**Journal of Mathematical Modeling**
Vol. 12, No. 1, 2024, pp. 131–144. Research Article

JMM

# A new $(I+P)$-like preconditioner for the SOR method for solving multi-linear systems with $\mathscr{M}$-tensors

**Afsaneh Hasanpour, Maryam Mojarrab***

*Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran*
*Email(s): a_hasanpour@pgs.usb.ac.ir, ma_mojarrab@math.usb.ac.ir*

**Abstract.** The use of preconditioning techniques has been shown to offer significant advantages in solving multi-linear systems involving nonsingular $\mathscr{M}$-tensors. In this paper, we introduce a new preconditioner that employs $(I+P)$-like preconditioning techniques, and give the proof of its convergence. We also present numerical examples and comparison results that demonstrate the superior efficiency of our preconditioner compared to both the original SOR method and the previously proposed preconditioned SOR method.

*Keywords*: Preconditioner, multi-linear systems, $\mathscr{M}$-tensor, SOR.
*AMS Subject Classification 2010*: 15A06, 65F10, 65F20.

## 1 Introduction

Tensor equations found numerous applications in engineering and scientific computing [1–4]. These applications span various fields, ranging from evolutionary game dynamics [5, 6] and partial differential equations to data mining [7–12] and image processing [2, 13, 14].

Suppose that $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is an $n$-dimension real tensor of $m$-order and $b$ is a vector in $\mathbb{R}^n$. Consider the following tensor equation

$$\mathscr{A}x^{m-1} = b. \tag{1}$$

The tensor-vector product is a vector where the entries are defined by

$$(\mathscr{A}x^{m-1})_i = \sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{ii_2i_3\cdots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}, \qquad i = 1,2,\ldots,n,$$

where $x_i$ denotes the $i$th component of $x$.

*Corresponding author
Received: 23 August 2023 / Revised: 01 Dcember 2023 / Accepted: 03 Dcember 2023
DOI: 10.22124/jmm.2023.25368.2253

© 2024 University of Guilan 

Various algorithms have been developed for solving the multi-linear system represented by equation (1). It has been proven that if $\mathscr{A}$ is a nonsingular $\mathscr{M}$-tensor and $b$ is a positive vector, then equation (1) has a unique positive solution [15]. In [3], certain conditions were established for the existence and uniqueness of the solution to (1). In the case where $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, continuous-time neural networks were proposed in [16] to obtain the unique positive solution of (1). Iterative methods have also been applied to solve multi-linear systems. In [4], the authors presented a new tensor-based method for solving symmetric $\mathscr{M}$-tensor systems. In [15], Newton's method, the Jacobi method, and the Gauss-Seidel method were proposed as methods for solving (1). Han introduced the homotopy method for solving (1) for $\mathscr{M}$-tensors [17]. In [18], He presented a Newton-type method for solving tensor equations. Liu et al. introduced variant tensor splittings by considering $\mathscr{A} = \mathscr{E} - \mathscr{F}$ and extended it for strong $\mathscr{M}$-tensors [3], such that

$$x_k = (M(\mathscr{E})^{-1}\mathscr{F}x_{k-1}^{m-1} + M(\mathscr{E})^{-1}b)^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \ldots,$$

where the iterative tensor of the splitting method is $M(\mathscr{E})^{-1}\mathscr{F}$ and $M(\mathscr{E})$ is the majorization matrix of tensor $\mathscr{E}$.

It is recalled that if $M(\mathscr{A})$ denotes the majorization matrix of $\mathscr{A} \in \mathbb{R}^{[m,n]}$, then it is an $n \times n$ matrix whose entries are given by $M(\mathscr{A})_{ij} = a_{ij\ldots j}$ for $i, j = 1, \ldots, n$ [26–28]. If $M(\mathscr{A})$ is nonsingular, then $\mathscr{A}$ is said to be a left-invertible tensor or a left-nonsingular tensor. In this case, we can express $\mathscr{A}$ as the product of $M(\mathscr{A})$ and the identity tensor $\mathscr{I}_m$ as follows: $\mathscr{A} = M(\mathscr{A})\mathscr{I}_m$. The order 2 left-inverse of $\mathscr{A}$ is denoted as $M(\mathscr{A})^{-1}$, and it satisfies the property that $M(\mathscr{A})^{-1}\mathscr{A} = \mathscr{I}_m$ [29].

The Jacobi, Gauss-Seidel, and SOR iterative methods will be obtained by taking $\mathscr{A} = \mathscr{D} - \mathscr{L} - \mathscr{F}$, $\mathscr{E} = \mathscr{D}$, $\mathscr{E} = \mathscr{D} - \mathscr{L}$, and $\mathscr{E} = \frac{1}{\omega}(\mathscr{D} - \omega\mathscr{L})$ respectively, where $\mathscr{D} = D\mathscr{I}_m$, $\mathscr{L} = L\mathscr{I}_m$, such that $D$ is the positive diagonal matrix and $L$ is the strictly lower triangle nonnegative matrix [3].

Preconditioning techniques play a crucial role in solving linear and multi-linear systems, as they can significantly enhance the convergence rate of the method by employing a suitable preconditioner. While numerous efficient preconditioners have been proposed for solving linear systems, relatively few studies have focused on preconditioned methods for solving multi-linear systems. In [19], Li et al. proposed a preconditioned tensor splitting method for solving the following preconditioned multi-linear systems

$$P\mathscr{A}x^{m-1} = Pb,$$

where $P$ is the preconditioner and the iterative scheme is as follows:

$$x_k = (M(\mathscr{E}_P)^{-1}\mathscr{F}_P x_{k-1}^{m-1} + M(\mathscr{E}_P)^{-1}Pb)^{[\frac{1}{m-1}]}, \qquad k = 1, 2, \ldots,$$

so that $P\mathscr{A} = \mathscr{E}_P - \mathscr{F}_P$ is a tensor splitting of $P\mathscr{A}$.

A modified preconditioned Gauss-Seidel method was proposed in [21]. Another approach was presented in [22], where a new preconditioner was constructed using the components of the first column of the majorization matrix of $\mathscr{A}$. Liu et al. introduced a preconditioned SOR method for solving multi-linear systems with $\mathscr{M}$-tensors [23]. In [24], the authors proposed preconditioners for multi-linear systems based on the majorization matrix.

