Journal of Mathematical Modeling
Vol. 12, No. 1, 2024, pp. 33–49. Research Article

JMM

A fitted mesh method for a class of two-parameter nonlinear singular perturbation problems

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Abstract. A class of two-parameter singularly perturbed nonlinear second order ordinary differential equations is considered in this article. A fitted mesh method which is a combination of finite difference scheme and a Shishkin mesh is developed to solve the problems. The method is proved to be essentially first order parameter independent convergent. Numerical experiments support the established theoretical results.

Keywords: Two-parameter nonlinear singular perturbation problems, boundary layers, Shishkin mesh, parameter independent convergence.
AMS Subject Classification 2010: 65L11, 65L12, 65L20, 65L70.

1 Introduction

Physical problems are often related with Boundary Value Problems (BVPs) involving many small parameters. Precisely, a second order BVP whose derivatives are multiplied by different small parameters arise in chemical reactor theory [2] and lubrication theory [3]. The investigation of these problems was initiated by O'Malley [13].

Since Singular Perturbation Problems (SPPs) exclude exact solutions and further classical numerical methods fail to solve these problems, new methods are developed to analyze these problems. In the literature, several numerical methods are available for different types of two-parameter linear SPPs; some of the methods are mentioned below.

A first order convergent numerical method involving a Finite Difference Scheme (FDS) is constructed for a two-parameter linear SPP in [8]. For this problem, a finite element method is developed in [16]. Article [15] deals with a higher order compact numerical method for aforementioned problem. In [1],

*Corresponding author
Received: 08 August 2023 / Revised: 15 October 2023 / Accepted: 05 Nowember 2023
DOI: 10.22124/jmm.2023.25201.2237

a Haar wavelet multi-resolution method is developed for various types of SPPs which includes a two-parameter linear SPP also.

Gracia et al. [5] solved a two-parameter SPP by a second order monotone numerical method. For the same problem, a quintic B-spline method of order four is designed in [12]. For this type of problem with discontinuous convection coefficient and source term, a FDS with Shishkin-Bakhvalov type mesh is constructed in [17]. Tariku Birabasa et al. developed a second order computational method for a two-parameter SPP of parabolic type in [10].

A hybrid scheme which is a combination of central difference method and an upwind method is designed in [6] for a two-parameter SPP of elliptic type for which a finite element method involving Lagrange-type interpolation together with a Bakhvalov-type mesh is developed in [20]. Ram et al. [18] constructed a numerical method involving a FDS and a Shishkin mesh for aforementioned problem with a discontinuous source data.

In the literature, very few methods are available for nonlinear SPPs. It is highly complicated to establish a parameter independent method for nonlinear SPPs; the occurrence of small quantities in two-parameter SPPs increase the complexity in establishing a parameter independent method for such problems. Further, one of the reduced problems of a two-parameter nonlinear SPP itself is a nonlinear SPP which also increase the difficulty in solving a two-parameter nonlinear SPP.

No numerical method is available in the literature for a two-parameter nonlinear SPP except the following. In [19], a numerical method is developed to solve a second order semilinear two-parameter SPP through exponential spline technique in which the semilinear function $f(x,y)$ is expressed as the linear form $r(x)y-g(x)$ in order to know the behavior of the derivatives.

In the present article, a numerical method which is a combination of FDS and a Shishkin mesh is developed for a class of two-parameter nonlinear SPPs. Intermediate value theorem plays a vital role in establishing the theoretical results. In this work no artificial condition is imposed either on the problem or on the perturbation parameters. Figures included in the present article clearly exhibit the behaviour of the boundary layers and also different widths of the boundary layers. The two conditions based on which the present problem has been divided into two different cases are utilized cleverly to establish a parameter independent numerical method in both the cases.

2 The main problem

Precisely, the two-parameter nonlinear SPP under consideration is

$$\mathbb{T}u(t) = \varepsilon u''(t) + \mu\, a(t)u'(t) - f(t, u(t)) = 0 \text{ on } \Omega = (0,1), \tag{1}$$

$$\text{with } u(0) = u_0 \text{ and } u(1) = u_1, \tag{2}$$

where u_0 and u_1 are given constants, $0 < \varepsilon < 1$ and $0 < \mu < 1$. It is assumed that for all $t \in \overline{\Omega} = [0,1]$, $a \in C^3(\overline{\Omega})$ such that $a(t) \geq \alpha > 0$ and for all $(t, z(t)) \in \overline{\Omega} \times \mathbb{R}$, the nonlinear term $f(t, z(t)) \in C^3(\overline{\Omega} \times \mathbb{R})$ such that

$$\frac{\partial f(t, z(t))}{\partial z} \geq \beta > 0.$$

Further,

$$\gamma \leq \min_{t \in \overline{\Omega}} \left(\frac{\partial f(t, z(t))/\partial z}{a(t)} \right).$$

The above conditions and the implicit function theorem ensures the existence of a unique solution $u(t)$ to (1)-(2) such that $u \in C^3(\overline{\Omega})$ [14].

As reported in [13], based on the ratio of μ^2 to ε, the solution $u(t)$ of (1)-(2) exhibits boundary layers of different widths near both the boundaries $t=0$ and $t=1$. Thus the two cases $\frac{\mu^2}{\varepsilon} \leq \frac{\gamma}{\alpha}$ and $\frac{\mu^2}{\varepsilon} \geq \frac{\gamma}{\alpha}$ are considered separately in this article.

Case (1): $\frac{\mu^2}{\varepsilon} \leq \frac{\gamma}{\alpha}$
In this case, when $\varepsilon = 0$, (1)-(2) becomes

$$f(t, r_1(t)) = 0 \text{ on } \overline{\Omega}. \tag{3}$$

In this circumstance, for $u(t)$, a boundary layer of width $\mathscr{O}(\sqrt{\varepsilon})$ is expected at both the neighbourhoods of $t=0$ and $t=1$.

From (3), for $0 \leq k \leq 3$ and $t \in \overline{\Omega}$,

$$|r_1^{(k)}(t)| \leq C. \tag{4}$$

Case (2): $\frac{\mu^2}{\varepsilon} \geq \frac{\gamma}{\alpha}$
In this case, when $\varepsilon = 0$, (1)-(2) becomes

$$\mu\, a(t)\, r_2'(t) - f(t, r_2(t)) = 0 \text{ on } [0,1), \quad r_2(1) = u_1, \tag{5}$$

which is also a nonlinear SPP. In this circumstance, for $u(t)$, a boundary layer of width $\mathscr{O}(\varepsilon/\mu)$ is expected at the neighbourhood of $t=0$ and that of width $\mathscr{O}(\mu)$ is expected at the neighbourhood of $t=1$.

