Journal of Mathematical Modeling
Vol. 12, No. 1, 2024, pp. 145–155. Research Article

JMM

Moore-Penrose inverse of an interval matrix and its application

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Abstract. In this paper, we introduce the concept of Moore-Penrose inverse of a rectangular interval matrix based on a modified interval arithmetic. We determine the Moore-Penrose inverse in such a way that it satisfies all the four criteria similar to the real case. Also, we use the Moore-Penrose inverse for solving rectangular interval linear systems, algebraically.

Keywords: Inverse matrix, Moore-Penrose inverse, rectangular linear system.
AMS Subject Classification 2010: 15A09, 15A24.

1 Introduction

There are numerous applications of matrices both in mathematics and other sciences. Matrices play vital roles in solving problems related to eigenvalues and eigenvectors, robotics, projection of three-dimensional images into two-dimensional screens, creating a realistic decreeing motion, quantum mechanics and electrical circuits, see [10, 21, 22]. Some of them merely take advantage of the compact representation of a set of numbers in a matrix. But due to the rounding and measurement errors, often we do not know the exact value of the measured quantities. In this case a common approach is using interval numbers which include these quantities, safety. Using this interval parameters in the structure of matrices, creates interval matrices.

Hansen and Smith [9] started the use of interval arithmetic in matrix computations. After this motivation, several authors such as Alefeld and Herzberger [1], Hansen [8], Jaulin et al. [14], Neumaier [19], Rohn [24], and Ganesan [7] studied interval matrices.

Matrices play an important role in forming the structure of system of linear equations. Systems of linear equations appear in various fields of science such as computer science, physics, technology,

*Corresponding author
Received: 14 July 2023 / Revised: 21 November 2023 / Accepted: 29 Dcember 2023
DOI: 10.22124/JMM.2023.24972.2219

business, engineering and economics. However, many applications deal with data that are not deterministic and should be determined by physical measurements. But due to the measurement errors, the measured parameters are accompanied by errors and uncertainties. One approach for uncertainty quantification is to consider interval parameters. So we will have the system of interval linear equations. In the existing literature, often interval linear systems with square coefficient matrix have been considered, see [2–6, 11–13, 17, 23, 26]. In contrast, interval linear systems with rectangular coefficient matrix have been given less attention. Here, we want to consider the rectangular interval linear systems and find the exact (algebraic) solution of them, using the Moore-Penrose inverse of the interval coefficient matrix under the modified interval arithmetic.

The rest of the paper is organized as follows. Section 2 gives an overview of the generalized intervals and modified interval arithmetic. In Section 3, we introduce the Moore-Penrose inverse of an interval matrix and then present an approach for finding the exact solution of a rectangular interval linear system under modified interval arithmetic introduced in Section 2. Finally, in Section 4 we complete the paper with a brief conclusion.

2 Preliminaries

In this paper, boldface letters stand for the interval quantities and ordinary letters denote the real quantities. The set of real interval numbers is denoted by $\mathbb{IR}$ and $\mathbb{IR}^{m\times n}$ stands for the set of m-by-n real interval matrices. If $S \subseteq \mathbb{R}^{m\times n}$ is a bounded set of real matrices, then the interval hull of S is denoted by $\square S$ and is defined as

$$\square S = \cap\{X \in \mathbb{R}^{m\times n} : X \supseteq S\},$$

in other words, $\square S$ is the tightest interval matrix which encloses S.

2.1 Interval linear systems

A system of interval linear equations with coefficient matrix $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and right-hand side vector $\mathbf{b} \in \mathbb{IR}^m$ is denoted by

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

and is interpreted as the family of the system of linear equations

$$Ax = b, \quad A \in \mathbf{A}, \ \ b \in \mathbf{b}.$$

The system of interval linear equations (1) frequently appears in the cases when the components of the input data are accompanied by some errors. These types of systems appear for instance in circuit analysis therein each circuit consists of resistance, inductance and capacitance and has been mathematically modeled as a system of linear equations, but due to the uncertainties in the problem, it should be modeled as an interval system of equations, see [25]. Also interval system of equations appears in automatic control, social sciences, astrophysics, traffic control and expert systems to ergonomics, economics, robotics and finding numerical solution of the boundary value problems, see [16, 27].

