation of $z(s,t_{j+1})$ at the $(j+1)^{th}$ time level.

The semi-discrete operator $\mathscr{L}$ satisfies the following lemma.

Lemma 5. *For $j = 0,1,2,\ldots,M-1$, suppose that $\Psi^{j+1}(s)$ be a sufficiently smooth function in D such that $\Psi^{j+1}(0) \geq 0$ and $\mathscr{K}\Psi^{j+1}(1) \geq 0$. Then $\mathscr{L}\Psi^{j+1}(s) \geq 0, \forall s \in (0,1)$, implies $\Psi^{j+1}(s) \geq 0$.*

Proof. Assume $s^* \in [0,1]$ be such that $\Psi^{j+1}(s^*) = \min_{s\in D} \Psi^{j+1}(s) < 0$. From the above assumption, it is known that $s^* \notin \{0,1\}$ implies that $s^* \in (0,1)$. By using properties from calculus, we have $\frac{d^2}{ds^2}\Psi^{j+1}(s) \geq 0$, which implies that $\mathscr{L}^{\Delta t}\Psi^{j+1}(s) < 0$. This contradicts with $\mathscr{L}^{\Delta t}\Psi^{j+1}(s) \geq 0$, $\forall\ s \in (0,1)$. Therefore, we conclude that $\Psi^{j+1}(s) \geq 0$, $\forall\ s \in [0,1]$. □

Following that, we investigate the estimation of local truncation error for temporal semi-discretization, denoted as $e^j := z(s,t_{j+1}) - Z^{j+1}(s)$, for $j = 0,1,2,\ldots,M$.

Lemma 6. *[26] The local truncation error associated with the implicit Euler method satisfies the bound*

$$\left|e^j\right| \leq C(\Delta t)^2.$$

The bound for the global truncation error of the semi-discrete scheme are given as follows.

Theorem 1. *The global error in the temporal direction satisfies the estimate*

$$\|TE^{j+1}\| \leq C\Delta t, \qquad \forall j \leq T/\Delta t. \tag{7}$$

Proof. The global error estimate at $(j+1)^{th}$ time step is obtained using the local error estimate up to j^{th} time step as follows. For $j \leq T/\Delta t$,

$$\begin{aligned} \left|E_{j+1}\right| = &\left|\sum_{i=1} e_j\right| \leq |e_1| + |e_2| + |e_3| + |e_4| + \cdots + \left|e_j\right| \\ \leq &C_1(j\Delta t)\Delta t \leq C_1 T\Delta t \leq C\Delta t. \end{aligned}$$

Hence, $\|E^{j+1}\| = \max_i \left|Z(s,t_{j+1}) - Z^{j+1}(s)\right|_D \leq C\Delta t$, where C is a positive constant independent of ε and Δt. □

The solution $Z^{j+1}(s)$ of the problem (5) can be decomposed as

$$Z^{j+1}(s) = V^{j+1}(s) + W^{j+1}(s),$$

where $V^{j+1}(s)$ and $W^{j+1}(s)$ are the regular and singular component respectively, and $V^{j+1}(s)$ is the solution of

$$\begin{cases} -\varepsilon \frac{d^2 V^{j+1}(s)}{ds^2} + r(s)V^{j+1}(s) = g_1^{j+1}(s), \quad s \in (0,1), \\ V^{j+1}(0) = V_0^{j+1}(0), \\ V^{j+1}(1) = r(s)^{-1}(1)\left(g^{j+1}(1)\right), \\ \mathscr{K} V^{j+1}(1) = \mathscr{K} V_0^{j+1}(1). \end{cases}$$

And also $W^{j+1}(s)$ is the solution of

$$\begin{cases} -\varepsilon \frac{d^2 W^{j+1}(s)}{ds^2} + r(s)W^{j+1}(s) = 0, & s \in (0,1), \\ W^{j+1} = Z^{j+1}(s) - V_0^{j+1}(s), \\ \mathscr{K} W^{j+1}(1) = \mathscr{K} Z^{j+1}(1) - \mathscr{K} V_0^{j+1}(1), \end{cases}$$

where $V_0^{j+1}(s)$ is the solution of the reduced problem.

Lemma 7. *The derivatives of $V^{j+1}(s)$ and $W^{j+1}(s)$ satisfy the estimates*

$$\left|\frac{d^k V^{j+1}(s)}{ds^k}\right| \le C(1 + \varepsilon^{-\frac{(k-2)}{2}} d_1(s,\alpha)), \qquad s \in (0,1),$$
$$\left|\frac{d^k W^{j+1}(s)}{ds^k}\right| \le \varepsilon^{-\frac{k}{2}} d_1(s,\alpha), \qquad s \in (0,1),$$

for $k = 0,1,2,3,4$, where $d_1(s,\alpha) = e^{-s\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}} + e^{-(1-s)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}}$.

Proof. For the proof, see [17]. □

4.2 Spatial semi-discretization

The construction of the piecewise-uniform mesh, with $N \ge 4$ mesh elements on the interval $[0,1]$, involves partitioning the interval $[0,1]$ into three subintervals: $[0,\mu]$, $(\mu, 1-\mu]$, and $(1-\mu, 1]$. To achieve a piecewise-uniform mesh, we assume $\frac{N}{2}$ mesh elements in the regular region and $\frac{N}{4}$ mesh elements in the boundary layer region. Consequently, the piecewise-uniform mesh is defined as follows:

$$s_i = \begin{cases} 0, \quad i = 0 \\ s_{i-1} + h_i, \quad i = 1,2,3,\ldots,N, \end{cases} \qquad \text{where } h_i = \begin{cases} \frac{4\mu}{N}, & i = 1,2,3,\ldots,\frac{N}{4}, \\ \frac{2(1-2\mu)}{N}, & i = \frac{N}{4}+1,\ldots,\frac{3N}{4}, \\ \frac{4\mu}{N}, & i = \frac{3N}{4}+1,\ldots,N. \end{cases}$$

The transition parameter μ, which separates the boundary between the uniform and non-uniform portions of the mesh, is defined as $\mu = \min\left\{\frac{1}{4},\, 2\sqrt{\varepsilon}\ln(N)\right\}$, where N represents the number of mesh elements in the spatial direction. The subsequent difference formula was employed to discretize the problem:

$$D_s^- Z^{j+1}(s) = \frac{Z_i^{j+1} - Z_{i-1}^{j+1}}{h_i}, \quad D_s^+ Z^{j+1}(s) = \frac{Z_{i+1}^{j+1} - Z_i^{j}}{h_{i+1}}, \quad D_s^+ D_s^- Z^{j+1}(s) = 2\frac{(D_s^+ - D_s^-)}{h_i + h_{i+1}} Z_i^{j+1},$$

where $h_i = s_i - s_{i-1}$, $h_{i+1} = s_{i+1} - s_i$ and Z_i^{j+1} be denoted for the numerical approximation for the exact solution $z(s,t)$ at (s_i, t_{j+1}).