Decompose $r_2(t)$ of (5) into $v_{r_2}(t)$ and $w_{r_2}(t)$ such that $r_2(t) = v_{r_2}(t) + w_{r_2}(t)$, where

$$\mu\, a(t)\, v_{r_2}'(t) - f(t, v_{r_2}(t)) = 0 \text{ on } [0,1),\ v_{r_2}(1) = r_2(1), \tag{6}$$

and

$$\mu\, a(t) w_{r_2}'(t) - f(t,\, v_{r_2}(t) + w_{r_2}(t)) + f(t,\, v_{r_2}(t)) = 0 \ \text{ on } [0,1), \tag{7}$$

$$w_{r_2}(1) = r_2(1) - v_{r_2}(1). \tag{8}$$

Following the arguments in [9] for an equation whose derivative is multiplied by a perturbation parameter, it can be established that, for $t \in \overline{\Omega}$ and for $0 \leq k \leq 7$,

$$|v_{r_2}^{(k)}(t)| \leq C\mu^{-(k-1)} \ \text{ and } \ |w_{r_2}^{(k)}(t)| \leq C\mu^{-k}. \tag{9}$$

In this article, C denotes a positive constant, which is free from t, ε, μ and N.

3 Analytical results

Decompose $u(t)$ of (1)-(2) into $v(t)$ and $w(t)$ such that $u(t) = v(t) + w(t)$ where

$$\varepsilon v''(t) + \mu\, a(t) v'(t) - f(t,\, v(t)) = 0 \ \text{ on } \Omega, \tag{10}$$

$$v(0), v(1) \ \text{ are suitably chosen} \tag{11}$$

and

$$\varepsilon w''(t) + \mu\, a(t) w'(t) - f(t,\, v(t) + w(t)) + f(t,\, v(t)) = 0 \ \text{ on } \Omega, \tag{12}$$

$$w(0) = u_0 - v(0), \quad w(1) = u_1 - v(1). \tag{13}$$

3.1 Bounds on $v(t)$ and its derivatives

Case (1): Using (10) and (3), we have

$$\varepsilon v''(t)+\mu a(t)v'(t)-b(t)v(t)=b(t)r_1(t)=g(t) \text{ on } \Omega, \tag{14}$$

$$v(0),v(1) \text{ are suitably chosen,} \tag{15}$$

where $b(t)=\frac{\partial f(t,\theta(t))}{\partial u}$ is an intermediate value. Equation (14) can be written as

$$\mathbb{T}'v(t)=\varepsilon v''(t)+\mu a(t)v'(t)-b(t)v(t)=g(t). \tag{16}$$

Decompose $v(t)$ as

$$v(t)=v_0(t)+\sqrt{\varepsilon}v_1(t)+(\sqrt{\varepsilon})^2v_2(t)+(\sqrt{\varepsilon})^3v_3(t), \tag{17}$$

where

$$-b(t)v_0(t)= g(t),\ t\in\overline{\Omega}, \tag{18}$$

$$b(t)v_1(t)= \sqrt{\varepsilon}v_0''(t)+\frac{\mu}{\sqrt{\varepsilon}}a(t)v_0'(t), \quad t\in\overline{\Omega}, \tag{19}$$

$$b(t)v_2(t)= \sqrt{\varepsilon}v_1''(t)+\frac{\mu}{\sqrt{\varepsilon}}a(t)v_1'(t), \quad t\in\overline{\Omega}, \tag{20}$$

$$\mathbb{T}'v_3(t)= -\sqrt{\varepsilon}v_2''(t)-\frac{\mu}{\sqrt{\varepsilon}}a(t)v_2'(t), \quad t\in\Omega, \tag{21}$$

$$v_3(0)= 0=v_3(1). \tag{22}$$

From (18), (19) and (20) and using the condition $\frac{\mu^2}{\varepsilon}\le\frac{\gamma}{\alpha}$, for $t\in\overline{\Omega}$,

$$|v_0^{(k)}(t)|\le C,\ 0\le k\le 7, \tag{23}$$

$$|v_1^{(k)}(t)|\le C,\ 0\le k\le 5, \tag{24}$$

$$|v_2^{(k)}(t)|\le C,\ 0\le k\le 3. \tag{25}$$

By utilizing Lemma 2 in [5], for $t\in\overline{\Omega}$,

$$|v_3(t)|\le C. \tag{26}$$

Also, by utilizing Theorem 1 in [7] and the condition $\frac{\mu^2}{\varepsilon}\le\frac{\gamma}{\alpha}$, for $t\in\overline{\Omega}$,

$$|v_3'(t)|\le C\varepsilon^{-1/2}. \tag{27}$$

From (21),

$$|v_3^{(k)}(t)|\le C\varepsilon^{-k/2},\ k=2,3, \tag{28}$$

and from (26), (27) and (28), for $t\in\overline{\Omega}$ and for $k=0,1,2,3$, we have

$$|v_3^{(k)}(t)|\le C\varepsilon^{-k/2}. \tag{29}$$

Finally (17), (23), (24), (25) and (29), for $t\in\overline{\Omega}$ and for $k=0,1,2,3$ implies that

$$|v^{(k)}(t)|\le C. \tag{30}$$

Case (2): Using (10) and (6), we have

$$\varepsilon v''(t) + \mu a(t) v'(t) - c(t)v(t) = \mu a(t) v'_{r_2}(t) - c(t)v_{r_2}(t) \text{ on } \Omega, \tag{31}$$

$$v(0), v(1) \text{ are suitably chosen,} \tag{32}$$

where $c(t) = \frac{\partial f(t,\eta(t))}{\partial u}$ is an intermediate value. Equation (31) can be written as

$$\mathbb{T}' v(t) = \varepsilon v''(t) + \mu a(t) v'(t) - c(t)v(t) = \mu a(t) v'_{r_2}(t) - c(t)v_{r_2}(t). \tag{33}$$

Let

$$\mathbb{T}'_1 \psi(t) = \mu a(t) \psi'(t) - c(t)\psi(t). \tag{34}$$

Decompose $v(t)$ as

$$v(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t), \tag{35}$$

where

$$\begin{aligned}
\mathbb{T}'_1 v_0(t) &= \mu a(t) v'_{r_2}(t) - c(t)v_{r_2}(t), \quad t \in [0,1), && (36)\\
& v_0(1) \quad \text{is suitably chosen} \\
\mathbb{T}'_1 v_1(t) &= -v''_0(t), \quad t \in [0,1), && (37)\\
& v_1(1) \quad \text{is suitably chosen} \\
\mathbb{T}'_1 v_2(t) &= -v''_1(t), \quad t \in [0,1), \quad v_2(1) = 0, && (38)\\
\mathbb{T}' v_3(t) &= -v''_2(t), \quad t \in \Omega, \quad v_3(0) = 0 = v_3(1). && (39)
\end{aligned}$$