The objective of this paper is to present an extension of the $(I+P)$-type preconditioner for the SOR method to solve multi-linear systems. We apply the newly proposed preconditioned SOR method to several numerical examples and compare its performance against the original SOR method. The results of the numerical experiments and comparisons demonstrate the effectiveness of the proposed preconditioner.

This paper is structured as follows. Section 2 presents some preliminary information, including related definitions and lemmas. In Section 3, we introduce a new preconditioner for the SOR method. Section 4 contains several numerical examples that demonstrate the effectiveness of the proposed preconditioned iterative method. The final section is the concluding remark.

## 2 Preliminaries

A real tensor $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a multidimensional array consists of $n^m$ real entries:

$$(a_{i_1 i_2 i_3 \cdots i_m}) \in \mathbb{R}, \qquad i_j = 1, \ldots, n, \;\; j = 1, \ldots, m.$$

We denote the set of all $n$-dimension real tensors of $m$-order by $\mathbb{R}^{[m,n]}$. By putting $m = 2$, $\mathbb{R}^{[2,n]}$ shows all $n \times n$ real matrices, and when $m = 1$, $\mathbb{R}^{[1,n]}$ is as $\mathbb{R}^n$.

**Definition 1** ([30, 31]). *The $a_{i\cdots i}$, $i = 1, \ldots, n$ is a diagonal entry of $\mathscr{A} \in \mathbb{R}^{[m,n]}$. In addition, the identity tensor is defined as $\mathscr{I}_m = (\delta_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ where*

$$\delta_{i_1 i_2 \cdots i_m} = \begin{cases} 1, & if \;\; i_1 = i_2 = \cdots = i_m, \\ 0, & otherwise. \end{cases}$$

**Definition 2** ([23]). *For $\mathscr{A} \in \mathbb{R}^{[m,n]}$, a pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector of $\mathscr{A}$ if we have*

$$\mathscr{A} x^{m-1} = \lambda x^{[m-1]}, \tag{2}$$

*where $x^{[m-1]} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. Also,*

$$\rho(\mathscr{A}) = max\{|\lambda| \,|\lambda \in \sigma(\mathscr{A})\}$$

*is the spectral radius of $\mathscr{A}$, where $\sigma(\mathscr{A})$ is the set of all eigenvalues of $\mathscr{A}$.*

**Definition 3** ([32, 33]). *The $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a $\mathscr{Z}$-tensor when its off-diagonal entries are non-positive. If there exists a nonnegative tensor $\mathscr{B}$ and a positive real number $\xi \geq \rho(\mathscr{B})$ satisfied in $\mathscr{A} = \xi \mathscr{I}_m - \mathscr{B}$, then $\mathscr{A}$ is an $\mathscr{M}$-tensor. If $\xi > \rho(\mathscr{B})$, then $\mathscr{A}$ is called a strong $\mathscr{M}$-tensor.*

**Definition 4** ([3]). *Consider $\mathscr{A}$, $\mathscr{E}$, $\mathscr{F} \in \mathbb{R}^{[m,n]}$, and $\mathscr{O}$ denotes the null tensor. Then*
*(i) $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a splitting of $\mathscr{A}$ if $\mathscr{E}$ is a left-nonsingular.*
*(ii)$\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a regular splitting of $\mathscr{A}$ if $\mathscr{E}$ is left-nonsingular with $M(\mathscr{E})^{-1} \geq \mathscr{O}$, and $\mathscr{F} \geq \mathscr{O}$.*
*(iii) $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a weak regular splitting of $\mathscr{A}$ if $\mathscr{E}$ is left non-singular with $M(\mathscr{E})^{-1} \geq \mathscr{O}$, and $M(\mathscr{E})^{-1}\mathscr{F} \geq \mathscr{O}$.*
*(iv) $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a convergent splitting if $\rho(M(\mathscr{E})^{-1}\mathscr{F}) < 1$.*

**Definition 5** ([25]). *The tensor $\mathscr{C} = A\mathscr{B} \in \mathbb{R}^{[m,n]}$ is the product of $A \in \mathbb{R}^{[2,n]}$ and $\mathscr{B} \in \mathbb{R}^{[m,n]}$ that is defined as follows*

$$c_{j i_2 \cdots i_m} = \sum_{j_2=1}^{n} a_{j j_2} b_{j_2 i_2 \cdots i_m}, \qquad j, i_r = 1, 2, \ldots, n, \; (r = 2, \ldots, m). \tag{3}$$

*It can be written as*

$$\mathscr{C}_1 = (A\mathscr{B})_1 = A\mathscr{B}_1,$$

*where the matrices* $\mathscr{C}_1$ *and* $\mathscr{B}_1$ *are obtained from flattening* $\mathscr{C}$ *and* $\mathscr{B}$ *along with the first index. Let* $\mathscr{B} \in \mathbb{R}^{[3,n]}$*, then*

$$\mathscr{B}_1 = \begin{pmatrix} b_{111} & \dots & b_{1n1} & b_{112} & \dots & b_{1n2} & \dots & b_{11n} & \dots & b_{1nn} \\ b_{211} & \dots & b_{2n1} & b_{212} & \dots & b_{2n2} & \dots & b_{21n} & \dots & b_{2nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n11} & \dots & b_{nn1} & b_{n12} & \dots & b_{nn2} & \dots & b_{n1n} & \dots & b_{nnn} \end{pmatrix}.$$

**Lemma 1** ([19]). *If* $\mathscr{A}$ *is a* $\mathscr{Z}$*-tensor, then we have the following equivalent conditions*
*(i)* $\mathscr{A}$ *is a strong* $\mathscr{M}$*-tensor.*
*(ii) There exists some* $x \geq 0$*, such that* $\mathscr{A}x^{m-1} > 0$*, where* $x \in \mathbb{R}^n$*.*
*(iii)* $\mathscr{A}$ *has a convergent (weak) regular splitting.*
*(iv) All (weak) regular splittings of* $\mathscr{A}$ *are convergent.*