The solution set of (1) is defined as

$$\Xi(\mathbf{A},\mathbf{b}) := \{x \in \mathbb{R}^n : (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b)\}.$$

Often the solution set of an interval system of linear equations is very complicated, so we are interested in finding some enclosures for $\Xi(\mathbf{A},\mathbf{b})$. The interval vector with smallest radius containing $\Xi(\mathbf{A},\mathbf{b})$ is the hull of the solution set and is denoted by $\mathbf{A}^H\mathbf{b}$, i.e.,

$$\mathbf{A}^H\mathbf{b} := \square\Xi(\mathbf{A},\mathbf{b}). \tag{2}$$

If $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is a regular matrix, i.e., every matrix within $\mathbf{A}$ is nonsingular then the inverse of $\mathbf{A}$ is defined as

$$\mathbf{A}^{-1} := \square\{A^{-1} : A \in \mathbf{A}\}, \tag{3}$$

that is $\mathbf{A}^{-1}$ is the smallest interval matrix containing the set $\{A^{-1} : A \in \mathbf{A}\}$. If $\mathbf{A} \in \mathbb{IR}^{n\times n}$ is regular then we have

$$\mathbf{A}^H\mathbf{b} \subseteq \mathbf{A}^{-1}\mathbf{b},$$

therein $\mathbf{A}^{-1}\mathbf{b}$ denotes the multiplication of $\mathbf{A}^{-1}$ by $\mathbf{b}$ which is calculated by the operations of classical interval arithmetic.

If $\mathbf{A} \in \mathbb{IR}^{m\times n}$ is a rectangular matrix with $m < n$ then $\Xi(\mathbf{A},\mathbf{b})$ is either empty or unbounded and $\mathbf{A}^H\mathbf{b}$ is not defined. If $m \geq n$ and $\mathbf{A}$ is regular then $\Xi(\mathbf{A},\mathbf{b})$ may still be empty, so that $\mathbf{A}^H\mathbf{b}$ is not necessarily an interval matrix. However, sometimes it is more appropriate to consider the following family of least square problems for all $A \in \mathbf{A}$ and $b \in \mathbf{b}$:

$$\text{Find } x \in \mathbb{R}^n \text{ such that } ||Ax-b||_2 \text{ is minimum.} \tag{4}$$

The solution set (4) is the set

$$\Xi^L(\mathbf{A},\mathbf{b}) := \{x \in \mathbb{R}^n : (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(||Ax-b||_2 \text{ is minimum})\}.$$

Similar to $\Xi(\mathbf{A},\mathbf{b})$, the structure of $\Xi^L(\mathbf{A},\mathbf{b})$ is very complicated. The hull of $\Xi^L(\mathbf{A},\mathbf{b})$ is denoted by $\mathbf{A}^L\mathbf{b}$, i.e.,

$$\mathbf{A}^L\mathbf{b} := \square\Xi^L(\mathbf{A},\mathbf{b}). \tag{5}$$

By the above definitions, for $\mathbf{A} \in \mathbb{IR}^{m\times n}$ and $\mathbf{b} \in \mathbb{IR}^m$ it is obvious that

$$\mathbf{A}^H\mathbf{b} \subseteq \mathbf{A}^L\mathbf{b}.$$

In the classical interval arithmetic, the Moore-Penrose inverse of an interval matrix $\mathbf{A} \in \mathbb{IR}^{m\times n}$ is denoted by $\mathbf{A}^\dagger$ and is defined as

$$\mathbf{A}^\dagger := \square\{A^\dagger : A \in \mathbf{A}\}. \tag{6}$$

The Moore-Penrose inverse of an interval matrix introduced in relation (6), helps us to find a subset of the solution set of problem (4).

As mentioned in [18], we can reduce the problem (4) ($m > n$) to the square case by observing that

$$\mathbf{A}^H\mathbf{b} \subseteq \mathbf{A}^L\mathbf{b} \subseteq \mathbf{x}, \tag{7}$$

in which $\mathbf{x}$ satisfies in

$$\begin{pmatrix} \mathbf{r} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} I & \mathbf{A} \\ \mathbf{A}^\top & 0 \end{pmatrix}^H \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix},$$

and $\mathbf{r}$ satisfies in the coupled system

$$\begin{cases} \mathbf{r}+\mathbf{A}\mathbf{x}=\mathbf{b}, \\ \mathbf{A}^\top \mathbf{r}=0, \end{cases}$$

and I is the identity matrix, or

$$\mathbf{A}^H\mathbf{b} \subseteq \mathbf{A}^L\mathbf{b} \subseteq (\mathbf{A}^\top\mathbf{A})^H(\mathbf{A}^\top\mathbf{b}). \tag{8}$$

But the approach via (7) and (8) is not recommended, since $\mathbf{A}^L\mathbf{b}$ depends very sensitively on scaling of the problem.