Also, decompose $v_0(t)$ as

$$v_0(t) = v_{0,0}(t) + \mu v_{0,1}(t) + \mu^2 v_{0,2}(t) + \mu^3 v_{0,3}(t), \tag{40}$$

where

$$\begin{aligned}
c(t)v_{0,0}(t) &= c(t)v_{r_2}(t), \quad t \in \overline{\Omega}, && (41)\\
c(t)v_{0,1}(t) &= a(t)v'_{0,0}(t) - a(t)v'_{r_2}(t), \quad t \in \overline{\Omega}, && (42)\\
c(t)v_{0,2}(t) &= a(t)v'_{0,1}(t), \quad t \in \overline{\Omega}, && (43)\\
\mathbb{T}'_1 v_{0,3}(t) &= -a(t)v'_{0,2}(t), \quad t \in [0,1), \quad v_{0,3}(1) = 0. && (44)
\end{aligned}$$

From (41), (42) and (43), for $t \in \overline{\Omega}$ and for $0 \le k \le 7$, we have

$$|v^{(k)}_{0,0}(t)| \le C\mu^{-(k-1)}, \quad |v^{(k)}_{0,1}(t)| \le C\mu^{-k}, \quad |v^{(k)}_{0,2}(t)| \le C\mu^{-(k+1)}, \tag{45}$$

and by utilizing Lemma 4 in [7], for $t \in \overline{\Omega}$, we get

$$|v_{0,3}(t)| \le C\mu^{-2}. \tag{46}$$

Also, from (44), for $t \in \overline{\Omega}$ and for $0 \le k \le 7$,

$$|v^{(k)}_{0,3}(t)| \le C\mu^{-(k+2)}, \tag{47}$$

and from (40), (45) and (47), for $t \in \overline{\Omega}$ and for $0 \le k \le 7$,

$$|v_0^{(k)}(t)| \le C\mu^{-(k-1)}. \tag{48}$$

Now, decompose $v_1(t)$ as

$$v_1(t) = v_{1,0}(t) + \mu v_{1,1}(t) + \mu^2 v_{1,2}(t), \tag{49}$$

where

$$c(t)v_{1,0}(t) = v_0''(t), \quad t \in \overline{\Omega}, \tag{50}$$

$$c(t)v_{1,1}(t) = a(t)v_{1,0}'(t), \quad t \in \overline{\Omega}, \tag{51}$$

$$\mathbb{T}_1' v_{1,2}(t) = -a(t)v_{1,1}'(t), \quad t \in [0,1), \quad v_{1,2}(1) = 0. \tag{52}$$

From (50), (51) and (52), for $t \in \overline{\Omega}$ and for $0 \le k \le 4$, we have

$$|v_{1,0}^{(k)}(t)| \le C\mu^{-(k+1)}, \quad |v_{1,1}^{(k)}(t)| \le C\mu^{-(k+2)}, \quad |v_{1,2}^{(k)}(t)| \le C\mu^{-(k+3)}, \tag{53}$$

and from (49) and (53), for $t \in \overline{\Omega}$ and for $0 \le k \le 4$, one has

$$|v_1^{(k)}(t)| \le C\mu^{-(k+1)}. \tag{54}$$

By utilizing Lemma 4 in [7] with (38), for $t \in \overline{\Omega}$, we have

$$|v_2(t)| \le C\mu^{-3}, \tag{55}$$

and from (38), for $t \in \overline{\Omega}$ and for $0 \le k \le 4$, we get

$$|v_2^{(k)}(t)| \le C\mu^{-(k+3)}. \tag{56}$$

Also, by utilizing Lemma 2 in [5] with (39), for $t \in \overline{\Omega}$, one has

$$|v_3(t)| \le C\mu^{-5}. \tag{57}$$

Finally, from (39), for $t \in \overline{\Omega}$ and for $0 \le k \le 3$, we get

$$|v_3^{(k)}(t)| \le C\mu^{-(k+5)}. \tag{58}$$

Now, using the fact that $\frac{\varepsilon}{\mu^2} \le \frac{\alpha}{\gamma}$ and from (35), (48), (54), (56) and (58), for $t \in \overline{\Omega}$ and for $k = 0,1,2,3$, we have

$$|v^{(k)}(t)| \le C\mu^{-(k-1)}. \tag{59}$$

3.2 Bounds on $w(t)$ and its derivatives

Using (12), implies

$$\varepsilon w''(t) + \mu a(t) w'(t) - d(t)w(t) = 0, \tag{60}$$

where $d(t) = \frac{\partial f(t,\phi(t))}{\partial u}$ is an intermediate value. Equation (60) can be written as

$$\mathbb{T}'w(t) = \varepsilon w''(t) + \mu a(t) w'(t) - d(t)w(t) = 0. \tag{61}$$

The component $w(t)$ is decomposed further as $w(t) = w_L(t) + w_R(t)$, where

$$\mathbb{T}' w_L(t) = 0, \ t \in \Omega, \quad w_L(0) = w(0) \quad \text{and} \quad w_L(1) = 0, \tag{62}$$

and

$$\mathbb{T}' w_R(t) = 0, \ t \in \Omega, \quad w_R(0) = 0 \quad \text{and} \quad w_R(1) = w(0). \tag{63}$$

Theorem 1. *For all $t \in \overline{\Omega}$ and for $0 \le k \le 3$, in Case (1),*

$$|w_L^{(k)}(t)| \le C\varepsilon^{-k/2}, \quad |w_R^{(k)}(t)| \le C\varepsilon^{-k/2},$$

and in Case (2),

$$|w_L^{(k)}(t)| \le C\left(\frac{\mu}{\varepsilon}\right)^k, \quad |w_R^{(k)}(t)| \le C\mu^{-k}.$$

Proof. From equations (62) and (63), the bounds on w_L, w_R and their derivatives can be derived in both cases by following a similar procedure in [7] on the interval $\overline{\Omega}$. □

4 Mesh and discrete problem

On $\overline{\Omega}$, a Shishkin mesh is constructed as follows. Let $\Omega^N = \{t_j\}_{j=1}^{N-1}$ then $\overline{\Omega}^N = \{t_j\}_{j=0}^{N}$. The $\overline{\Omega}$ is divided into $[0,\tau_1]$, $(\tau_1, 1-\tau_2]$ and $(1-\tau_2, 1]$ such that $\overline{\Omega} = [0,\tau_1] \cup (\tau_1, 1-\tau_2] \cup (1-\tau_2, 1]$. Parameters τ_1 and τ_2, separating the uniform meshes are defined by

$$\tau_1 = \begin{cases} \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln N\right\}, & \text{if} \quad \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\alpha}, \\ \min\left\{\frac{1}{4}, \frac{2\varepsilon}{\mu\alpha}\ln N\right\}, & \text{if} \quad \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\alpha}, \end{cases}$$

and

$$\tau_2 = \begin{cases} \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln N\right\}, & \text{if} \quad \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\alpha}, \\ \min\left\{\frac{1}{4}, \frac{2\mu}{\gamma}\ln N\right\}, & \text{if} \quad \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\alpha}. \end{cases}$$

From the total N mesh points, $\frac{N}{4}$ mesh points are placed uniformly on each of the sub-domains $[0,\tau_1]$ and $[1-\tau_2, 1]$. The remaining $\frac{N}{2}$ mesh points are placed on the sub-domain $[\tau_1, 1-\tau_2]$. Let h_1, h_2 and h_3 denote the step size in $[0,\tau_1]$, $[\tau_1, 1-\tau_2]$ and $[1-\tau_2, 1]$ respectively. Then $h_1 = \frac{4\tau_1}{N}$, $h_2 = \frac{2(1-\tau_1-\tau_2)}{N}$ and $h_3 = \frac{4\tau_2}{N}$.