**Lemma 2** ([3]). *If* $\mathscr{A}$ *is a strong* $\mathscr{M}$*-tensor, then* $M(\mathscr{A})$ *is a nonsingular M-matrix.*

**Lemma 3** ([19]). *Let* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *be a strong* $\mathscr{M}$*-tensor. For any weak regular splitting* $\mathscr{A} = \mathscr{E} - \mathscr{F}$ *of* $\mathscr{A}$*, if* $(\rho, x)$ *is perron eigenpair of* $\mathscr{T} = M(\mathscr{E})^{-1}\mathscr{F}$*, then* $\mathscr{A}x^{m-1} \geq 0$*.*

**Lemma 4** ([19]). *Suppose that* $\mathscr{A}$ *is a strong* $\mathscr{M}$*-tensor and* $\mathscr{A} = \mathscr{E}_1 - \mathscr{F}_1 = \mathscr{E}_2 - \mathscr{F}_2$ *are two weak regular splittings with* $M(\mathscr{E}_2)^{-1} \geq M(\mathscr{E}_1)^{-1}$*. If the Perron vector x of* $M(\mathscr{E}_2)^{-1}\mathscr{F}_2$ *satisfies* $\mathscr{A}x^{m-1} \geq 0$*, then* $\rho(M(\mathscr{E}_2)^{-1}\mathscr{F}_2) < \rho(M(\mathscr{E}_1)^{-1}\mathscr{F}_1)$*.*

**Lemma 5** ([15]). *Assume* $\mathscr{A}$ *is a strong* $\mathscr{M}$*-tensor. Then the multi-linear system* (1) *has a unique positive solution for every positive vector b.*

# 3 The preconditioner structure

Li et al. [12] proposed a preconditioner of the form $P = I + S$, with

$$S = \begin{pmatrix} 0 & -\alpha_1 a_{12\dots 2} & 0 & \dots & 0 \\ 0 & 0 & -\alpha_2 a_{23\dots 3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\alpha_{n-1} a_{(n-1)n\dots n} \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

where $a_{ij\dots j},\ i,j = 1,\dots,n$ are the components of $\mathscr{A}$ in (1), and $\alpha_i \in [0,1]$, $i = 1,\dots,n-1$. Using this preconditioner and [20], we propose a preconditioner as follows

$$P_s = (I + K_s) = (I + S)[(I - S) + (L + U)(I + S)],$$

where $L$ and $U$ are the positive strictly lower and strictly upper triangular parts of $M(\mathscr{A}) = D - L - U$, respectively.

Without loss of generality, we take all the diagonal entries of the tensor $\mathscr{A}$ in (1) equal to 1. Applying a nonsingular matrix $P_s$ as a preconditioner, we get a new preconditioned multi-linear system

$$\hat{\mathscr{A}}x^{m-1} = \hat{b},$$

where $\hat{\mathscr{A}} = P_s\mathscr{A}$ and $\hat{b} = P_s b$. Consider $\hat{\mathscr{A}} = \hat{\mathscr{D}} - \hat{\mathscr{L}} - \hat{\mathscr{F}}$, with $\hat{\mathscr{D}} = \hat{D}\mathscr{I}_m$, $\hat{\mathscr{L}} = \hat{L}\mathscr{I}_m$, where $\hat{D}, -\hat{L}$ are the positive diagonal matrix and the strictly lower triangular matrix of $M(\hat{\mathscr{A}})$, respectively. We take the preconditioned SOR method as

$$x_k = (\mathscr{T}_p x_{k-1}^{m-1} + q_p)^{[\frac{1}{m-1}]}, \qquad k = 1,2,\ldots,$$

where

$$\begin{aligned}
\mathscr{T}_p &= M(\hat{\mathscr{E}}_p)^{-1}\hat{\mathscr{F}},\\
\mathscr{E}_p &= \frac{1}{\omega}(\hat{\mathscr{D}} - \omega\hat{\mathscr{L}}),\\
\mathscr{F}_p &= \frac{1}{\omega}((1-\omega)\hat{\mathscr{D}} + \omega\hat{\mathscr{F}}),\\
q_p &= M(\hat{\mathscr{E}})^{-1}\hat{b}.
\end{aligned}$$

**Theorem 1.** *Let $\mathscr{A}$ be a $\mathscr{Z}$-tensor. Then $P_s = (I + K_s)$ is nonnegative for $\alpha_i \in [0,1]$, $i = 1,2,\ldots,n-1$. Moreover, $(I + K_s) \geq (I + S)$.*

*Proof.* We have

$$\begin{aligned}
P_s = (I + K_s) &= (I+S)[(I-S) + (L+U)(I+S)]\\
&= (I+S)[I - S + L + U + US + LS]\\
&= \underbrace{(I+S)}_{\geq 0}[\underbrace{I+L}_{\geq 0} + \underbrace{(U-S)}_{\geq 0} + \underbrace{(U+L)S}_{\geq 0}] \geq 0.
\end{aligned}$$

Furthermore,

$$P_s = (I + K_s) = \underbrace{(I+S)}_{\geq 0} + \underbrace{(I+S)[L + (U-S) + (U+L)S]}_{\geq 0}.$$

Thus, $(I + K_s) \geq (I + S)$. $\square$

**Theorem 2.** *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. Then for the new preconditioner $P_s$, $\hat{\mathscr{A}} = P_s\mathscr{A}$ is a strong $\mathscr{M}$-tensor.*