2.2 Modified interval arithmetic

We want to consider the generalized interval numbers which for the first time were introduced by Kaucher [15]. Kaucher extended the set of proper intervals $\mathbb{IR}=\{\mathbf{x}=[\underline{x},\overline{x}]:\underline{x}\leq\overline{x},\underline{x},\overline{x}\in\mathbb{R}\}$ by the set $\overline{\mathbb{IR}}=\{\mathbf{x}=[\underline{x},\overline{x}]:\underline{x}\geq\overline{x},\underline{x},\overline{x}\in\mathbb{R}\}$ of improper intervals, resulting in a more flexible set of generalized intervals $\mathbb{KR}=\{\mathbf{x}=[\underline{x},\overline{x}]:\underline{x},\overline{x}\in\mathbb{R}\}$. For example $[-1,1]$ and $[1,-1]$ are generalized intervals. Now, we want to consider a modified interval arithmetic on $\mathbb{KR}$ which was introduced by Nirmala et al. [20]. The set of m-by-n generalized interval matrices is denoted by $\mathbb{KR}^{m\times n}$.

The midpoint and width of an interval number $\mathbf{x}=[\underline{x},\overline{x}]$ are denoted by $m(\mathbf{x})$ and $r(\mathbf{x})$, respectively, and are defied as

$$m(\mathbf{x})=\frac{\overline{x}+\underline{x}}{2},\quad r(\mathbf{x})=\frac{\overline{x}-\underline{x}}{2}.$$

Also the magnitude of $\mathbf{x}$ is denoted by $|\mathbf{x}|$ and is defined as

$$|\mathbf{x}|=\max\{|x|:x\in\mathbf{x}\}.$$

The concepts of midpoint, radius and magnitude of interval matrices are defined, componentwise.

For generalized interval numbers $\mathbf{x},\mathbf{y}\in\mathbb{KR}$, the modified interval operation $\circledast\in\{+,-,*,/\}$ is defined as follows

$$\mathbf{x}\circledast\mathbf{y}=[m(\mathbf{x})\circledast m(\mathbf{y})-k,m(\mathbf{x})\circledast m(\mathbf{y})+k],$$

therein

$$k=\min\{(m(\mathbf{x})\circledast m(\mathbf{y}))-\alpha,\beta-(m(\mathbf{x})\circledast m(\mathbf{y}))\},$$

in which α and β, respectively, are lower bound and upper bound of the interval $\mathbf{x}\odot\mathbf{y}$, where $\odot$ denotes the classical interval operation. In particular, we have

Addition:

$$\mathbf{x}+\mathbf{y}=[\underline{x},\overline{x}]+[\underline{y},\overline{y}]=[m(\mathbf{x})+m(\mathbf{y})-k,m(\mathbf{x})+m(\mathbf{y})+k],$$
$$\text{where } k=\frac{(\overline{x}+\overline{y})-(\underline{x}+\underline{y})}{2}.$$

Subtraction:

$$\mathbf{x}-\mathbf{y}=[\underline{x},\overline{x}]-[\underline{y},\overline{y}]=[m(\mathbf{x})-m(\mathbf{y})-k,m(\mathbf{x})-m(\mathbf{y})+k],$$
$$\text{where } k=\frac{(\overline{x}+\overline{y})-(\underline{x}+\underline{y})}{2},$$

also if $\mathbf{x}=\mathbf{y}$ then $\mathbf{x}-\mathbf{y}=[0,0]$.

Multiplication:

$$\begin{aligned}
&\mathbf{x} * \mathbf{y} = \mathbf{x}\mathbf{y} = [\underline{x}, \overline{x}][\underline{y}, \overline{y}] = [m(\mathbf{x})m(\mathbf{y}) - k, m(\mathbf{x})m(\mathbf{y}) + k],\\
&\text{where } k = \min\{(m(\mathbf{x})m(\mathbf{y})) - \alpha, \beta - (m(\mathbf{x})m(\mathbf{y}))\}\ ,\\
&\alpha = \min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \quad \beta = \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}.
\end{aligned}$$