By replacing the second-order derivative with a central difference scheme, we apply an ordinary finite difference approach to the spatial semi-discrete. From (5), the discrete operator $\mathscr{L}^{\Delta t}$ is defined as

$$\mathscr{L}^{\Delta t} Z^{j+1}(s_i) = -\varepsilon D_s^+ D_s^- Z_i^{j+1} + r_i Z_i^{j+1}, \qquad i = 1,2,\ldots,N-1, \tag{8}$$

with initial and boundary conditions

$$\begin{cases} Z_0 = \phi_l(0), \\ Z_i^0 = \phi_b, \qquad\qquad i = 1,2,3,\ldots,N-1, \\ \mathscr{K}^N Z_N^{j+1} = Z_N^{j+1} - \varepsilon \sum_{i=1}^N \dfrac{g_{i-1}Z_{i-1}^{j+1} + 4g_i Z_i^{j+1} + g_{i+1}Z_{i+1}^{j+1}}{3} h_i = \phi_r, \end{cases} \tag{9}$$

where $Z_i = Z(s_i), r_i = r(s_i), g_i = g(s_i)$.
For $i = N$, composite Simpsons $\frac{1}{3}$ integration rule is applied to treat the integral boundary condition $\int_0^1 g(s)z(s,t)ds$, which is given as

$$\begin{aligned} \int_{i=1}^N g(s)z(s,t)ds &\approx \sum_{i=1}^N \frac{g_{i-1}Z_{i-1}^{j+1} + 4g_i Z_i^{j+1} + g_{i+1}Z_{i+1}^{j+1}}{3} h_i. \\ &= \frac{g_0 Z_0^{j+1} + 4g_1 Z_1^{j+1} + g_2 Z_2^{j+1}}{3} h_1 + \cdots + \frac{g_{N-2}Z_{N-2}^{j+1} + 4g_{N-1}Z_{N-1}^{j+1} + g_N Z_N^{j+1}}{3} h_N. \end{aligned} \tag{10}$$

Substituting (10) into (9)(c), we obtain

$$Z_N^{j+1} - \varepsilon \left[\frac{g_0 Z_0^{j+1} + 4g_1 Z_1^{j+1} + g_2 Z_2^{j+1}}{3} h_1 + \cdots + \frac{g_{N-2}Z_{N-2}^{j+1} + 4g_{N-1}Z_{N-1}^{j+1} + g_N Z_N^{j+1}}{3} h_N \right] = \phi_r,$$

which implies that

$$Z_N^{j+1}\left(1 - \varepsilon \frac{g_N}{3} h_N\right) - \varepsilon \left[\frac{g_0 Z_0^{j+1} + 4g_1 Z_1^{j+1} + g_2 Z_2^{j+1}}{3} h_1 + \cdots + \frac{g_{N-2}Z_{N-2}^{j+1} + 4g_{N-1}Z_{N-1}^{j+1}}{3} h_N \right] = \phi_r.$$

As a result, the integral boundary condition at the spatial domain's right end is provided as

$$Z_N^{j+1} = \frac{\varepsilon}{1 - \frac{\varepsilon g_N}{3} h_N} \left[\frac{g_0 Z_0^{j+1} + 4g_1 Z_1^{j+1} + g_2 Z_2^{j+1}}{3} h_1 + \cdots + \frac{g_{N-2}Z_{N-2}^{j+1} + 4g_{N-1}Z_{N-1}^{j+1}}{3} h_N + \phi_r \right].$$

The proposed method satisfies the following semi-discrete maximum principle.

Lemma 8 (Semi-discrete Maximum Principle)**.** *Assume that* $\sum_{i=1}^N \dfrac{g_{i-1} + 4g_i + g_{i+1}}{3} h_i = \rho < 1$ *and* ψ^{j+1} *is any mesh function satisfying* $\psi_0^{j+1} \geq 0$, $\psi_i^{j+1} \geq 0$, $\mathscr{K}^N \psi_N^{j+1} \geq 0$, $\mathscr{L}\Psi_i^{j+1} \geq 0, \forall i \in D^N$, *then* $\psi_i^{j+1} \geq 0$, *for all* $i = 0,1,2,\ldots,N$.

Proof. Define a test function $v_1^{j+1}(s_i) = 1 + s_i$. Note that $v_1^{j+1}(s_i) \geq 0 \quad \forall s_i \in \bar{D}^N$, $\mathscr{L}^N v_1^{j+1}(s_i) > 0$, $\forall s_i \in D^N$, $v_1^{j+1}(s_0) > 0$, and $\mathscr{K}^N v_1^{j+1}(s_N) > 0$. Let

$$\lambda_1 = \max\left\{\frac{-\psi^{j+1}(s_i)}{v_1^{j+1}(s_i)}\right\}, \quad i = 1,2,3,\dots,N.$$

Then there exist s_i^* such that $\psi^{j+1}(s_i^*) + \lambda_1 v_1^{j+1}(s_i^*) = 0$ and $\psi^{j+1}(s_i) + \lambda_1 v_1^{j+1}(s_i) \geq 0$, for all $i = 1,2,3,\dots,N$, $j = 1,2,3,\dots,M-1$. Hence the function $\psi^{j+1}(s_i) + \lambda_1 v_1^{j+1}(s_i)$ attains a minimum value at s_i^*. Assume that the theorem does not hold true, then $\lambda_1 > 0$.