*Proof.* At first, we show that under the certain conditions, the preconditioned tensor is a $\mathscr{Z}$-tensor. We take $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$, where $\mathscr{L} = L\mathscr{I}_m$, and $-L$ is the strictly lower triangular matrix corresponding to $M(\mathscr{A})$. Let $\mathscr{A}$ be a $\mathscr{Z}$-tensor and consider $\tilde{\mathscr{A}} = (I+S)\mathscr{A}$. We have

$$\begin{aligned}
\tilde{\mathscr{A}} &= (I+S)\mathscr{A}\\
&= \mathscr{I}_m - \mathscr{L} - \mathscr{F} + S\mathscr{I}_m - S\mathscr{L} - S\mathscr{F}\\
&= (\mathscr{I}_m - \mathscr{D}_1) - (\mathscr{L} + \mathscr{L}_1) - (\mathscr{F} - S\mathscr{I}_m + \mathscr{F}_1 + S\mathscr{F})\\
&= \tilde{\mathscr{D}} - \tilde{\mathscr{L}} - \tilde{\mathscr{F}},
\end{aligned}$$

where $\mathscr{D}_1 = D_1\mathscr{I}_m$, $\mathscr{L}_1 = L_1\mathscr{I}_m$, $\mathscr{F}_1 = F_1\mathscr{I}_m$ such that $D_1, L_1$ are the diagonal matrix and the strictly lower triangular matrix corresponding to $M(S\mathscr{L})$, respectively, and $S\mathscr{L} = \mathscr{D}_1 + \mathscr{L}_1 + \mathscr{F}_1$. Furthermore, $\tilde{\mathscr{D}}$ and $\tilde{\mathscr{L}}$ are diagonal and strictly lower triangular parts of tensor $\tilde{\mathscr{A}}$.

For $\hat{\mathscr{A}} = (I+K_s)\mathscr{A}$, and considering $SL = D_1 + L_1 + F_1$, we have

$$\begin{aligned}
\hat{\mathscr{A}} &= (I+K_s)\mathscr{A} \\
&= (I+S)\mathscr{A} + (I+S)[L(I+S) + U(I+S) - S]\mathscr{A} \\
&= [I + (I+S)L + (I+S)U - S](I+S)\mathscr{A} \\
&= [I + L + SL + (I+S)U - S](\tilde{\mathscr{D}} - \tilde{\mathscr{L}} - \tilde{\mathscr{F}}) \\
&= [(I+D_1) + (L+L_1) + ((I+S)U - S + F_1)](\tilde{\mathscr{D}} - \tilde{\mathscr{L}} - \tilde{\mathscr{F}}) \\
&= \hat{\mathscr{D}} - \hat{\mathscr{L}} - \hat{\mathscr{F}},
\end{aligned}$$

where $\hat{\mathscr{D}}$ and $\hat{\mathscr{L}}$ are diagonal and strictly lower triangular parts of tensor $\hat{\mathscr{A}}$, and

$$\begin{aligned}
\hat{\mathscr{D}} &= (I+D_1)\tilde{\mathscr{D}} - \mathscr{D}_2 - \mathscr{D}_3 \\
&= \mathscr{I}_m - D_1\mathscr{D}_1 - \mathscr{D}_2 - \mathscr{D}_3, \\
\hat{\mathscr{L}} &= \underbrace{\underbrace{D_1}_{\geq O}\tilde{\mathscr{L}}}_{\geq \mathscr{O}} + \underbrace{\underbrace{(L+L_1)}_{\geq O}\tilde{\mathscr{L}}}_{\geq \mathscr{O}} + \mathscr{L}_2 + \mathscr{L}_3, \\
\hat{\mathscr{F}} &= (I+D_1)\tilde{\mathscr{F}} - ((I+S)U - S + F_1)\tilde{\mathscr{D}} + ((I+S)U - S + F_1)\tilde{\mathscr{F}} + \mathscr{F}_2 + \mathscr{F}_3 \\
&= \underbrace{\tilde{\mathscr{F}} + D_1\tilde{\mathscr{F}} - ((I+S)U - S + F_1)(\mathscr{I}_m - D_1)}_{\geq \mathscr{O}} + ((I+S)U - S + F_1)\tilde{\mathscr{F}} + \mathscr{F}_2 + \mathscr{F}_3,
\end{aligned}$$

assuming that

$$\begin{aligned}
(L+L_1)\tilde{\mathscr{F}} &= \mathscr{D}_2 + \mathscr{L}_2 + \mathscr{F}_2, \\
((I+S)U - S + F_1)\tilde{\mathscr{L}} &= \mathscr{D}_3 + \mathscr{L}_3 + \mathscr{F}_3,
\end{aligned}$$

where $\mathscr{D}_2 = D_2\mathscr{I}_m$, $\mathscr{L}_2 = L_2\mathscr{I}_m$ such that $D_2$, $L_2$ are the diagonal matrix and the strictly lower triangular matrix corresponding to $M((L+L_1)\tilde{\mathscr{F}})$, respectively. In addition, $\mathscr{D}_3 = D_3\mathscr{I}_m$, $\mathscr{L}_3 = L_3\mathscr{I}_m$ such that $D_3, L_3$ are the diagonal matrix and the strictly lower triangular matrix corresponding to $M(((I+S)U - S + F_1)\tilde{\mathscr{L}})$, respectively. It can be seen that $\hat{\mathscr{L}}, \hat{\mathscr{F}} \geq \mathscr{O}$. Thus, $\hat{\mathscr{A}}$ is a $\mathscr{M}$-tensor.

We take

$$\begin{aligned}
\mathscr{E} &= (\mathscr{I}_m - \mathscr{L}), \\
\hat{\hat{\mathscr{E}}} &= (I+K_s)\mathscr{E}, \\
\hat{\hat{\mathscr{F}}} &= (I+K_s)\mathscr{F}.
\end{aligned}$$

This can be easily confirmed that $\hat{\mathscr{A}} = \hat{\hat{\mathscr{E}}} - \hat{\hat{\mathscr{F}}}$, and $\hat{\hat{\mathscr{F}}} \geq \mathscr{O}$. According to

$$M(\hat{\hat{\mathscr{E}}})^{-1}\hat{\hat{\mathscr{F}}} = (I-L)^{-1}(I+K_s)^{-1}(I+K_s)\mathscr{F} = (I-L)^{-1}\mathscr{F} \geq \mathscr{O}, \tag{4}$$

$\hat{\mathscr{A}} = \hat{\hat{\mathscr{E}}} - \hat{\hat{\mathscr{F}}}$ is a weak regular splitting. Since $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, using (4), we have

$$\rho(M(\hat{\hat{\mathscr{E}}})^{-1}\hat{\hat{\mathscr{F}}}) = \rho((I-L)^{-1}\mathscr{F}) < 1.$$

Therfore, by using Lemma 1, $\hat{\mathscr{A}}$ is a strong $\mathscr{M}$-tensor. □