Division:

$$\begin{aligned}
&1/\mathbf{x} = \frac{1}{[\underline{x}, \overline{x}]} = [\frac{1}{m(\mathbf{x})} - k, \frac{1}{m(\mathbf{x})} + k],\\
&\text{where } k = \min\{\frac{1}{\overline{x}}(\frac{\overline{x} - \underline{x}}{\overline{x} + \underline{x}}), \frac{1}{\underline{x}}(\frac{\overline{x} - \underline{x}}{\overline{x} + \underline{x}})\}, \quad m(\mathbf{x}) \neq 0\ ,\\
&\mathbf{x}/\mathbf{y} = \mathbf{x} * \frac{1}{\mathbf{y}}, \quad m(\mathbf{x}) \neq 0,\\
&\text{also, if } \mathbf{x} = \mathbf{y}, \text{ then } \mathbf{x}/\mathbf{y} = [1, 1].
\end{aligned}$$

If $\odot$ and $\circledast$, respectively, stand for the classical and modified interval operations, then for $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$ we have

$$\mathbf{x} \circledast \mathbf{y} \subseteq \mathbf{x} \odot \mathbf{y}.$$

It is to be noted that under the modified interval arithmetic, the set of generalized interval numbers $\mathbb{KR}$ possesses group properties with respect to addition and multiplication operations and satisfies the distributive law and hence many other important results.

In the modified interval arithmetic, there is a specific index for comparing two interval numbers. According to this index, two interval numbers $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$ are said to be equivalent when $m(\mathbf{x}) = m(\mathbf{y})$ and in this case we write $\mathbf{x} \approx \mathbf{y}$. In particular, if $m(\mathbf{x}) = m(\mathbf{y})$ and $r(\mathbf{x}) = r(\mathbf{y})$ then $\mathbf{x} = \mathbf{y}$. If $m(\mathbf{x}) = 0$ then $\mathbf{x}$ is considered as zero interval number. Similar relations hold in a completely similar way for matrices, i.e., two interval matrices $\mathbf{A}, \mathbf{B} \in \mathbb{KR}^{m \times n}$ are said to be equivalent when $m(\mathbf{A}) = m(\mathbf{B})$ and is denoted by $\mathbf{A} \approx \mathbf{B}$.

3 Moore-Penrose inverse of an interval matrix and rectangular interval linear systems

In this section, we first introduce a new concept for $\mathbf{A}^\dagger$ similar to the real case and then using $\mathbf{A}^\dagger$, find the exact solution of the rectangular interval linear systems under the modified interval arithmetic. In Subsection 3.1, all arithmetic operations are implemented under the modified interval arithmetic.

3.1 Moore-Penrose inverse of an interval matrix

Definition 1. *For $\mathbf{A} \in \mathbb{KR}^{m \times n}$, the Moore-Penrose inverse of $\mathbf{A}$ is defined as an interval matrix $\mathbf{A}^\dagger \in \mathbb{KR}^{n \times m}$ satisfying all of the following four criteria:*

$$(1)\ \mathbf{A}\mathbf{A}^\dagger\mathbf{A} \approx \mathbf{A}, \qquad (2)\ \mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger \approx \mathbf{A}^\dagger, \qquad (3)\ (\mathbf{A}\mathbf{A}^\dagger)^\top \approx \mathbf{A}\mathbf{A}^\dagger, \qquad (4)\ (\mathbf{A}^\dagger\mathbf{A})^\top \approx \mathbf{A}^\dagger\mathbf{A}.$$

Theorem 1. *The Moore-Penrose inverses of an interval matrix are equivalent.*

Proof. Let $\mathbf{A} \in \mathbb{KR}^{m\times n}$ and $\mathbf{B}, \mathbf{C} \in \mathbb{KR}^{n\times m}$ are two Moore-Penrose inverses for $\mathbf{A}$. We have

$$\mathbf{AB} \approx (\mathbf{AB})^\top \approx \mathbf{B}^\top\mathbf{A}^\top \approx \mathbf{B}^\top(\mathbf{ACA})^\top \approx \mathbf{B}^\top\mathbf{A}^\top\mathbf{C}^\top\mathbf{A}^\top \approx (\mathbf{AB})^\top(\mathbf{AC})^\top \approx \mathbf{ABAC} \approx \mathbf{AC}.$$

So, $\mathbf{AB} \approx \mathbf{AC}$. Similarly, we conclude that $\mathbf{BA} \approx \mathbf{CA}$. The proof is completed by observing that

$$\mathbf{B} \approx \mathbf{BAB} \approx \mathbf{BAC} \approx \mathbf{CAC} \approx \mathbf{C}.$$

□

Note that in the above argument, we utilized the associative law of multiplication between interval matrices which holds under the modified interval arithmetic. Now, we present some results which provide us efficient formulas for constructing the Moore-Penrose inverse of an interval matrix. In the upcoming results, by $\mathbf{A}^{-1}$, we mean the inverse of the regular matrix $\mathbf{A}$, which is constructed by the proposed technique in [20].