Case 1. $s_i^* = s_0$**,** $\quad 0 < (\psi^{j+1} + \lambda_1 v_1^{j+1})(s_0) = 0.$

Case 2. $s_i^* = s_i$, $\quad i = 1,2,3,\dots,N$**,**

$$0 < \mathscr{L}(\psi^{j+1} + \lambda_1 v_1^{j+1})(s_i^*) = \left(-\frac{\varepsilon}{2} D_s^+ D_s^- + \frac{r_i}{2}\right)\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(s_i) \leq 0.$$

Case 3. $s_i^* = s_N$**,**

$$\begin{aligned} 0 <& \mathscr{K}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(s_N) \\ =& \left(\psi j+1 + \lambda_1 v_1^{j+1}\right)(s_N) - \frac{\varepsilon}{3}\sum_{i=1}^{N}\left[g_{i-1}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(s_{i-1}) + 4g_i\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(s_i)\right]h_i \\ &- \frac{\varepsilon}{3}\sum_{i=1}^{N} g_{i+1}\left(\psi^{j+1} + \lambda_1 v_1^{j+1}\right)(s_{i+1})h_i \leq 0. \end{aligned} \tag{11}$$

This contradicts our assumption. Therefore $\lambda_1 > 0$ is not possible. Hence $\psi_i^{j+1} \geq 0$, $\forall s_i \in \bar{D}^N$. □

An immediate result of the aforementioned semi-discrete maximum principle provided in Lemma 8 is the following discrete stability result.

Lemma 9. *Let ψ_i^{j+1} be any mesh function for $i = 0,1,\dots,N$. Then*

$$\|\psi_i^{j+1}\| \leq \max\left\{\|\psi_0^{j+1}\|, \|\mathscr{K}\psi_N^{j+1}\|, \|\mathscr{L}\psi_i^{j+1}\|\right\}.$$

Proof. One can prove this lemma, by constructing a barrier function

$$\left(\Theta_i^{j+1}\right)^{\pm} = \max\left\{\left|\psi_0^{j+1}\right|, \left|\mathscr{K}\psi_N^{j+1}\right|, \left|\mathscr{L}\psi_i^{j+1}\right|\right\} \pm \psi_i^{j+1}.$$

Then, the results follows by applying semi-discrete maximum principle. □

5 Uniform convergence analysis

We estimate ε-uniform error by decomposing the solution Z_i^{j+1} into smooth and singular components as $Z_i^{j+1} = V_i^{j+1} + W_i^{j+1}$, where V_i^{j+1} is the solution of

$$\begin{cases} \mathscr{L}V_i^{j+1} = g_i^{j+1}, & i = 1,2,\dots,N \\ V_0^{j+1} = \phi_{l_0}^{j+1}, & \\ V_i^{j+1} = \phi_{b_i}^{j+1}, & \\ \mathscr{K}^N V_N^{j+1} = \mathscr{K}V_0^{j+1}(s_N), & \end{cases} \tag{12}$$

and W_i^{j+1} must satisfy

$$\begin{cases} \mathscr{L}^N W_i^{j+1} = 0, & i = 1,2,\ldots,N \\ W_i^{j+1} = Z_i^{j+1} - v_i^{j+1}, \\ \mathscr{K}^N W_N^{j+1} = \mathscr{K}^N Z_N^{j+1} - \mathscr{K}^N V_N^{j+1}. \end{cases} \tag{13}$$

Theorem 2. *Suppose $Z^{j+1}(s)$ and Z_i^{j+1} are the solutions of the problem (5)-(6) and (8)-(9) respectively, and assume that the coefficients $r(s), a(s), g(s) \in C^{4+\alpha_1}(0,1)$, and the boundary conditions satisfy $\phi_l^{j+1} \in C^{3+\alpha_1/2}(0,T)$, $\phi_b^{j+1} \in C^{6+\alpha_1}(\Gamma_b)$, $\phi_r^{j+1} \in (0,T)$, where $\alpha_1 \in (0,1)$. Then we have*

$$\sup_{0<\varepsilon\leq 1} \|Z^{j+1}(s) - z_i^{j+1}\| \leq CN^{-2}\ln^2 N,$$

and the error can be written in the form of

$$Z_i^{j+1} - Z^{j+1}(s_i) = \left(V_i^{j+1} - V^{j+1}(s_i)\right) + \left(W_i^{j+1} - W^{j+1}(s_i)\right).$$

Proof. Error estimates for both the smooth and singular components are established independently. The subsequent error estimates pertain to the smooth component:

At the point $s_i = s_N$,

$$\begin{aligned}
\mathscr{K}^N\left(V_i^{j+1} - V^{j+1}(s_i)\right) &= \mathscr{K}^N V_i^{j+1} - \mathscr{K}^N V^{j+1}(s_i) \\
&= \phi_r - \mathscr{K}^N V_i^{j+1} \\
&= \mathscr{K} V^{j+1}(s_i) - \mathscr{K}^N V^{j+1}(s_i) \\
&= V^{j+1}(s_i) - \varepsilon \int_{s_0}^{s_N} g(s) V^{j+1}(s) ds - V^{j+1}(s_i) \\
&\quad + \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}V_{i-1}^{j+1} + 4g_i V_i^{j+1} + g_{i+1}V_{i+1}^{j+1}}{3} h_i \\
&= \varepsilon \frac{g_0 V_0^{j+1} + 4g_1 V_1^{j+1} + g_2 V_2^{j+1}}{3} h_1 + \cdots \\
&\quad + \varepsilon \frac{g_{N-1}V_{N-1}^{j+1} + 4g_N V_N^{j+1} + g_{N+1}V_{N+1}^{j+1}}{3} h_N \\
&\quad - \varepsilon \int_{s_0}^{s_1} g(x) V^{j+1}(s) ds - \cdots - \varepsilon \int_{s_N}^{s_{N+1}} g(s) V^{j+1}(s) ds \\
&= -\varepsilon \frac{h_1^4}{90} g^{iv}(\xi_1) \frac{\partial^4 V^{j+1}}{\partial s^4}(\xi_1) - \cdots - \frac{h_1^4}{90} g^{iv}(\xi_N) \frac{\partial^4 V^{j+1}(\xi_N)}{\partial s^4}, \\
\left|\mathscr{K}^N\left(V_i^{j+1} - V^{j+1}(x_i)\right)\right| &= \left| C\varepsilon \left(h_1^4 \frac{\partial^4 V^{j+1}}{\partial s^4}(\xi_1) + \cdots + h_N^4 \frac{\partial^4 V^{j+1}}{\partial s^4}(\xi_N) \right) \right| \\
&\leq C\varepsilon \left(h_1^4 \frac{\partial^4 V^{j+1}}{\partial s^4}(\xi_1) + \cdots + h_N^4 \frac{\partial^4 V^{j+1}}{\partial s^4}(\xi_N) \right) \\
&\leq CN^{-2},
\end{aligned}$$