**Lemma 6.** *Suppose $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is a strong $\mathscr{M}$-tensor. If $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a splitting such that $\mathscr{E}$ is a $\mathscr{Z}$-tensor and $\mathscr{F}$ is nonnegative, then the splitting is a convergent regular splitting.*

*Proof.* Since $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, there exist $\hat{x} > 0$ such that $\mathscr{A}\hat{x}^{m-1} > 0$. As $\mathscr{A} = \mathscr{E} - \mathscr{F}$ and $\mathscr{F}$ is nonnegative, we have $\mathscr{E} \geq \mathscr{A}$. Thus $\mathscr{E}\hat{x}^{m-1} \geq \mathscr{A}\hat{x}^{m-1} > 0$. Since $\mathscr{E}$ is a $\mathscr{Z}$-tensor and $\mathscr{E}\hat{x}^{m-1} > 0$, according to Lemma 1 $\mathscr{E}$ is a strong $\mathscr{M}$-tensor. In addition by Lemma 2 $M(\mathscr{E})$ is a nonsingular $M$-matrix and consequently $M(\mathscr{E})^{-1} \geq 0$. Therefore, $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is a convergence regular splitting. □

**Theorem 3.** *If $\hat{\mathscr{A}} \in \mathbb{R}^{[m,n]}$ is a strong $\mathscr{M}$-tensor, then $\hat{\mathscr{A}} = \hat{\mathscr{E}} - \hat{\mathscr{F}}$ is a convergence regular splitting.*

*Proof.* Since $\hat{\mathscr{A}}$ is a strong $\mathscr{M}$-tensor, and $\mathscr{I}_m - D_1\mathscr{D}_1 - \mathscr{D}_2 - \mathscr{D}_3$ is the diagonal part of $\hat{\mathscr{A}}$, $\hat{a}_{ij_2j_3\cdots j_m} \leq 0$, $(i, j_2, j_3, \ldots, j_m) \neq (j, j, j, \ldots, j)$, $i = 1, 2, \ldots, n$. Thus

$$\hat{\mathscr{E}} = \mathscr{I}_m - D_1\mathscr{D}_1 - \mathscr{D}_2 - \mathscr{D}_3 - (D_1\tilde{\mathscr{L}} + (L+L_1)\tilde{\mathscr{L}}\mathscr{L} + \mathscr{L}_2 + \mathscr{L}_3),$$

is a $\mathscr{Z}$-tensor. According to Lemma 6 and this fact that $\mathscr{F}$ is nonnegative, $\hat{\mathscr{E}}$ is a strong $\mathscr{M}$-tensor and $\hat{\mathscr{A}} = \hat{\mathscr{E}} - \hat{\mathscr{F}}$ is a convergence regular splitting. □

**Theorem 4.** *The preconditioned multi-linear system (2) for proposed preconditioner $P_s$ with any $\alpha_i \in [0,1], i = 1, 2, \ldots, n-1$, has the same unique positive solution of (1).*

*Proof.* Using Lemma 5, Theorem 2, and the fact that $\hat{b} > b > 0$, it is easy to prove. □

**Theorem 5.** *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. For $\mathscr{A} = \mathscr{E} - \mathscr{F}$, $\hat{\mathscr{A}} = \hat{\mathscr{E}} - \hat{\mathscr{F}}$, $\mathscr{T} = M(\mathscr{E})^{-1}\mathscr{F}$, $\mathscr{T}_p = M(\hat{\mathscr{E}})^{-1}\hat{\mathscr{F}}$, and $\omega \in (0,1)$, we can conclude*

$$\rho(\mathscr{T}_p) \leq \rho(\mathscr{T}) < 1.$$

*Proof.* From Theorem 2, we have

$$\begin{aligned}
&\mathscr{E} = \mathscr{I}_m - \mathscr{L}, \\
&\hat{\mathscr{E}} = \hat{\mathscr{D}} - \hat{\mathscr{L}} = \mathscr{I}_m - D_1\mathscr{D}_1 - \mathscr{D}_2 - \mathscr{D}_3 - (D_1\tilde{\mathscr{L}} + (L+L_1)\tilde{\mathscr{L}}\mathscr{L} + \mathscr{L}_2 + \mathscr{L}_3).
\end{aligned}$$

Since $\hat{\mathscr{A}} = P_s\mathscr{A} = \hat{\mathscr{E}} - \hat{\mathscr{F}}$, $P_s$ is an invertible matrix, so $\mathscr{A} = P_s^{-1}\hat{\mathscr{E}} - P_s^{-1}\hat{\mathscr{F}}$ is a weak regular splitting because

$$\begin{aligned}
&M(P_s^{-1}\hat{\mathscr{E}})^{-1} = M(\hat{\mathscr{E}})^{-1}P_s \geq 0, \\
&M(P_s^{-1}\hat{\mathscr{E}})^{-1}P_s^{-1}\hat{\mathscr{F}} = M(\hat{\mathscr{E}})^{-1}\hat{\mathscr{F}} \geq \mathscr{O}.
\end{aligned}$$

Let $(\hat{\rho},\hat{x})$ be the Perron eigenpair of $\hat{\mathscr{T}}$ as the iterative tensor of $\mathscr{A}=P_s^{-1}\hat{\mathscr{E}}-P_s^{-1}\hat{\mathscr{F}}$. By Lemma 3, we have $\mathscr{A}\hat{x}^{m-1}\geq 0$. Furthermore

$$\begin{aligned}\hat{\mathscr{D}} &= \mathscr{I}_m - D_1\mathscr{D}_1-\mathscr{D}_2-\mathscr{D}_3\leq \mathscr{I}_m,\\ \hat{\mathscr{L}} &= D_1\tilde{\mathscr{L}}+(L+L_1)\tilde{\mathscr{L}}\mathscr{L}+\mathscr{L}_2+\mathscr{L}_3\\ &= D_1\tilde{\mathscr{L}}+L\mathscr{L}+L_1\mathscr{L}+(L+L_1)\mathscr{L}_1+\mathscr{L}_2+\mathscr{L}_3\geq \mathscr{L}.\end{aligned}$$

Thus $\hat{\mathscr{D}}-\hat{\mathscr{L}}\leq \mathscr{I}_m-\mathscr{L}$ and $\hat{\mathscr{E}}\leq\mathscr{E}$.