The discrete problem corresponding to (1)-(2) is defined to be

$$\mathbb{T}^N U(t_j) = \varepsilon\,\delta^2 U(t_j) + \mu\, a(t_j) D^+ U(t_j) - f(t_j, U(t_j)) = 0, \quad \text{for} \quad t_j \in \Omega^N, \tag{64}$$

$$U(t_0) = u(t_0) \quad \text{and} \quad U(t_N) = u(t_N). \tag{65}$$

Here

$$\delta^2 Z(t_j) = \frac{(D^+ - D^-)Z(t_j)}{\overline{h}_j}, \quad D^+ Z(t_j) = \frac{Z(t_{j+1}) - Z(t_j)}{h_{j+1}}, \quad D^- Z(t_j) = \frac{Z(t_j) - Z(t_{j-1})}{h_j},$$

where $h_j = t_j - t_{j-1}$, $\overline{h}_j = \frac{h_{j+1}+h_j}{2}$, $\overline{h}_0 = \frac{h_1}{2}$ and $\overline{h}_N = \frac{h_N}{2}$.

5 Error analysis

Let Θ_1 and Θ_2 be mesh functions such that $\Theta_1(t_0) = \Theta_2(t_0)$ and $\Theta_1(t_N) = \Theta_2(t_N)$. For $t_j \in \Omega^N$,

$$\begin{aligned}(\mathbb{T}^N\Theta_1 - \mathbb{T}^N\Theta_2)(t_j) &= \varepsilon\,\delta^2(\Theta_1-\Theta_2)(t_j) + \mu\,a(t_j)D^+(\Theta_1-\Theta_2)(t_j)\\ &\quad - f(t_j,\Theta_1(t_j)) + f(t_j,\Theta_2(t_j))\\ &= \varepsilon\,\delta^2(\Theta_1-\Theta_2)(t_j) + \mu\,a(t_j)D^+(\Theta_1-\Theta_2)(t_j) - e(t_j)(\Theta_1-\Theta_2)(t_j)\\ &= (\mathbb{T}^N)'(\Theta_1-\Theta_2)(t_j), \end{aligned} \tag{66}$$

where $e(t_j) = \frac{\partial f(t_j,\zeta(t_j))}{\partial u}$ is an intermediate value and $(\mathbb{T}^N)'$ is the Frechet derivative of $\mathbb{T}^N$.

Theorem 2. *If Ψ is a mesh function such that $\Psi(t_0) \geq 0$, $\Psi(t_N) \geq 0$ and $(\mathbb{T}^N)'\Psi \leq 0$ on Ω^N then $\Psi \geq 0$ on $\overline{\Omega}^N$.*

Proof. Let j^* be such that $\Psi(t_{j^*}) = \min_j \Psi(t_j)$ and suppose $\Psi(t_{j^*}) < 0$. Then $j^* \neq 0, N$. Let $t_{j^*} \in \Omega^N$. We have

$$(\mathbb{T}^N)'\Psi(t_{j^*}) = \varepsilon\,\delta^2\Psi(t_{j^*}) + \mu\,a(t_{j^*})D^+\Psi(t_{j^*}) - e(t_{j^*})\Psi(t_{j^*}) > 0,$$

which is a contradiction. Hence $\Psi \geq 0$ on $\overline{\Omega}^N$. □

Theorem 3. *If Ψ is any mesh function on $\overline{\Omega}^N$, then for any $t_j \in \overline{\Omega}^N$*

$$|\Psi(t_j)| \leq \max\left\{|\Psi(t_0)|,\ |\Psi(t_N)|,\ |(\mathbb{T}^N)'\Psi(t_j)|_{\Omega^N}\right\}.$$

Proof. Let $t_j \in \overline{\Omega}^N$. Consider

$$\Phi^{\pm}(t_j) = \max\left\{|\Psi(t_0)|,\ |\Psi(t_N)|,\ |(\mathbb{T}^N)'\Psi(t_j)|_{\Omega^N}\right\} \pm \Psi(t_j).$$

Then $\Phi^{\pm}(t_j) \geq 0$ for $j = 0, N$. Using the properties of $a(t_j)$ and $e(t_j)$, it is not hard to verify that $(\mathbb{T}^N)'\Phi^{\pm} \leq 0$ on Ω^N. Hence by Theorem 2, $\Phi^{\pm} \geq 0$ on $\overline{\Omega}^N$. □

Since $(\mathbb{T}^N)'$ is linear, using Theorem 3 with (66), implies that

$$|(\Theta_1-\Theta_2)(t_j)| \;\leq\; C\,|(\mathbb{T}^N)'(\Theta_1-\Theta_2)(t_j)| = C\,|\mathbb{T}^N\Theta_1(t_j) - \mathbb{T}^N\Theta_2(t_j)|. \tag{67}$$

Similar to the continuous case, U can be decomposed into V and W such that for $t_j \in \Omega^N$, in Case (1),

$$\varepsilon\,\delta^2V(t_j) + \mu\,a(t_j)D^+V(t_j) - b(t_j)V(t_j) = g(t_j), \tag{68}$$

$$V(t_0) = v(t_0), \quad V(t_N) = v(t_N), \tag{69}$$

in Case (2),

$$\varepsilon\,\delta^2V(t_j) + \mu\,a(t_j)D^+V(t_j) - c(t_j)V(t_j) = \mu\,a(t_j)D^+v_{r_2}(t_j) - c(t_j)v_{r_2}(t_j), \tag{70}$$

$$V(t_0) = v(t_0), \quad V(t_N) = v(t_N) \tag{71}$$

and for $t_j \in \Omega^N$,

$$\varepsilon\,\delta^2 W(t_j) + \mu\, a(t_j) D^+ W(t_j) - d(t_j) W(t_j) = 0, \tag{72}$$

$$W(t_0) = w(t_0), \quad W(t_N) = w(t_N). \tag{73}$$

Equations (68), (70) and (72) can be written in operator form as follows

$$(\mathbb{T}^N)' V(t_j) = \varepsilon\,\delta^2 V(t_j) + \mu\, a(t_j) D^+ V(t_j) - b(t_j) V(t_j) = g(t_j), \tag{74}$$

$$\begin{aligned}(\mathbb{T}^N)' V(t_j) &= \varepsilon\,\delta^2 V(t_j) + \mu\, a(t_j) D^+ V(t_j) - c(t_j) V(t_j) \\ &= \mu\, a(t_j) D^+ v_{r_2}(t_j) - c(t_j) v_{r_2}(t_j),\end{aligned} \tag{75}$$

and

$$(\mathbb{T}^N)' W(t_j) = \varepsilon\,\delta^2 W(t_j) + \mu\, a(t_j) D^+ W(t_j) - d(t_j) W(t_j) = 0. \tag{76}$$