where $s_{i-1} \leq \xi_i \leq s_i, \quad 1 \leq i \leq N$, and C is chosen to be an arbitrary positive constant. From the discrete and difference equations, we have

$$\mathscr{L}\left(V_i^{j+1} - V^{j+1}(s_i)\right) = g^{j+1} - \mathscr{L}V^{j+1}(s_i) = \left(\mathscr{L} - \mathscr{L}^N\right)V^{j+1}(s_i).$$

Then, it implies that

$$\mathscr{L}^N\left(V_i^{j+1} - V^{j+1}(s_i)\right) = -\varepsilon\left(\frac{\partial^2}{\partial s^2} - D_s^+ D_s^-\right)V^{j+1}(s_i).$$

By using the result obtained in [18], we have the following classical estimates at $s_i \in D^N$:

$$\begin{aligned}\mathscr{L}^N\left(V_i^{j+1} - V^{j+1}(s_i)\right) &\leq \begin{cases} \frac{\varepsilon}{3}(s_{i+1} - s_{i-1})\,\|\frac{\partial^3 V^{j+1}(s)}{\partial s^3}\|, & \text{if } s_i = \mu \text{ or } s_i = 1-\mu, \\ \frac{\varepsilon}{12}(s_i - s_{i-1})^2\,\|\frac{\partial^4 V^{j+1}(s)}{\partial s^4}\|, & \text{otherwise,} \end{cases} \\ &\leq C\begin{cases} \sqrt{\varepsilon}\,N^{-1}, & \text{if } \; s_i = \mu \; \text{ or } \; s_i = 1-\mu, \\ N^{-2}, & \text{otherwise.} \end{cases}\end{aligned}$$

Now, let us define a barrier function as

$$\Phi^{j+1}(s_i) = C\frac{\mu}{\sqrt{\varepsilon}}\vartheta(s_i)N^{-2} + N^{-2},$$

where ϑ is a piecewise linear polynomial

$$\vartheta(s) = \begin{cases} \frac{s}{\mu}, & \text{for } \; 0 \leq s \leq \mu, \\ 1, & \text{for } \; \mu \leq s \leq 1-\mu, \\ \frac{1-s}{\mu}, & \text{for } \; 1-\mu \leq s \leq 1. \end{cases}$$

Then, $\forall\, s_i, i = 1, 2, 3, \ldots, N$,

$$0 \leq \Phi^{j+1}(s_i) \leq CN^{-2}\ln(N).$$

And

$$\mathscr{L}^N\Phi^{j+1}(s_i) \geq \begin{cases} C\sqrt{\varepsilon}N^{-1} + N^{-2}, & \text{if } \; s_i = \mu \; \text{ or } \; s_i = 1-\mu, \\ CN^{-2}, & \text{otherwise,} \end{cases}$$

where the observations that $\frac{\mu}{\sqrt{\varepsilon}} \leq 2\ln(N)$ and

$$\mathscr{L}^N\vartheta(s_i) = \begin{cases} \frac{\varepsilon N}{\mu} + r(s_i), & \text{if } \; s_i = \mu \; \text{ or } \; s_i = 1-\mu, \\ r(s_i)\vartheta(s_i), & \text{otherwise.} \end{cases}$$

Also for all $s_i \in \Gamma^N$, we obtain $\phi_r^{j+1}(s_i) \geq 0$, which implies that $\mathscr{K}^N\vartheta^{j+1}(s_i) \geq 0$. Define a comparison function

$$\pi^{\pm}(s_i) = \Phi(s_i) \pm \left(V_i^{j+1} - V^{j+1}(s_i)\right).$$

For each point $s_i,\ i=1,2,\ldots,N$, we have $\mathscr{L}^N\pi^{\pm}(s_i)\geq 0$. Then, from the semi-discrete maximum principle, $\pi^{\pm}(s_i)\geq 0, \forall i=1,2,\ldots,N$, and $V_i^{j+1}-V^{j+1}(s_i)\leq \Phi(s_i)\leq CN^{-2}\ln(N)$. Hence,

$$\left|V_i^{j+1}-V^{j+1}(s_i)\right|\leq CN^{-2}\ln(N). \tag{14}$$

To estimate the singular component error, we decompose $W^{j+1}(s)$ into $W_l^{j+1}(s)$ and $W_r^{j+1}(s)$ as follows:

$$\begin{cases}\mathscr{L}^N W_l^{j+1}(s_i)=0, & i=1,2,\ldots,N,\\ W_l^{j+1}(s_i)=\phi_l^{j+1}(s_i)-v_0^{j+1}(s_i), & s_i\in\Gamma_l^N,\\ W_l^{j+1}(s_i)=0, & s_i\in\Gamma_r^N,\\ W_l^{j+1}(s_i)=0, & s_i\in\Gamma_b^N,\end{cases}$$

and

$$\begin{cases}\mathscr{L}^N W_r^{j+1}(s_i)=0, & i=1,2,\ldots,N,\\ W_r^{j+1}(s_i)=0, & s_i\in\Gamma_l^N,\\ \mathscr{K}^N W_r^{j+1}(s_i)=\mathscr{K}^N W^{j+1}(s_i), & s_i\in\Gamma_r^N,\\ W_r^{j+1}(s_i)=0, & s_i\in\Gamma_b^N.\end{cases}$$

The error of singular component is equivalent to

$$W_i^{j+1}-W^{j+1}(s_i)=\left(W_{l_i}^{j+1}-W_l{}^{j+1}(s_i)\right)+\left(W_{r_i}^{j+1}-W_r{}^{j+1}(s_i)\right).$$

$$\mathscr{L}^N\left(W_i^{j+1}-W^{j+1}(s_i)\right)\leq -\varepsilon\left(\frac{\partial^2}{\partial s^2}-D^+D^-\right)W^{j+1}(s_i).$$

By using a classical estimate given in [18], at each point $s_i,\ \ i=1,2,\ldots,N$, it follows that

$$\mathscr{L}^N\left(W_i^{j+1}-W^{j+1}(s_i)\right)\leq C(N^{-1}\ln(N))^2, \qquad \text{for } \ i=1,2,\ldots,N.$$

The error estimate for $W_{r_i}^{j+1}-W_r{}^{j+1}(s_i)$ is given as follows. The explanation depends on whether $\mu=\frac{1}{4}$ or $\mu=2\sqrt{\varepsilon}\ln(N)$.