Since $\mathscr{E}$ and $\hat{\mathscr{E}}$ are strong $\mathscr{M}$-tensors, we have

$$M(\hat{\mathscr{E}})\leq M(\mathscr{E}).$$

So

$$M(\hat{\mathscr{E}})^{-1}\geq M(\mathscr{E})^{-1}.$$

We have $M(\hat{\mathscr{E}})^{-1}\hat{\mathscr{F}}=M(\frac{1}{\omega}(\hat{\mathscr{D}}-\omega\hat{\mathscr{L}}))^{-1}\hat{\mathscr{F}}$. Furthermore

$$M(P_s^{-1}\hat{\mathscr{E}})^{-1}=M(\hat{\mathscr{E}})^{-1}P_s\geq M(\hat{\mathscr{E}})^{-1}\geq M(\mathscr{E})^{-1},$$

and

$$\rho(\hat{\mathscr{T}})=\rho(M(P_s^{-1}\hat{\mathscr{E}})^{-1}P_s^{-1}\hat{\mathscr{F}})=\rho(M(\hat{\mathscr{E}})^{-1}\hat{\mathscr{F}})=\rho(\mathscr{T}_p).$$

Now Lemma 4 leads to $\rho(\mathscr{T}_p)=\rho(\hat{\mathscr{T}})\leq\rho(\mathscr{T})<1$. $\square$

# 4 Numerical examples

In this section, numerical examples are given to show the efficiency of the preconditioned SOR method. The stopping criterion $\parallel\mathscr{A}x^{m-1}-b\parallel_2\leq 10^{-10}$ is used and a maximum of 1000 iterations is allowed. In all examples, we take the starting vector $x_0$ equal to $ones(n,1)$. To find the optimal parameter $\omega$, we search $[0.01,2)$ with a step length of 0.01. In this case, the best SOR performance for each value $\omega$ will determine the optimal $\omega$. We explore the value of $\alpha$ at $[0.1,1]$ with a step length of 0.1. All examples were executed in double precision in MATLAB R2014a.

We show the number of iterations by "*Iter*", the logarithm of $\parallel\mathscr{A}x_k^{m-1}-b\parallel_2$ in base 10 ($x_k$ is the $k$th approximate solution) by "*Error*" and the CPU time in seconds by "*time*" for the new preconditioned *SOR* ($P_sSOR$), the *SOR* method, and the former preconditioned *SOR* (*PSOR*) [12], respectively.

The product $\mathscr{A}x^{m-1}$ denoted in (1) can be computed by transforming into the following matrix-vector product:

$$\mathscr{A}x^{m-1}=\mathscr{A}\underbrace{(x\otimes x\otimes\ldots\otimes x)}_{m-1},$$

where $\otimes$ shows the Kronecker product. Also the matrix-tensor product $B\mathscr{A}$ is defined in (3).

**Example 1.** Consider $\mathscr{A} \in \mathbb{R}^{[3,n]}$ and $b \in \mathbb{R}^n$ in which

$$\begin{cases} a_{111} = a_{nnn} = 1, \\ a_{122} = a_{n(n-1)(n-1)} = -0.5, \\ a_{iii} = \dfrac{\theta^2}{h^2} + \dfrac{\mu_1}{h} + \eta, & i = 2,3,\ldots,n-1, \\ a_{i(i-1)i} = a_{i(i-1)(i-1)} = -\dfrac{\theta^2}{4h^2} + \dfrac{\mu_2^2}{2h}, & i = 2,3,\ldots,n-1, \\ a_{i(i+1)i} = a_{i(i+1)(i+1)} = -\dfrac{\theta^2}{4h^2} + \dfrac{\mu_2^2}{2h}, & i = 2,3,\ldots,n-1, \end{cases}$$

where

$$\theta = 0.2, \qquad \mu_1 = 0.04, \qquad \eta = 0.04, \qquad \mu_2 = -0.04, \qquad h = \frac{2}{n}.$$

From [34], it is found that $\mathscr{A}$ is a strong $\mathscr{M}$-tensor. The right hand-side vector $b$ is considered equal to $ones(n,1)$. Numerical results in Table 1 with different sizes of $\mathscr{A}$ represent that the new preconditioned method is better than original ones and former preconditioned *SOR* methods for solving $\mathscr{M}$-tensor equation.

Table 1: *Numerical results for Example 1 with* $\omega_{opt} = 1.1$.

| | $P_S SOR$ | | *SOR* | | *PSOR* | |
|---|---|---|---|---|---|---|
| $n$ | iter | time | iter | time | iter | time |
| 10 | 40 | 0.007 | 54 | 0.010 | 39 | 0.008 |
| 20 | 62 | 0.010 | 117 | 0.016 | 85 | 0.013 |

Fig. 1 shows the performance of the new preconditioned, old preconditioned and the original *SOR* methods in reducing residual norm. It can be seen that the $P_s SOR$ method converges faster than the others.

**Example 2.** Let $\mathscr{A} \in \mathbb{R}^{[3,n]}$ and $b \in \mathbb{R}^n$ with

$$\begin{cases} a_{111} = (2+n)/2, \\ a_{nnn} = 1, \\ a_{1ii} = -1/2, \\ a_{iii} = 2, & i = 2,3,\ldots,n-1, \\ a_{ii(i-1)} = -1/2, & i = 2,3,\ldots,n-1, \\ a_{i(i-1)(i-1)} = -1/2, & i = 2,3,\ldots,n-1, \\ a_{i(i+1)(i+1)} = -1/2, & i = 2,3,\ldots,n-1, \end{cases}$$

and

$$\begin{cases} b_1 = c_0^2, \\ b_i = a/(n-1)^2, & i = 2,3,\ldots,n-1, \\ b_n = c_n^2, \end{cases}$$

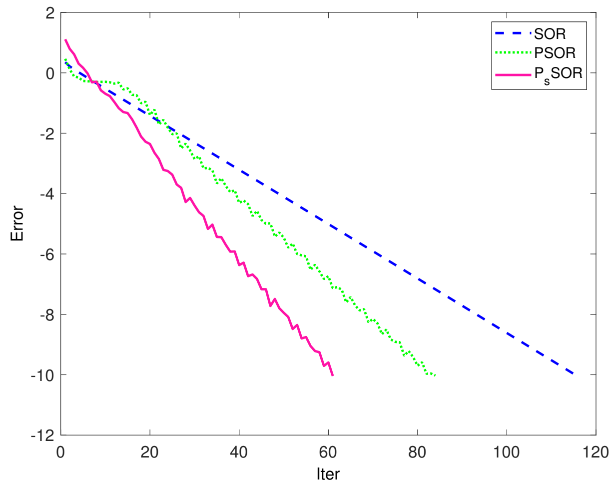


Figure 1: Performance of methods in reducing residual norm of methods for Example 1 ($n = 20$).