Theorem 2. *Let $\mathbf{A} \in \mathbb{KR}^{m\times n}$ with $m < n$. If $\mathbf{AA}^\top$ is regular then $\mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}$.*

Proof. We have to show $\mathbf{B} = \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}$ satisfies in all mentioned criteria in Definition 1. We can write

(1) $\mathbf{ABA} \approx \mathbf{A}(\mathbf{A}^\top(\mathbf{AA}^\top)^{-1})\mathbf{A} \approx (\mathbf{AA}^\top)(\mathbf{AA}^\top)^{-1}\mathbf{A} \approx \mathbf{IA} \approx \mathbf{A}$,

(2) $\mathbf{BAB} \approx (\mathbf{A}^\top(\mathbf{AA}^\top)^{-1})\mathbf{A}(\mathbf{A}^\top(\mathbf{AA}^\top)^{-1}) \approx \mathbf{A}^\top((\mathbf{AA}^\top)^{-1}(\mathbf{AA}^\top))(\mathbf{AA}^\top)^{-1} \approx \mathbf{A}^\top\mathbf{I}(\mathbf{AA}^\top)^{-1}$
$\approx \mathbf{A}^\top(\mathbf{AA}^\top)^{-1} \approx \mathbf{B}$,

(3) $(\mathbf{AB})^\top \approx (\mathbf{A}(\mathbf{A}^\top(\mathbf{AA}^\top)^{-1}))^\top \approx ((\mathbf{AA}^\top)(\mathbf{AA}^\top)^{-1})^\top \approx \mathbf{I}^\top \approx \mathbf{I} \approx \mathbf{AB}$,

(4) $(\mathbf{BA})^\top \approx ((\mathbf{A}^\top(\mathbf{AA}^\top)^{-1})\mathbf{A})^\top \approx \mathbf{A}^\top((\mathbf{AA}^\top)^\top)^{-1}\mathbf{A} \approx \mathbf{A}^\top(\mathbf{AA}^\top)^{-1}\mathbf{A} \approx (\mathbf{A}^\top(\mathbf{AA}^\top)^{-1})\mathbf{A}$
$\approx \mathbf{BA}$.

So, $\mathbf{B} = \mathbf{A}^\dagger$. □

Theorem 3. *Let $\mathbf{A} \in \mathbb{KR}^{m\times n}$ with $m > n$. If $\mathbf{A}^\top\mathbf{A}$ is regular, then $\mathbf{A}^\dagger = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$.*

Proof. We must show $\mathbf{B} = (\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top$ satisfies in four conditions mentioned in Definition 1. We have

(1) $\mathbf{ABA} \approx \mathbf{A}((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top)\mathbf{A} \approx \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}(\mathbf{A}^\top\mathbf{A}) \approx \mathbf{AI} \approx \mathbf{A}$,

(2) $\mathbf{BAB} \approx ((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top)\mathbf{A}((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top) \approx ((\mathbf{A}^\top\mathbf{A})^{-1}(\mathbf{A}^\top\mathbf{A}))((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top) \approx \mathbf{IB} \approx \mathbf{B}$,

(3) $(\mathbf{AB})^\top \approx (\mathbf{A}((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top))^\top \approx \mathbf{A}((\mathbf{A}^\top\mathbf{A})^\top)^{-1}\mathbf{A}^\top \approx \mathbf{A}(\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top \approx \mathbf{A}((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top) \approx \mathbf{AB}$,

(4) $(\mathbf{BA})^\top \approx ((\mathbf{A}^\top\mathbf{A})^{-1}\mathbf{A}^\top\mathbf{A})^\top \approx ((\mathbf{A}^\top\mathbf{A})^{-1}(\mathbf{A}^\top\mathbf{A}))^\top \approx \mathbf{I}^\top \approx \mathbf{I} \approx \mathbf{BA}$.

So, we conclude that $\mathbf{B} = \mathbf{A}^\dagger$. □

For some special cases, we have the following remarks.