Case-I: When $\mu=\frac{1}{4}$, the mesh is uniform and $\mu=2\sqrt{\varepsilon}\ln(N)\geq\frac{1}{4}$. It is clear that $s_i-s_{i-1}=N^{-1}$ and $\varepsilon^{-\frac{1}{2}}\leq C\ln(N)$. From [18], we have

$$\begin{aligned}\mathscr{K}^N\left(W_{r_i}^{j+1}-W_r{}^{j+1}(s_i)\right)&=\mathscr{K}^N W_{r_i}^{j+1}-\mathscr{K}^N W_r{}^{j+1}(s_i)\\ &=\phi_r-\mathscr{K}^N W_r{}^{j+1}(s_i)\\ &=\mathscr{K}W_r{}^{j+1}(s_i)-\mathscr{K}^N W_r{}^{j+1}(s_i),\\ \left|\mathscr{K}^N\left(W_{r_i}^{j+1}-W_r{}^{j+1}(s_i)\right)\right|&\leq C\varepsilon\left(h_1^4\frac{\partial^4}{\partial^4}W_r{}^{j+1}(\xi_i)+\cdots+h_N^4\frac{\partial^4}{\partial^4}W_r{}^{j+1}(\xi_N)\right)\\ &\leq C\varepsilon^{-1}\left(h_1^4+\cdots+h_N^4\right)\\ &\leq CN^{-2}\ln^2 N,\end{aligned}$$

where $s_{i-1} \leq \xi_i \leq s_i$. By applying semi-discrete uniform stability in Lemma 9, we obtain

$$\left| W_{r_i}^{j+1} - W_r^{j+1}(s_i) \right| \leq CN^{-2}\ln^2 N.$$

Case-II: When $\mu < \frac{1}{4}$, the mesh is piecewise uniform in the subinterval $[\mu, 1-\mu]$ and the mesh elements are $2(1-2\mu)/N$, while the rest of the intervals $[0,\mu]$ and $[1-\mu,1]$ have mesh elements of $4\mu/N$. By [18], we have

$$\mathscr{K}^N \left(W_{r_i}^{j+1} - W_r^{j+1}(x_i) \right) \leq CN^{-2}\ln^2 N,$$

and

$$\begin{aligned} \left| \mathscr{K}^N \left(W_{r_i}^{j+1} - W_r^{j+1}(s_i) \right) \right| \leq & C\varepsilon \left(h_1^4 \frac{\partial^4}{\partial^4} W_r^{j+1}(\xi_i) + \cdots + h_N^4 \frac{\partial^4}{\partial^4} W_r^{j+1}(\xi_N) \right) \\ \leq & C\varepsilon^{-1} \left(h_1^4 + \cdots + h_N^4 \right) \\ \leq & CN^{-2}\ln^2 N, \end{aligned}$$

where $s_{i-1} \leq \xi_i \leq s_i$. By applying Lemma 7 and semi-discrete uniform stability in Lemma 9, we obtain

$$\left| W_{r_i}^{j+1} - W_r^{j+1}(s_i) \right| \leq CN^{-2}\ln^2 N. \tag{15}$$

Similar reasoning are used to compute W_l's error estimate. By combining equations (14) and (15), we have

$$\left| Z_i^{j+1} - Z^{j+1}(s_i) \right| \leq C \left(N^{-2}\ln(N) + N^{-2}\ln^2 N \right) \leq CN^{-2}\ln^2 N. \tag{16}$$

□

The semidiscrete error estimate produced in (7) and (16) is used to summarize the outcomes of this work, which is concluded by the following theorem.

Theorem 3. *The error estimate for the solution of the continuous and fully discrete problems is provided by*

$$\sup_{\varepsilon} \max_{i,j} \left| z(s,t) - Z_i^{j+1} \right| \leq C \left(N^{-2}\ln^2 N + \Delta t \right),$$

where $z(s,t)$ and Z_i^{j+1} are the solutions of problem (1) and (8)-(9) respectively.

Proof. The proof follows from Theorem 1 and Theorem 2. □

Theorem 3 establishes the ε-uniform convergence of the developed method with a first-order convergence in the temporal direction and an almost second-order convergence in the spatial direction.

6 Numerical examples, results and discussions

In this section, we illustrate the proposed method by using two numerical examples.

Example 1.

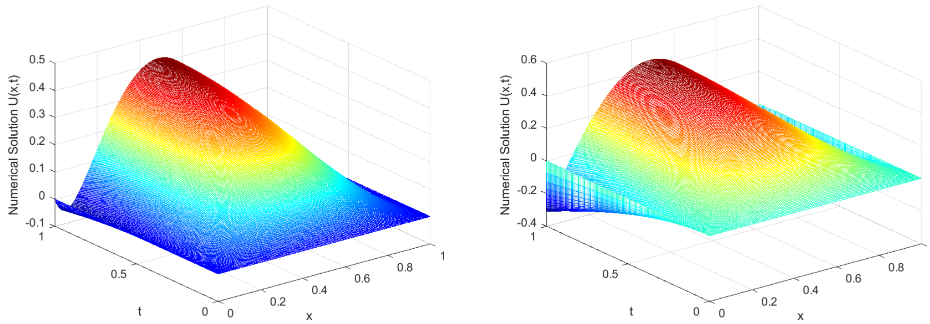
$$\begin{cases} \frac{\partial z(s,t)}{\partial t} - \varepsilon \frac{\partial^2 z(s,t)}{\partial s^2} + \frac{1+s^2}{2} z(s,t) = e^{-t} - 1 + \sin(\pi s), & (s,t) \in (0,1) \times (0,1] \\ z(s,0) = 0, & s \in [0,1], \\ z(0,t) = 0, & t \in (0,1], \\ \mathscr{K} z(1,t) = z(1,t) - \varepsilon \int_0^1 \frac{s}{6} z(s,t) ds = 0, & t \in (0,1]. \end{cases}$$