Table 2: *Numerical results for Example 2 with* $\omega_{opt} = 1.3$.

| | $P_S SOR$ | | $SOR$ | | $PSOR$ | |
|---|---|---|---|---|---|---|
| $n$ | iter | time | iter | time | iter | time |
| 10 | 23 | 0.007 | 35 | 0.012 | 26 | 0.010 |
| 50 | 25 | 0.017 | 41 | 0.034 | 32 | 0.020 |
| 100 | 26 | 0.117 | 42 | 0.121 | 35 | 0.123 |
| 150 | 26 | 0.380 | 42 | 0.402 | 37 | 0.480 |

where $c_0 = 1/2$, $c_1 = 1/3$, and $a = 2$. Numerical results in Table 2 with different sizes of $\mathscr{A}$ indicate that $P_s SOR$ converges faster than $SOR$ and $PSOR$ for the optimal value of $\omega$.

Furthermore, from Fig. 2, we can observe that $P_s SOR$ converges faster than the others.

**Example 3.** Let $\mathscr{B} \in \mathbb{R}^{[3,n]}$ be a nonnegative tensor with $M(\mathscr{B}) = rand(n,n)$, where "*rand*" is a function in MATLAB that produces an n by n matrix with entries in $[0,1]$ uniformly. The other entries of $\mathscr{B}$ are also considered zero. For $i = 2,3,\ldots,n$, $b_{ii-1i} = b_{iii-1} = 1/6$, and for $i = 2,3,\ldots,n-1$, $b_{ii+1i} = b_{iii+1} = 1/6$. Consider $\mathscr{A} = n^2 \mathscr{I}_m - 0.01\mathscr{B}$. The right-hand side vector $b$ is taken equal to $ones(n,1)$. In this example, we also compare $PSOR$ with the Newton method in [15]. Numerical results in Table 3 with different sizes of $\mathscr{A}$ represent that $P_s SOR$ converges faster than $PSOR$, $SOR$, and the $Newton$ method for the optimal value of $\omega$, especially when $n$ is large.

In addition, the number of iterations required for $P_s SOR$ to converge for different values of $\omega$ from 0.1 to 2 with a step length of 0.1 is shown in Figure 3. It can be observed that the optimal value of $\omega$ is 1.01.

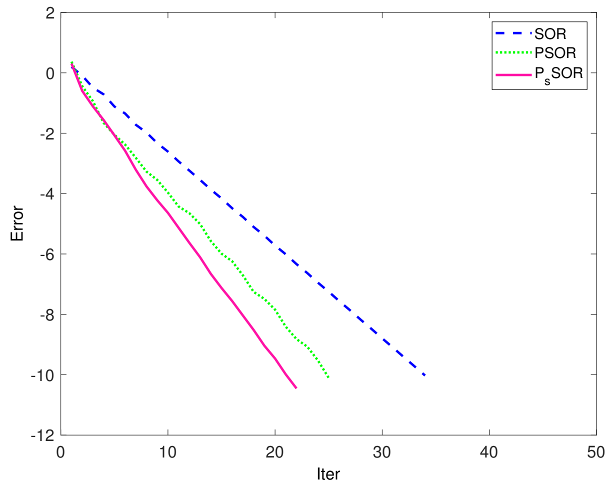


Figure 2: Performance of methods in reducing residual norm of methods for Example 2 ($n = 10$).

Table 3: *Numerical results of Example 3 with* $\omega_{opt} = 1.01$.

| | $P_S SOR$ | | $SOR$ | | $PSOR$ | | $Newton$ | |
|---|---|---|---|---|---|---|---|---|
| $n$ | iter | time | iter | time | iter | time | iter | time |
| 5 | 7 | 0.006 | 7 | 0.009 | 7 | 0.006 | 7 | 0.006 |
| 50 | 8 | 0.011 | 9 | 0.014 | 9 | 0.012 | 11 | 0.046 |
| 150 | 9 | 0.016 | 9 | 0.156 | 9 | 0.018 | 12 | 0.167 |

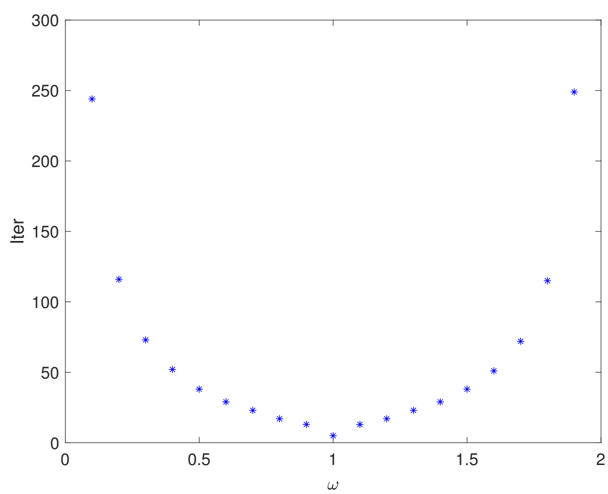


Figure 3: Search to find the optimal parameter $\omega$ for Example 3 by taking $n = 5$.