Theorem 4. *Let v be the solution of (14)-(15) and V be that of (68)-(69). Then for $t_j \in \overline{\Omega}^N$,*

$$|(V - v)(t_j)| \;\le\; CN^{-1}.$$

Proof. Let $t_j \in \overline{\Omega}^N$. From (67) we have

$$|(V - v)(t_j)| \;\le\; C\,|(\mathbb{T}^N)'(V - v)(t_j)|. \tag{77}$$

Since the operator $(\mathbb{T}^N)'$ is linear, from [11] (Chapter 8, p.70) one has

$$|(\mathbb{T}^N)'(V - v)(t_j)| \le CN^{-1}\left(\varepsilon\,|v|_3 + \mu\,|v|_2\right). \tag{78}$$

Using (30) with (78),

$$|(\mathbb{T}^N)'(V - v)(t_j)| \le CN^{-1}. \tag{79}$$

Now, from (77) and (79), for $t_j \in \overline{\Omega}^N$, we have

$$|(V - v)(t_j)| \;\le\; CN^{-1}, \tag{80}$$

which completes the proof. □

Theorem 5. *Let v be the solution of (31)-(32) and V be that of (70)-(71). Then for $t_j \in \overline{\Omega}^N$,*

$$|(V - v)(t_j)| \;\le\; CN^{-1}.$$

Proof. Let $t_j \in \overline{\Omega}^N$. Using (59) with (78) we get

$$|(\mathbb{T}^N)'(V - v)(t_j)| \le CN^{-1}\left(\frac{\varepsilon}{\mu^2} + 1\right). \tag{81}$$

In this case using the fact that $\frac{\varepsilon}{\mu^2} \le C$, (81) becomes

$$|(\mathbb{T}^N)'(V - v)(t_j)| \le CN^{-1}. \tag{82}$$

Now, from (77) and (82), for $t_j \in \overline{\Omega}^N$, we have

$$|(V - v)(t_j)| \;\le\; CN^{-1}. \tag{83}$$

□

Analogous to the continuous case, W can be decomposed into W_L and W_R such that $W = W_L + W_R$ where for $t_j \in \Omega^N$,

$$(\mathbb{T}^N)' W_L(t_j) = 0, \quad W_L(t_0) = w_L(t_0), \quad W_L(t_N) = 0, \tag{84}$$

and

$$(\mathbb{T}^N)' W_R(t_j) = 0, \quad W_R(t_0) = 0, \quad W_R(t_N) = w_R(t_N). \tag{85}$$

Theorem 6. *Let w_L, w_R, W_L and W_R be the solutions of (62), (63), (84) and (85), respectively. Then for $t_j \in \overline{\Omega}^N$,*

$$|(W_L - w_L)(t_j)| \le \begin{cases} CN^{-1}\ln N, & \text{if } \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\alpha} \\ CN^{-1}(\ln N)^2, & \text{if } \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\alpha} \end{cases} \tag{86}$$

and in both cases

$$|(W_R - w_R)(t_j)| \le CN^{-1}\ln N. \tag{87}$$

Proof. The results follow by using similar procedure in [7]. □

Theorem 7. *Let u be the solution of (1)-(2) and U be that of (64)-(65). Then for $t_j \in \overline{\Omega}^N$, we have*

$$|(U - u)(t_j)| \le \begin{cases} CN^{-1}\ln N, & \text{if } \frac{\mu^2}{\varepsilon} \le \frac{\gamma}{\alpha} \\ CN^{-1}(\ln N)^2, & \text{if } \frac{\mu^2}{\varepsilon} \ge \frac{\gamma}{\alpha}. \end{cases}$$

Proof. The result follows by using the triangle inequality, Theorems 4, 5 and 6. □

6 Numerical experiments

Three examples are presented in this section in which both Case (1) and Case (2) are considered separately. A variant of the continuation method in [4] is used to calculate the numerical approximations. The $\varepsilon\mu$-uniform order of convergence and the $\varepsilon\mu$-uniform error constant are computed using the general methodology from [4].

Example 1. Consider

$$\varepsilon u''(t) + \mu(2 - t^2 + \sin(t))u'(t) - \left((u(t))^5 + 2u(t) - \frac{1}{1+\sqrt{\pi}}\right) = 0, \quad t \in (0,1),$$

with $u(0) = e^{-0.5}$ and $u(1) = \cos(0.7)$.

Example 2. Consider

$$\varepsilon u''(t) + \mu(3+t)u'(t) - \left((u(t))^7 + 3u(t) - 1\right) = 0, \quad t \in (0,1),$$

with $u(0) = 1 - e^{-1.5}$ and $u(1) = \sin(0.7)$.

Example 3. Consider

$$\varepsilon u''(t) + \mu\left(2 + e^{-t}\right)u'(t) - \left((u(t))^7 + 2u(t) - t\right) = 0, \quad t \in (0,1),$$

with $u(0) = 0.5$ and $u(1) = e^{-0.5} + \sin(0.5)$.

Table 1: $\alpha = 0.9$, $\gamma = 1$ and $\varepsilon = 2^{-6}$.

μ	N				
	64	128	256	512	1024
2^{-14}	3.0559e-04	7.8080e-05	1.9978e-05	5.1886e-06	1.3931e-06
2^{-15}	3.0406e-04	7.7322e-05	1.9595e-05	4.9977e-06	1.2973e-06
2^{-16}	3.0329e-04	7.6943e-05	1.9403e-05	4.9021e-06	1.2496e-06
2^{-17}	3.0291e-04	7.6754e-05	1.9307e-05	4.8544e-06	1.2258e-06
2^{-18}	3.0272e-04	7.6659e-05	1.9260e-05	4.8307e-06	1.2139e-06
2^{-19}	3.0262e-04	7.6612e-05	1.9236e-05	4.8188e-06	1.2080e-06
2^{-20}	3.0257e-04	7.6588e-05	1.9224e-05	4.8129e-06	1.2050e-06
2^{-21}	3.0255e-04	7.6576e-05	1.9218e-05	4.8100e-06	1.2035e-06
2^{-22}	3.0254e-04	7.6570e-05	1.9215e-05	4.8085e-06	1.2028e-06
2^{-23}	3.0253e-04	7.6567e-05	1.9213e-05	4.8077e-06	1.2024e-06
2^{-24}	3.0253e-04	7.6566e-05	1.9212e-05	4.8074e-06	1.2022e-06
2^{-25}	3.0253e-04	7.6565e-05	1.9212e-05	4.8072e-06	1.2021e-06
2^{-26}	3.0253e-04	7.6565e-05	1.9212e-05	4.8071e-06	1.2021e-06
D^N	3.0559e-04	7.8080e-05	1.9978e-05	5.1886e-06	1.3931e-06
p^N	1.8971e+00	1.9665e+00	1.9450e+00	1.8971e+00	
C_p^N	1.1154e+00	1.0615e+00	1.0116e+00	9.7855e-01	9.7855e-01