Example 2.

$$\begin{cases} \frac{\partial z(s,t)}{\partial t} - \varepsilon \frac{\partial^2 z(s,t)}{\partial s^2} + \frac{1+s^2}{2} z(s,t) = t^3, & (s,t) \in (0,1) \times (0,1] \\ z(s,0) = 0, & s \in [0,1], \\ z(0,t) = 0, & t \in (0,1], \\ \mathscr{K} z(1,t) = z(1,t) - \varepsilon \int_0^1 \cos(s) z(s,t) ds = 0, & t \in (0,1]. \end{cases}$$

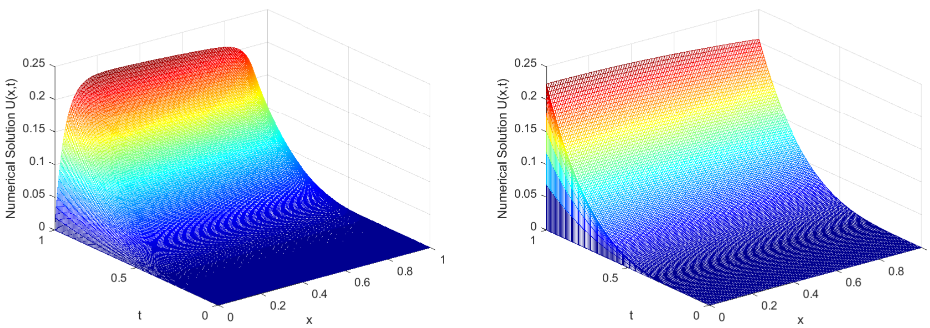
Since the analytical solution for the given model problems are unknown, we devised the double mesh principle to determine the maximum pointwise error for the presented method. Let $Z^{N,\Delta t}$ denote a calculated solution of a problem with mesh points N and a time step size Δt, and $Z_{i,j}^{2N,\Delta t/2}$ be a calculated solution with double mesh points of $2N$ and half of the time step size $\Delta t/2$. The maximum absolute error is calculated as $E_\varepsilon^{N,\Delta t} = \max_{i,j} |Z_{i,j}^{N,\Delta t} - Z_{i,j}^{2N,\Delta t/2}|$ and the parameter uniform error estimate is computed $E^{N,\Delta t} = \max_\varepsilon (E_\varepsilon^{N,\Delta t})$. The rate of convergence of the developed numerical scheme is calculated as $R_\varepsilon^{N,\Delta t} = \log_2(E_\varepsilon^{N,\Delta t}) - \log_2(E_\varepsilon^{2N,\Delta t/2})$, and the parameter rate of convergence computed $R^{N,\Delta t} = \log_2(E^{N,\Delta t}) - \log_2(E^{2N,\Delta t/2})$.

The solutions to the above two examples exhibit strong parabolic boundary layers at the left and right ends of the domain. Figures 1 and 2 indicate the surface graphs for the numerical solution of Examples 1 and 2 respectively, demonstrating the existence of a boundary layer formation on the left and right sides of the spatial domain for different values of the perturbation parameter ε. Figures 3 and 4 show numerical solutions profiles for different time levels respectively for Examples 1 and Examples 2. Tables 1 and 2 show the maximum pointwise errors and the parameter uniform rate of convergence for the proposed method for Examples 1 and 2, for different values of the perturbation parameter ε, the mesh number N, and the time step size Δt. From these tables, one can observe that the presented method is uniformly convergent independent of the perturbation parameter ε, with almost second-order uniform convergence in the spatial direction. Tables 3 and 4 show the comparison of the developed method with a method that exists in the literature for Examples 1 and 2 respectively. The parameter-uniform convergence is also confirmed by the loglog plot drawn in figures 5. From these figures, one can observe that the maximum absolute error decreases monotonically as N increases.



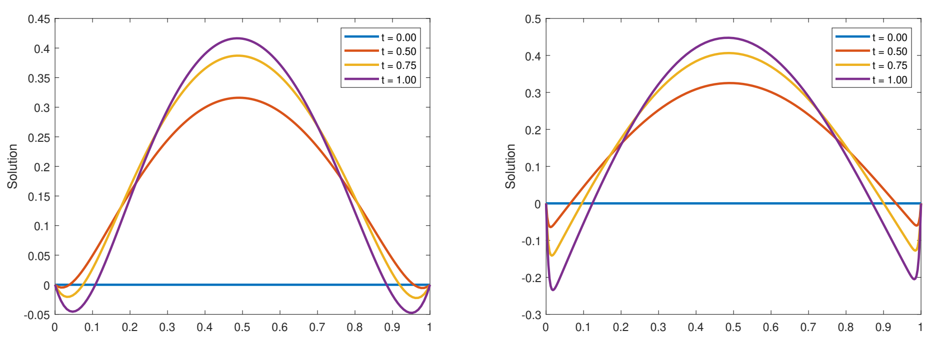
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-12}$

Figure 1: Numerical solution behavior for Example 1.



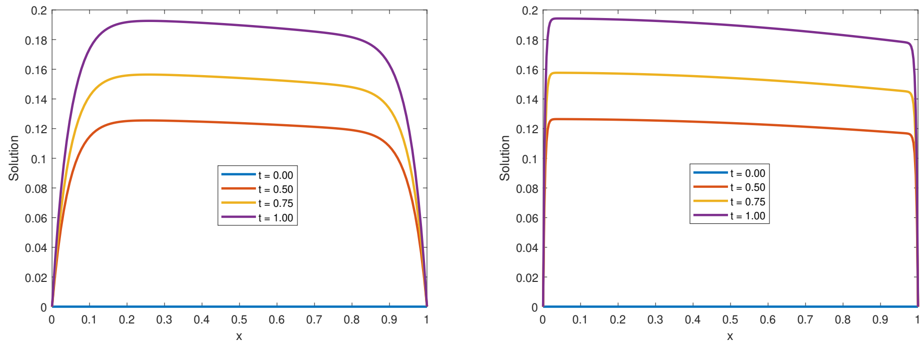
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-12}$

Figure 2: Numerical solution behavior for Example 2.