# 5 Conclusions

The purpose of this paper is to introduce a novel tensor splitting SOR preconditioner that can effectively solve multi-linear systems. To evaluate the performance of the proposed method, we applied both the standard SOR method and its preconditioned versions by a new preconditioner that we proposed and a preconditioner that was introduced before, on several numerical examples. Our analysis of the results indicate that the presented preconditioner leads to a significant reduction in the number of iterations and CPU time required for convergence compared to the standard SOR method.

# References

[1] L. Cui, C. Chen, W. Li, M. Ng, *An eigenvalue problem for even order tensors with its applications*, Linear Multilinear Algebra **64** (2016) 602–621.

[2] L. Cui, W. Li, M. Ng, *Primitive Tensors and Directed Hypergraphs*, Linear Algebra Appl. **471** (2015) 96–108.

[3] D. Liu, W. Li, W. Vong, *The tensor splitting with application to solve multi-linear systems*, J. Comput. Appl. Math. **330** (2018) 75–94.

[4] J. Xie, Q. Jin, M. Wei, *Tensor methods for solving symmetric $\mathscr{M}$-tensor systems*, J. Sci. Comput. **74** (2018) 412–425.

[5] J. Hofbauer, K. Sigmund, *Evolutionary game dynamics*, Bull. Am. Math. Soc. **40** (2003) 479–519.

[6] P.D. Taylor, L.B. Jonker, *Evolutionary stable strategies and game dynamics*, Math. Biosci. **40** (1987) 145–156.

[7] M. Che, L. Qi, Y. Wei, *Positive-definite tensors to nonlinear complementarity problems*, J. Optim. Theory Appl. **168** (2016) 475–487.

[8] M. Che, Y. Wei, *Theory and Computation of Complex Tensors and its Applications*, Singapore, Asia: Springer, 2020.

[9] W. Ding, Z. Luo, L.Qi, *P-tensors, $P_0$-tensors, and tensor complementarity problem*, Linear Algebra Appl. **555** (2018) 336–354.

[10] J.M. Smith, G.R. Price, *The logic of animal conflict*, Nature. **246** (1973) 15–18.

[11] Z. Luo, L. Qi, N. Xiu, *The sparsest solutions to $\mathscr{Z}$-tensor complementarity problems*, Optim. Lett. **11** (2017) 471-482.

[12] X. Li, M. Ng, *Solving sparse non-negative tensor equations: algorithms and applications*, Front Math. China. **10** (2015) 649–680.

[13] J. Yang, X. Zhao X, T. Ji, T. Ma, T. Huang, *Low-rank tensor train for tensor robust principal component analysis*, Appl. Math. Comput. **367** (2020) 124783.

[14] J. Yang, X. Zhao, J. Mei, S. Wang, T. Ma, T. Huang, *Total variation and high-order total variation adaptive model for restoring blurred images with Cauchy noise*, Comput. Math. Appl. **77** (2019) 1255–1272.

[15] W. Ding, Y. Wei, *Solving multi-linear systems with $\mathscr{M}$-tensors*, J. Sci. Comput. **68** (2016) 689–715.

[16] X. Wang, M. Che, Y. Wei, *Neural networks based approach solving multi-linear systems with M-tensors*, Neurocomputing **351** (2019) 33–42.

[17] L. Han, *A homotopy method for solving multi-linear systems with $\mathscr{M}$-tensors*, Appl. Math. Lett. **69** (2017) 49–54.

[18] H. He, C. Ling, L. Qi, G. Zhou, *A globally and quadratically convergent algorithm for solving ulti linear systems with $\mathscr{M}$-tensors*, J. Sci. Comput. **76** (2018) 1718–1741.

[19] W. Li, D. Liu, S.W. Vong, *Comparison results for splitting iterations for solving multi-linear systems*, Appl. Numer. Math. **134** (2018) 105–121.

[20] H.S. Najafi, S.A. Edalatpanah, *Iterative methods with analytical preconditioning technique to linear complementarity problems: application to obstacle problems*, RAIRO-Oper. Res. **47** (2013) 59–71.

[21] L. Cui, M. Li, Y. Song, *Preconditioned tensor splitting iterations method for solving multi-linear systems*, Appl. Math. Lett. **96** (2019) 89–94.

[22] Y. Zhang, Q. Liu, Zh. Chen, *Preconditioned Jacobi type method for solving multi-linear systems with $\mathscr{M}$-tensors*, Appl. Math. Lett. **104** (2020) 21–31.

[23] D. Liu, W. Li, S.W. Vong, *A new preconditioned SOR method for solving multi-linear systems with an $\mathscr{M}$-tensor*, Calcolo. **57** (2020) 15.

[24] F.P.A. Beik, M. Najafi-Kalyani, K. Jbilou, *Preconditioned iterative methods for multi-linear systems based on the majorization matrix*, Linear Multilinear Algebra **70** (2021) 5827–5846.

[25] T.G. Kolda, B.W. Bader, *Tensor decompositions and applications*, SIAM Rev. **51** (2009) 455–500.

[26] D. Liu, W. Li, S.W. Vong, *Tensor complementarity problems: the GUS-property and an algorithm*, Linear Multilinear Algebra **66** (2018) 1726–1749.

[27] K. Pearson, *Essentially positive tensors*, Int. J. Algebra **4** (2010) 421–427.

[28] J. Shao, L. You, *On some properties of three different types of triangular blocked tensors*, Linear Algebra Appl. **511** (2016) 110–140.

[29] W. Liu, W. Li, *On the inverse of a tensor*, Linear Algebra Appl. **495** (2016) 199–205.

[30] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symb. Comput. **40** (2005) 1302–1324.

[31] L.H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing, CAMSAP 05, IEEE Computer Society Press, Piscataway, NJ. **1** (2005) 129–132

[32] W. Ding, L. Qi, Y. Wei, *M-tensors and nonsingular M-tensors*, Linear Algebra Appl. **439** (2013) 3264–3278.

[33] L. Zhang, L. Qi, G. Zhou, *M-tensors and some applications*, SIAM J. Matrix Anal. Appl. **35** (2014) 437–452.

[34] P. Azimzadeh, E. Bayraktar, *High order bellman equations and weakly chained diagonally dominant tensors*, SIAM J. Matrix Anal. Appl. **40** (2019) 276–298.