Remark 1. *The Moore-Penrose inverse of a regular square interval matrix is its inverse.*

Remark 2. *The Moore-Penrose inverse of an interval number $\mathbf{x} \in \mathbb{KR}$ is*

$$\mathbf{x}^\dagger = \begin{cases} 0, & \mathbf{x} \approx 0, \\ \frac{1}{\mathbf{x}}, & \textit{otherwise}. \end{cases}$$

Remark 3. *The Moore-Penrose inverse of an interval vector* $\mathbf{x} \in \mathbb{KR}^n$ *is*

$$\mathbf{x}^\dagger = \begin{cases} 0^\top, & \mathbf{x} \approx 0, \\ \frac{\mathbf{x}^\top}{\mathbf{x}^\top \mathbf{x}}, & \textit{otherwise}. \end{cases}$$

Remark 4. *The Moore-Penrose inverse of a zero interval matrix is its transpose.*

It is to be noted that the introduced Moore-Penrose inverse in (6) in classical interval arithmetic, does not satisfy in four mentioned criteria in Definition 1. However, our proposed formulas for constructing Moore-Penrose inverse lead us to an interval matrix which satisfies in all criteria mentioned in Definition 1.

3.2 Rectangular interval linear systems

Consider the interval system of linear equations

$$\mathbf{A}x = \mathbf{b}. \tag{9}$$

As previously mentioned, the solution set $\Xi(\mathbf{A}, \mathbf{b})$ in some cases is empty or unbounded and in the latter case, $\mathbf{A}^H\mathbf{b}$ is not defined. So, sometimes it is more appropriate to consider the interval lease square problem (4). Some characterizations of the solution set $\Xi(\mathbf{A}, \mathbf{b})$ are presented in Theorems 4 and 5 below, which have been expressed under the classical interval arithmetic.

Theorem 4. *[19] Let* $\mathbf{A} \in \mathbb{IR}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{IR}^m$*. Then we have*

$$\Xi(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n : \mathbf{A}x \cap \mathbf{b} \neq \emptyset\} = \{x \in \mathbb{R}^n : 0 \in \mathbf{A}x - \mathbf{b}\}.$$

Theorem 5. *[19] Let* $\mathbf{A} \in \mathbb{IR}^{m\times n}$ *and* $\mathbf{b} \in \mathbb{IR}^m$*. Then we have*

$$x \in \Xi(\mathbf{A}, \mathbf{b}) \Leftrightarrow |m(\mathbf{A})x - m(\mathbf{b})| \leq r(\mathbf{A})|x| + r(\mathbf{b}).$$

Example 1. Consider the interval linear system $\mathbf{A}x = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} [-1,1] & [1,3] \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} [0,10] \\ [1,3] \\ [100,500] \end{pmatrix}.$$

The midpoints of $\mathbf{A}$ and $\mathbf{b}$ are

$$m(\mathbf{A}) = \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m(\mathbf{b}) = \begin{pmatrix} 5 \\ 2 \\ 300 \end{pmatrix},$$

so for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, we obtain

$$|m(\mathbf{A})x - m(\mathbf{b})| = \left| \begin{pmatrix} 0 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 5 \\ 2 \\ 300 \end{pmatrix} \right| = \begin{pmatrix} |2x_2 - 5| \\ |x_1 - 2| \\ |x_2 - 300| \end{pmatrix}.$$

On the other hand, we have

$$r(\mathbf{A}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad r(\mathbf{b}) = \begin{pmatrix} 5 \\ 1 \\ 200 \end{pmatrix},$$

thus

$$r(\mathbf{A})|x| + r(\mathbf{b}) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |x_1| \\ |x_2| \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ 200 \end{pmatrix} = \begin{pmatrix} |x_1| + |x_2| + 5 \\ 1 \\ 200 \end{pmatrix}.$$

By Theorem 5, if $x \in \Xi(\mathbf{A}, \mathbf{b})$ then it should satisfies in the following system of inequalities

$$\begin{cases} |2x_2 - 5| \leq |x_1| + |x_2| + 5, \\ |x_1 - 2| \leq 1, \\ |x_2 - 300| \leq 200. \end{cases}$$

But the above system does not have any solution, so according to Theorem 5, we conclude that $\Xi(\mathbf{A}, \mathbf{b}) = \emptyset$.

The above example and many others motivate us to consider the interval linear system $\mathbf{A}x \approx \mathbf{b}$ instead of $\mathbf{A}x = \mathbf{b}$. If $\mathbf{A} \in \mathbb{KR}^{m\times n}$ with $m < n$, then for obtaining the solution set of the system $\mathbf{A}x \approx \mathbf{b}$, we utilize the Moore-Penrose inverse of $\mathbf{A}$ proposed by Theorem 2 and so we will have the following theorem in the modified interval arithmetic.