Case (1): $\frac{\mu^2}{\varepsilon} \leq \frac{\gamma}{\alpha}$

In this case, for Example 1, the maximum error for particular values of N, μ, ε, the maximum pointwise error D^N, the $\varepsilon\mu$-uniform order of convergence p^N and the $\varepsilon\mu$-uniform error constant C_p^N are presented in Table 1 and a graph of the numerical solution for $N = 256$, $\varepsilon = 2^{-6}$ and $\mu = 2^{-14}$ is provided in Figure 1a. The solution $u(t)$ is provided in Figure 1b for the values $N = 256$, $\varepsilon = 2^{-6}$ and $\mu = 2^{-8}, 2^{-13}$.

Further, the solution $u(t)$ is also provided in Figure 2a and Figure 2b for the values $N = 256$, $\varepsilon = 2^{-10}$, $\mu = 2^{-6}$ and $N = 256$, $\varepsilon = 2^{-15}$, $\mu = 2^{-14}$ respectively, such that the condition $\frac{\mu^2}{\varepsilon} \leq \frac{\gamma}{\alpha}$ is satisfied.

As mentioned in Section 2, from Figure 2b, it is clear that the solution $u(t)$ of Example 1 exhibits a boundary layer of width $\mathscr{O}(\sqrt{\varepsilon})$ at both the neighbourhoods of $t = 0$ and $t = 1$.

For Example 2 and Example 3, the maximum error for particular values of N, μ, ε, the maximum pointwise error D^N, the $\varepsilon\mu$-uniform order of convergence p^N and the $\varepsilon\mu$-uniform error constant C_p^N are presented in Table 3 and Table 5, respectively.

Case (2): $\frac{\mu^2}{\varepsilon} \geq \frac{\gamma}{\alpha}$

In this case, for Example 1, the maximum error for particular values of N, μ, ε, the maximum pointwise error D^N, the $\varepsilon\mu$-uniform order of convergence p^N and the $\varepsilon\mu$-uniform error constant C_p^N are presented in Table 2 and a graph of the numerical solution for $N = 256$, $\mu = 2^{-6}$ and $\varepsilon = 2^{-14}$ is shown in Figure 3a. Further, the changes in the solution $u(t)$ for $N = 256$, $\mu = 2^{-6}$ and $\varepsilon = 2^{-8}, 2^{-13}$ are shown in Figure 3b.

As mentioned in Section 2, from Figure 3a, $u(t)$ of Example 1 exhibits a boundary layer of width $\mathscr{O}\left(\frac{\varepsilon}{\mu}\right)$ at the neighbourhood of $t = 0$ and a boundary layer of width $\mathscr{O}(\mu)$ at the neighbourhood of $t = 1$.

For Example 2 and Example 3, the maximum error for particular values of N, μ, ε, the maximum pointwise error D^N, the $\varepsilon\mu$-uniform order of convergence p^N and the $\varepsilon\mu$-uniform error constant C_p^N are presented in Table 4 and Table 6, respectively.

Table 2: $\alpha = 0.9$, $\gamma = 1$ and $\mu = 2^{-6}$.

ε	N				
	64	128	256	512	1024
2^{-14}	1.3417e-02	1.1886e-02	8.7671e-03	5.9443e-03	3.4693e-03
2^{-15}	1.5855e-02	1.2789e-02	9.2887e-03	6.2409e-03	3.6318e-03
2^{-16}	2.6798e-02	1.6659e-02	9.6073e-03	6.4206e-03	3.7418e-03
2^{-17}	3.6282e-02	2.4066e-02	1.4276e-02	7.7081e-03	3.9762e-03
2^{-18}	4.2894e-02	2.9743e-02	1.8314e-02	1.0190e-02	5.3730e-03
2^{-19}	4.6897e-02	3.3390e-02	2.1041e-02	1.1953e-02	6.4394e-03
2^{-20}	4.9119e-02	3.5482e-02	2.2656e-02	1.3037e-02	7.1241e-03
2^{-21}	5.0293e-02	3.6606e-02	2.3540e-02	1.3684e-02	7.5322e-03
2^{-22}	5.0896e-02	3.7190e-02	2.4003e-02	1.4028e-02	7.7520e-03
2^{-23}	5.1202e-02	3.7487e-02	2.4240e-02	1.4206e-02	7.8661e-03
2^{-24}	5.1357e-02	3.7637e-02	2.4360e-02	1.4296e-02	7.9243e-03
2^{-25}	5.1434e-02	3.7713e-02	2.4421e-02	1.4342e-02	7.9537e-03
2^{-26}	5.1473e-02	3.7751e-02	2.4451e-02	1.4365e-02	7.9685e-03
2^{-27}	5.1492e-02	3.7770e-02	2.4466e-02	1.4376e-02	7.9759e-03
2^{-28}	5.1502e-02	3.7779e-02	2.4474e-02	1.4382e-02	7.9796e-03
2^{-29}	5.1507e-02	3.7784e-02	2.4477e-02	1.4385e-02	7.9814e-03
2^{-30}	5.1509e-02	3.7786e-02	2.4479e-02	1.4386e-02	7.9823e-03
2^{-31}	5.1510e-02	3.7787e-02	2.4480e-02	1.4387e-02	7.9828e-03
2^{-32}	5.1511e-02	3.7788e-02	2.4481e-02	1.4387e-02	7.9830e-03
2^{-33}	5.1511e-02	3.7788e-02	2.4481e-02	1.4387e-02	7.9832e-03
2^{-34}	5.1511e-02	3.7788e-02	2.4481e-02	1.4388e-02	7.9832e-03
D^N	5.1511e-02	3.7788e-02	2.4481e-02	1.4388e-02	7.9832e-03
p^N	4.4695e-01	6.2627e-01	7.6685e-01	8.4978e-01	
C_p^N	1.2406e+00	1.2406e+00	1.0956e+00	8.7771e-01	6.6388e-01

Table 3: $\alpha = 2.9$, $\gamma = 1$ and $\varepsilon = 2^{-6}$.