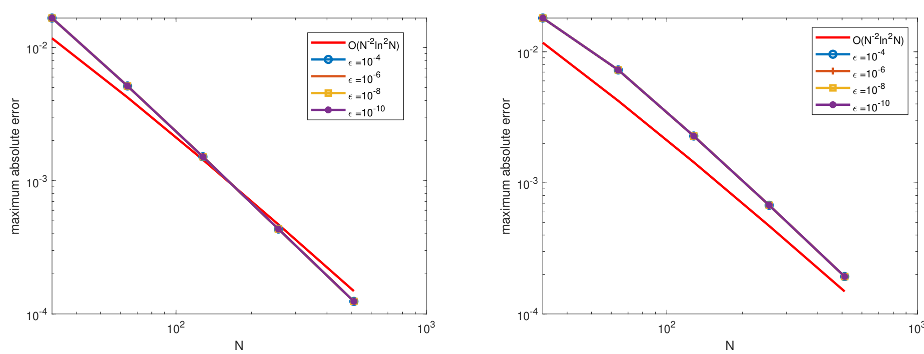
(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-4}$

Figure 3: Numerical solution behavior of Example 1 for different time level.



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-4}$

Figure 4: Numerical solution behavior for Example 2 for different time level.



(a) (b)

Figure 5: Log-Log plot of maximum absolute error for different values of ε (a) for Example 1, (b) for Example 2.

Table 1: Maximum absolute error of Example 1.

ε $\downarrow$	$N = 32$ $\Delta t = 0.1$	$N = 64$ $\Delta t = 0.1/4$	$N = 128$ $\Delta t = 0.1/4^2$	$N = 256$ $\Delta t = 0.1/4^3$	$N = 512$ $\Delta t = 0.1/4^4$
10^{-3}	1.5452e-02	4.7360e-03	1.2406e-03	3.1235e-04	7.8226e-05
10^{-4}	1.6278e-02	5.0844e-03	1.4961e-03	4.3023e-04	1.2322e-04
10^{-5}	1.6535e-02	5.1259e-03	1.5084e-03	4.3288e-04	1.2391e-04
10^{-6}	1.6616e-02	5.1390e-03	1.5123e-03	4.3372e-04	1.2412e-04
10^{-7}	1.6642e-02	5.1431e-03	1.5135e-03	4.3398e-04	1.2419e-04
10^{-8}	1.6650e-02	5.1444e-03	1.5139e-03	4.3406e-04	1.2421e-04
10^{-9}	1.6653e-02	5.1448e-03	1.5140e-03	4.3409e-04	1.2422e-04
10^{-10}	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$E^{N,\Delta t}$	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$R^{N,\Delta t}$	1.6946	1.7647	1.8024	1.8051	-

Table 2: Maximum absolute error of Example 2.

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
10^{-3}	1.8099e-02	7.1575e-03	2.0069e-03	5.1704e-04	1.3036e-04
10^{-4}	1.8099e-02	7.2697e-03	2.2771e-03	6.7485e-04	1.9289e-04
10^{-5}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-6}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-7}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-8}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-8}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-9}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
10^{-10}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10^{-20}	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$E^{N,\Delta t}$	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$R^{N,\Delta t}$	1.3159	1.6747	1.7545	1.8068	-

Table 3: Comparison of ε-uniform error $(E^{N,\Delta t})$ and ε-uniform rate of convergence $(E^{N,\Delta t})$ for Example 1.

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
Present Method					
$E^{N,\Delta t}$	1.6654e-02	5.1450e-03	1.5141e-03	4.3410e-04	1.2422e-04
$R^{N,\Delta t}$	1.6946	1.7647	1.8024	1.8051	-
Method in [26]					
$E^{N,\Delta t}$	1.2294e-02	3.3054e-03	8.4694e-04	2.1381e-04	5.3681e-05
$R^{N,\Delta t}$	1.8951	1.9645	1.9859	1.9938	-

Table 4: Comparison of ε-uniform error $(E^{N,\Delta t})$ and ε-uniform rate of convergence $(E^{N,\Delta t})$ for Example 2.

ε $\downarrow$	$N=32$ $\Delta t=0.1$	$N=64$ $\Delta t=0.1/4$	$N=128$ $\Delta t=0.1/4^2$	$N=256$ $\Delta t=0.1/4^3$	$N=512$ $\Delta t=0.1/4^4$
Present Method					
$E^{N,\Delta t}$	1.8099e-02	7.2697e-03	2.2771e-03	6.7486e-04	1.9289e-04
$R^{N,\Delta t}$	1.3159	1.6747	1.7545	1.8068	-
Method in [26]					
$E^{N,\Delta t}$	1.5809e-02	5.4540e-03	1.4696e-03	3.7419e-04	9.3970e-05
$R^{N,\Delta t}$	1.5354	1.8919	1.9736	1.9935	-

7 Conclusions

This paper introduces a robust and uniformly convergent numerical method for solving a class of singularly perturbed partial differential equations with integral boundary conditions. The proposed approach utilizes a finite difference method on a rectangular piecewise uniform Shishkin mesh for the spatial direction and employs the implicit Euler method for the temporal direction. The integration of the integral boundary condition is achieved through the composite Simpson's $\frac{1}{3}$ rule. Rigorous investigations into the uniform stability and convergence of the method affirm its robustness. Notably, the presented numerical method exhibits uniform convergence, showcasing its independence from the perturbation parameter ε. Furthermore, our analysis demonstrates that the developed method achieves almost second-order uniform convergence in the spatial direction and first-order convergence in the temporal direction. This attests to the method's efficiency in capturing accurate solutions while maintaining computational stability. To validate the practicality and effectiveness of the proposed method, two test examples are meticulously examined. The obtained numerical results not only align with the theoretical estimations but also highlight the method's reliability in capturing the intricate features of the considered singularly perturbed problems.