Theorem 6. *Consider the rectangular interval system of linear equations $\mathbf{A}x \approx \mathbf{b}$ with $\mathbf{A} \in \mathbb{KR}^{m\times n}$ ($m < n$) and $\mathbf{b} \in \mathbb{KR}^m$ and suppose that $\mathbf{A}\mathbf{A}^\top$ is regular. If $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b}$, where $\mathbf{A}^\dagger$ is the constructed Moore-Penrose inverse of $\mathbf{A}$ by Theorem 2 under the modified interval arithmetic, then $\mathbf{A}\mathbf{x} \approx \mathbf{b}$.*

Proof. Since $\mathbf{A}\mathbf{A}^\top$ is regular, so by Theorem 2 we have $\mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}$. Thus using the associative law of multiplication between interval matrices which holds under the modified interval arithmetic, we can write

$$\mathbf{A}\mathbf{x} \approx \mathbf{A}(\mathbf{A}^\dagger\mathbf{b}) \approx \mathbf{A}\big((\mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1})\mathbf{b}\big) \approx (\mathbf{A}\mathbf{A}^\top)(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{b} \approx \mathbf{I}\mathbf{b} \approx \mathbf{b}.$$

□

Example 2. Consider the rectangular interval system of linear equations $\mathbf{A}x \approx \mathbf{b}$, therein

$$\mathbf{A} = \begin{pmatrix} [0.2, 0.25] & [0, 0.1] & [0.4, 0.5] \\ [0.3, 0.5] & [0.1, 0.2] & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} [1,2] \\ [3,4] \end{pmatrix}.$$

We solve this system under modified interval arithmetic. We have

$$\mathbf{A}\mathbf{A}^\top = \begin{pmatrix} [0.2000, 0.3113] & [0.0599, 0.1351] \\ [0.0599, 0.1351] & [0.1000, 0.2651] \end{pmatrix},$$

so determinant of $\mathbf{A}\mathbf{A}^\top$ is

$$\det(\mathbf{A}\mathbf{A}^\top) = [0.2000, 0.3113][0.1000, 0.2651] - [0.0599, 0.1351][0.0599, 0.1351] = [0.0045, 0.0698].$$

Since $0 \notin \det(\mathbf{A}\mathbf{A}^\top)$, thus $\mathbf{A}\mathbf{A}^\top$ is invertible. The adjoint matrix $(\mathbf{A}\mathbf{A}^\top)^*$ of $\mathbf{A}\mathbf{A}^\top$ is

$$(\mathbf{A}\mathbf{A}^\top)^* = \begin{pmatrix} [\ 0.1000,\ 0.2651] & [\ \text{-}0.1351,\ \text{-}0.0599] \\ [\ \text{-}0.1351,\ \text{-}0.0599] & [\ 0.2000,\ 0.3113] \end{pmatrix}.$$

Thus by the proposed technique in [20], the inverse of $\mathbf{A}\mathbf{A}^\top$ is

$$(\mathbf{A}\mathbf{A}^\top)^{-1} = \frac{(\mathbf{A}\mathbf{A}^\top)^*}{\det(\mathbf{A}\mathbf{A}^\top)} = \begin{pmatrix} [\ 1.4346,\ 8.3917] & [\ \text{-}4.3889,\ \text{-}0.8607] \\ [\ \text{-}4.3889,\ \text{-}0.8607] & [\ 2.8693,\ 10.8943] \end{pmatrix}.$$

Finally using Theorem 2, the Moore-Penrose inverse of $\mathbf{A}$ is

$$\mathbf{A}^\dagger = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1} = \begin{pmatrix} [\text{-}1.5547,1.6658] & [\text{-}0.1483,4.4725] \\ [\text{-}0.7014,0.4053] & [0.0244,1.7776] \\ [0.5738,3.8480] & [\text{-}2.0181,\text{-}0.3443] \end{pmatrix}.$$

Now, if $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b}$, then

$$\mathbf{x} = \begin{pmatrix} [\text{-}3.7023,19.0037] \\ [\text{-}1.1814,7.0443] \\ [\text{-}6.6614,5.0260] \end{pmatrix}.$$

It is to be noted that $\mathbf{x}$ satisfies $\mathbf{A}\mathbf{x} \approx \mathbf{b}$.

4 Conclusion

In this paper, we introduced a new concept for Moore-Penrose inverse of an interval matrix which is similar to the real case. We then proposed some formula for constructing the Moore-Penrose inverse in the framework of the generalized intervals and under the modified interval arithmetic. Also, we investigated the rectangular interval system of linear equations and using the introduced Moore-Penrose inverse of the coefficient matrix, we obtained the exact solution of those systems.

Acknowledgment

The author would like to thank Shahid Chamran University of Ahvaz for financial support under grant number SCU.MM1402.33518.