μ	N				
	64	128	256	512	1024
2^{-14}	2.9873e-04	7.6593e-05	1.9689e-05	5.1618e-06	1.4079e-06
2^{-15}	2.9692e-04	7.5668e-05	1.9227e-05	4.9285e-06	1.2909e-06
2^{-16}	2.9602e-04	7.5206e-05	1.8996e-05	4.8119e-06	1.2325e-06
2^{-17}	2.9557e-04	7.4975e-05	1.8881e-05	4.7536e-06	1.2034e-06
2^{-18}	2.9534e-04	7.4860e-05	1.8823e-05	4.7245e-06	1.1888e-06
2^{-19}	2.9523e-04	7.4802e-05	1.8794e-05	4.7100e-06	1.1815e-06
2^{-20}	2.9517e-04	7.4773e-05	1.8780e-05	4.7028e-06	1.1779e-06
2^{-21}	2.9514e-04	7.4759e-05	1.8773e-05	4.6991e-06	1.1761e-06
2^{-22}	2.9513e-04	7.4751e-05	1.8769e-05	4.6973e-06	1.1752e-06
2^{-23}	2.9512e-04	7.4748e-05	1.8767e-05	4.6964e-06	1.1747e-06
2^{-24}	2.9512e-04	7.4746e-05	1.8766e-05	4.6960e-06	1.1745e-06
2^{-25}	2.9512e-04	7.4745e-05	1.8766e-05	4.6957e-06	1.1744e-06
2^{-26}	2.9511e-04	7.4745e-05	1.8766e-05	4.6956e-06	1.1743e-06
D^N	2.9873e-04	7.6593e-05	1.9689e-05	5.1618e-06	1.4079e-06
p^N	1.8743e+00	1.9598e+00	1.9315e+00	1.8743e+00	
C_p^N	9.9740e-01	9.3754e-01	8.8358e-01	8.4923e-01	8.4923e-01

Table 4: $\alpha = 2.9$, $\gamma = 1$ and $\mu = 2^{-6}$.

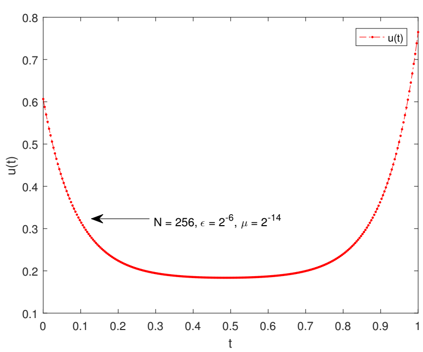
ε	N				
	64	128	256	512	1024
2^{-14}	1.3280e-02	8.2842e-03	5.2101e-03	3.1906e-03	1.8412e-03
2^{-15}	1.7279e-02	1.0414e-02	5.9364e-03	3.3006e-03	1.9016e-03
2^{-16}	1.9897e-02	1.2334e-02	7.2394e-03	3.8844e-03	2.0114e-03
2^{-17}	2.1408e-02	1.3471e-02	8.0438e-03	4.3732e-03	2.2890e-03
2^{-18}	2.2222e-02	1.4092e-02	8.4932e-03	4.6534e-03	2.4521e-03
2^{-19}	2.2644e-02	1.4417e-02	8.7310e-03	4.8062e-03	2.5407e-03
2^{-20}	2.2860e-02	1.4583e-02	8.8534e-03	4.8854e-03	2.5871e-03
2^{-21}	2.2968e-02	1.4667e-02	8.9155e-03	4.9258e-03	2.6111e-03
2^{-22}	2.3023e-02	1.4709e-02	8.9468e-03	4.9461e-03	2.6232e-03
2^{-23}	2.3050e-02	1.4730e-02	8.9625e-03	4.9563e-03	2.6293e-03
2^{-24}	2.3064e-02	1.4741e-02	8.9703e-03	4.9615e-03	2.6323e-03
2^{-25}	2.3071e-02	1.4746e-02	8.9743e-03	4.9640e-03	2.6339e-03
2^{-26}	2.3074e-02	1.4749e-02	8.9762e-03	4.9653e-03	2.6346e-03
2^{-27}	2.3076e-02	1.4750e-02	8.9772e-03	4.9659e-03	2.6350e-03
2^{-28}	2.3077e-02	1.4751e-02	8.9777e-03	4.9663e-03	2.6352e-03
2^{-29}	2.3077e-02	1.4751e-02	8.9780e-03	4.9664e-03	2.6353e-03
2^{-30}	2.3077e-02	1.4751e-02	8.9781e-03	4.9665e-03	2.6353e-03
2^{-31}	2.3077e-02	1.4751e-02	8.9782e-03	4.9666e-03	2.6354e-03
2^{-32}	2.3078e-02	1.4751e-02	8.9782e-03	4.9666e-03	2.6354e-03
2^{-33}	2.3078e-02	1.4751e-02	8.9782e-03	4.9666e-03	2.6354e-03
2^{-34}	2.3078e-02	1.4751e-02	8.9782e-03	4.9666e-03	2.6354e-03
D^N	2.3078e-02	1.4751e-02	8.9782e-03	4.9666e-03	2.6354e-03
p^N	6.4564e-01	7.1635e-01	8.5417e-01	9.1424e-01	
C_p^N	9.3773e-01	9.3773e-01	8.9287e-01	7.7271e-01	6.4144e-01

Table 5: $\alpha = 1.9$, $\gamma = 1$ and $\varepsilon = 2^{-6}$.

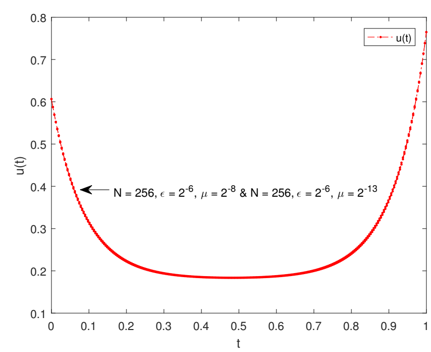
μ	N				
	64	128	256	512	1024
2^{-14}	8.6886e-04	2.3005e-04	5.8587e-05	1.4888e-05	3.8232e-06
2^{-15}	8.6715e-04	2.2926e-04	5.8192e-05	1.4690e-05	3.7246e-06
2^{-16}	8.6630e-04	2.2887e-04	5.7995e-05	1.4592e-05	3.6755e-06
2^{-17}	8.6587e-04	2.2867e-04	5.7896e-05	1.4542e-05	3.6511e-06
2^{-18}	8.6565e-04	2.2857e-04	5.7847e-05	1.4518e-05	3.6389e-06
2^{-19}	8.6555e-04	2.2852e-04	5.7822e-05	1.4505e-05	3.6329e-06
2^{-20}	8.6549e-04	2.2850e-04	5.7810e-05	1.4499e-05	3.6298e-06
2^{-21}	8.6547e-04	2.2849e-04	5.7804e-05	1.4496e-05	3.6283e-06
2^{-22}	8.6545e-04	2.2848e-04	5.7800e-05	1.4495e-05	3.6275e-06
2^{-23}	8.6545e-04	2.2848e-04	5.7799e-05	1.4494e-05	3.6271e-06
2^{-24}	8.6544e-04	2.2848e-04	5.7798e-05	1.4493e-05	3.6269e-06
2^{-25}	8.6544e-04	2.2847e-04	5.7798e-05	1.4493e-05	3.6269e-06
2^{-26}	8.6544e-04	2.2847e-04	5.7798e-05	1.4493e-05	3.6268e-06
D^N	8.6886e-04	2.3005e-04	5.8587e-05	1.4888e-05	3.8232e-06
p^N	1.9172e+00	1.9733e+00	1.9765e+00	1.9612e+00	
C_p^N	3.4299e+00	3.4299e+00	3.2991e+00	3.1662e+00	3.0710e+00

Table 6: $\alpha = 1.9$, $\gamma = 1$ and $\mu = 2^{-6}$.