In summary, this research contributes a powerful numerical tool that not only addresses the challenges posed by singularly perturbed partial differential equations with integral boundary conditions but also ensures robustness, accuracy, and efficiency in a uniformly convergent manner. The developed method stands as a valuable asset for researchers and practitioners engaged in the numerical treatment of such complex problems.

Acknowledgements

We extend our sincere gratitude to the editor and reviewers for their valuable and constructive comments, which greatly contributed to the improvement and refinement of this work. Their insightful feedback has been instrumental in enhancing the overall quality of the manuscript.

References

[1] D. Bahuguna, J. Dabas, *Existence and uniqueness of a solution to a semilinear partial delay differential equation with an integral condition*, Nonlinear Dyn. Syst. Theory **8** (2008) 7–19.

[2] A. Bouziani, *Mixed problem with boundary integral conditions for a certain parabolic equation*, Appl. Math. Stoch. Anal. **9** (1996) 323–330.

[3] J.R. Cannon, *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math. **21** (1963) 155–160.

[4] W.A. Day, *Extensions of a property of the heat equation to linear thermoelasticity and other theories*, Quart. Appl. Math. **40** (1982) 319–330.

[5] H.G. Debela, G.F. Duressa, *Exponentially fitted finite difference method for singularly perturbed delay differential equations with integral boundary condition*, Int. J. Appl. Sci. Eng. **11** (2019) 476–493.

[6] H.G. Debela, G.F. Duressa, *Accelerated fitted operator finite difference method for singularly perturbed delay differential equations with non-local boundary condition*, J. Egyptian Math. Soc. **28** (2020) 16.

[7] H.G. Debela, G.F. Duressa, *Uniformly convergent numerical method for singularly perturbed convection-diffusion type problems with nonlocal boundary condition*, Int. J. Numer. Methods Fluids **92** (2020) 1914–1926.

[8] S. Elango, A. Tamilselvan, R. Vadivel, N. Gunasekaran, H. Zhu, J. Cao, X. Li, *Finite difference scheme for singularly perturbed reaction diffusion problem of partial delay differential equation with nonlocal boundary condition*, Adv. Differ. Equ. **2021** (2021) 151.

[9] M. Feng, D. Ji, W. Ge, *Positive solutions for a class of boundary-value problem with integral boundary conditions in banach spaces*, J. Comput. Appl. Math. **222** (2008) 351–363.

[10] W.T. Gobena, G.F. Duressa, *Parameter-uniform numerical scheme for singularly perturbed delay parabolic reaction diffusion equations with integral boundary condition*, Int. J. Differ. Equ. Appl. **2021** (2021) 9993644.

[11] W.T. Gobena, G.F. Duressa, *Fitted operator average finite difference method for singularly perturbed delay parabolic reaction diffusion problems with non-local boundary conditions*, Tamkang J. Math. **54** (2023) 293–312.

[12] W.S. Hailu, G.F. Duressa, *Parameter-uniform cubic spline method for singularly perturbed parabolic differential equation with large negative shift and integral boundary condition*, Res. Math. **9** (2022) 2151080.

[13] M. Kudu, G. Amirali, G. Amiraliyev, *Uniform numerical approximation for parameter dependent singularly perturbed problem with integral boundary condition*, Miskolc Math. Notes. **19** (2018) 337–353.

[14] M. Kudu, G.M. Amiraliyev, *Finite difference method for a singularly perturbed differential equations with integral boundary condition*, Int. J. Math. Comput. **26** (2015) 71–79.

[15] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Uralceva, *Linear and Quasi- Linear Equations of Parabolic Type*, American Mathematical Society, 1968.

[16] H. Li, F. Sun, *Existence of solutions for integral boundary value problems of second-order ordinary differential equations*, Bound. Value Probl. **2012** (2012) 147.

[17] M. Manikandan, N. Shivaranjani, J. Miller, S. Valarmathi, *A parameter- uniform numerical method for a boundary value problem for a singularly perturbed delay differential equation*, Adv. Appl. Math. (2014) 71–88.

[18] J.J. Miller, E. Oriordan, G.I. Shishkin, *Fitted Numerical Methods For Singular Perturbation Problems: Error Estimates In The Maximum Norm For Linear Problems In One And Two Dimensions*, World World Scientific, 1996.

[19] V. Raja, A. Tamilselvan, *Fitted finite difference method for third order singularly perturbed convection diffusion equations with integral boundary condition*, Arab J. Math. Sci. **25** (2019) 231–242.

[20] V. Raja, A. Tamilselvan, *Numerical method for a system of singularly perturbed reaction diffusion equations with integral boundary conditions*, Int. J. Appl. Comput. Math. **5** (2019) 77.

[21] E. Sekar, A. Tamilselvan, *Finite difference scheme for third order singularly perturbed delay differential equation of convection diffusion type with integral boundary condition*, J. Appl. Math. Comput. **61** (2019) 73–86.

[22] E. Sekar, A. Tamilselvan, *Singularly perturbed delay differential equations of convectiondiffusion type with integral boundary condition*, J. Appl. Math. Comput. **59** (2019b) 701–722.

[23] E. Sekar, A. Tamilselvan, *Third order singularly perturbed delay differential equation of reaction diffusion type with integral boundary condition*, J. Appl. Math. Comput. Mech. **18** (2019) 99–110.

[24] G.M. Wondimu, M.M. Woldaregay, G.F. Duressa, T.G. Dinka, *Exponentially fitted numerical method for solving singularly perturbed delay reaction-diffusion problem with nonlocal boundary condition*, BMC Res. Notes. **16** (2023) 94.

[25] G.M. Wondimu, T.G. Dinka, M.M. Woldaregay, G.F. Duressa, *Fitted mesh numerical scheme for singularly perturbed delay reaction diffusion problem with integral boundary condition*, Comput. methods differ. equ. **11** (2023) 478-494.

[26] G.M. Wondimu, M.M. Woldaregay, T.G. Dinka, G.F. Duressa, *Numerical treatment of singularly perturbed parabolic partial differential equations with nonlocal boundary condition*, Front. Appl. Math. Stat. **8** (2022) 1005330.