References

[1] G. Alefeld, J. Herzberger, *Introduction to Interval Computations. Computer Science and Applied Mathematics*, Academic Press, New York, 1983.

[2] M. Dehghani-Madiseh, *On the interval generalized coupled matrix equations*, Electron. J. Linear Algebra **34** (2018) 695–717.

[3] M. Dehghani-Madiseh, *Fast enclosing the solution set of the parametric sylvester matrix equations*, Linear Multilinear Algebra **70** (2022) 1814–1830.

[4] M. Dehghani-Madiseh, *An efficient approach for enclosing the solution set of the interval coupled sylvester matrix equations*, Linear Multilinear Algebra, In press, https://doi.org/10.1080/03081087.2023.2236773.

[5] M. Dehghani-Madiseh, M. Hladík, *Efficient approaches for enclosing the united solution set of the interval generalized Sylvester matrix equation*, Appl. Numer. Math. **126** (2018) 18–33.

[6] M. Dehghani-Madiseh, M. Hladík, *Enclosing the solution set of the parametric generalised Sylvester matrix equation* $A(p)XB(p)+C(p)XD(p)=F(p)$, Int. J. Syst. Sci. **50** (2019) 2153–2167.

[7] K. Ganesan, *On some properties of interval matrices*, Int. J. Comput. Sci. **2** (2007) 92–99.

[8] E.R. Hansen, *On the solution of linear algebraic equations with interval coefficients*, Linear Algebra Appl. **2** (1969) 153–165.

[9] E.R. Hansen, R. Smith, *Interval arithmetic in matrix computations, part 2*, SIAM J. Numer. Anal. **4** (1967) 1–9.

[10] N.J. Higham, *Functions of Matrices: Theory and Computation*, SIAM, Philadelphia, PA, USA, 2008.

[11] M. Hladík, *Description of symmetric and skew-symmetric solution set*, SIAM J. Matrix Anal. Appl. **30** (2008) 509–521.

[12] M. Hladík, *Stability of the linear complementarity problem properties under interval uncertainty*, Cent. Eur. J. Oper. Res. **29** (2021) 875–889.

[13] M. Hladík, I. Skalna, *Relations between various methods for solving linear interval and para metric equations*, Linear Algebra Appl. **574** (2019) 1–21.

[14] L. Jaulin, M. Kieffer, O. Didrit, E.Walter, *Applied Interval Analysis*, Springer-Verlag, London, 2001.

[15] E.W. Kaucher, *Interval analysis in the extended interval space IR*, Computing Suppl. **2** (1980) 33–49.

[16] R.B. Kearfott, V. Kreinovich, *Applications of Interval Computations*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1996.

[17] W. Leela-apiradee, A. Gorka, K. Burimas, P. Thipwiwatpotjana, *Tolerance-localized and control-localized solutions of interval linear equations system and their application to course assignment problem*, Appl. Math. Comput. **421** (2022) 126930.

[18] A. Neumaier, *Linear interval equations*, Interval Mathematics **212** (1985) 109–120.

[19] A. Neumaier, *Interval Methods for Systems of Equations*, Cambridge University Press, Cambridge, 1990.

[20] T. Nirmala, D. Datta, H.S. Kushwaha, K. Ganesan, *Inverse interval matrix: A new approach*, Appl. Math. Sci. **5** (2011) 607–624.

[21] B.J. Olson, S.W. Shaw, C. Shi, C. Pierre, R.G. Parker, *Circulant matrices and their application to vibration analysis*, Appl. Mech. Rev. **66** (2014) 40–80.

[22] S. Parter, *On the eigenvalues and eigenvectors of a class of matrices*, SIAM **8** (1960) 376–388.

[23] A. Piegat, M. Pluciski, *The optimal tolerance solution of the basic interval linear equation and the explanation of the lodwicks anomaly*, Appl. Sci. **12** (2022) 4382.

[24] J. Rohn, *Inverse interval matrix*, SIAM J. Numer. Anal. **3** (1993) 864–870.

[25] L. Sahoo, *An application of interval system of linear equations in circuit analysis*, IJARIIE **3** (2017) 2395–4396.

[26] M. Salari, R. Ghanbari, K. Ghorbani-Moghadam, *New method for solving fuzzy LR interval linear systems using least squares models*, Fuzzy Inf. Eng. **14** (2022) 335–348.

[27] E. Zieniuk, M. Kapturczak, A. Kuelewski, *Solving interval systems of equations obtained during the numerical solution of boundary value problems*, Comp. Appl. Math. **35** (2016) 629–638.