ε	N				
	64	128	256	512	1024
2^{-14}	1.5641e-02	1.2120e-02	8.4836e-03	5.1766e-03	3.1124e-03
2^{-15}	2.5202e-02	1.5166e-02	8.7138e-03	5.2961e-03	3.1827e-03
2^{-16}	3.4236e-02	2.2755e-02	1.3407e-02	7.1542e-03	3.6552e-03
2^{-17}	4.1391e-02	2.9614e-02	1.8314e-02	1.0394e-02	5.4661e-03
2^{-18}	4.6316e-02	3.4736e-02	2.2192e-02	1.3296e-02	7.3085e-03
2^{-19}	4.9341e-02	3.8035e-02	2.4777e-02	1.5437e-02	8.8156e-03
2^{-20}	5.1049e-02	3.9946e-02	2.6305e-02	1.6923e-02	9.8280e-03
2^{-21}	5.1962e-02	4.0983e-02	2.7221e-02	1.7764e-02	1.0413e-02
2^{-22}	5.2434e-02	4.1523e-02	2.7774e-02	1.8213e-02	1.0729e-02
2^{-23}	5.2675e-02	4.1800e-02	2.8058e-02	1.8445e-02	1.0894e-02
2^{-24}	5.2797e-02	4.1940e-02	2.8202e-02	1.8563e-02	1.0987e-02
2^{-25}	5.2858e-02	4.2010e-02	2.8274e-02	1.8622e-02	1.1034e-02
2^{-26}	5.2888e-02	4.2045e-02	2.8311e-02	1.8652e-02	1.1058e-02
2^{-27}	5.2904e-02	4.2063e-02	2.8329e-02	1.8667e-02	1.1070e-02
2^{-28}	5.2911e-02	4.2071e-02	2.8338e-02	1.8674e-02	1.1076e-02
2^{-29}	5.2915e-02	4.2076e-02	2.8343e-02	1.8678e-02	1.1079e-02
2^{-30}	5.2917e-02	4.2078e-02	2.8345e-02	1.8680e-02	1.1080e-02
2^{-31}	5.2918e-02	4.2079e-02	2.8346e-02	1.8681e-02	1.1081e-02
2^{-32}	5.2918e-02	4.2080e-02	2.8347e-02	1.8681e-02	1.1082e-02
2^{-33}	5.2919e-02	4.2080e-02	2.8347e-02	1.8682e-02	1.1082e-02
2^{-34}	5.2919e-02	4.2080e-02	2.8347e-02	1.8682e-02	1.1082e-02
2^{-35}	5.2919e-02	4.2080e-02	2.8347e-02	1.8682e-02	1.1082e-02
D^N	5.2919e-02	4.2080e-02	2.8347e-02	1.8682e-02	1.1082e-02
p^N	3.3064e-01	5.6993e-01	6.0157e-01	7.5343e-01	
C_p^N	1.0220e+00	1.0220e+00	8.6578e-01	7.1755e-01	5.3528e-01



(a) $u(t)$ for $N = 256$, $\varepsilon = 2^{-6}$, $\mu = 2^{-14}$.

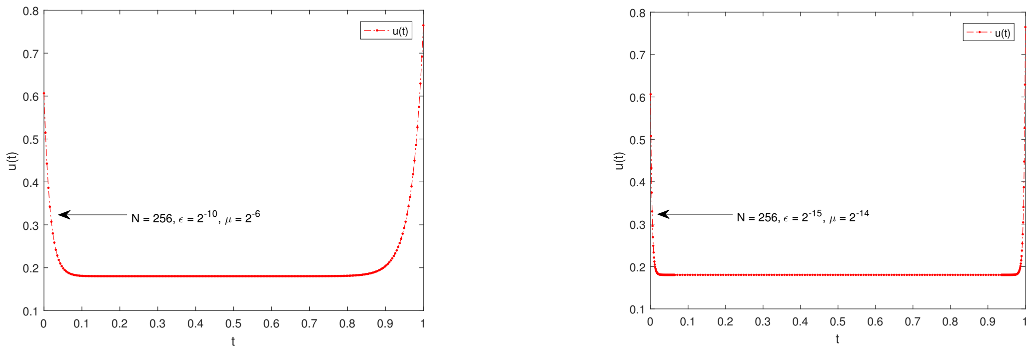


(b) $u(t)$ for $N = 256$, $\varepsilon = 2^{-6}$, $\mu = 2^{-8}, 2^{-13}$.

Figure 1: Solution profile of Example 1 in Case (1).

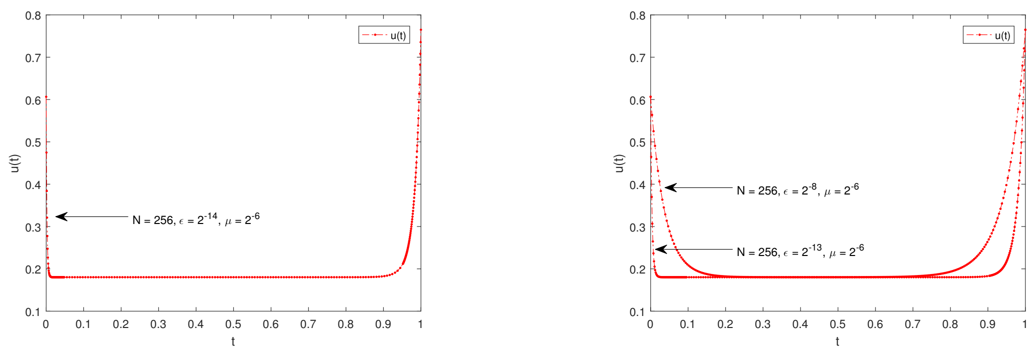
7 Conclusions

In this article, a computational method involving classical finite difference operators and a piecewise uniform Shishkin mesh is developed for a class of two-parameter singularly perturbed nonlinear differential



(a) $u(t)$ for $N = 256, \varepsilon = 2^{-10}, \mu = 2^{-6}$. (b) $u(t)$ for $N = 256, \varepsilon = 2^{-15}, \mu = 2^{-14}$.

Figure 2: Solution profile of Example 1 in Case (1).



(a) $u(t)$ for $N = 256$, $\varepsilon = 2^{-14}$, $\mu = 2^{-6}$. (b) $u(t)$ for $N = 256$, $\varepsilon = 2^{-8}, 2^{-13}$, $\mu = 2^{-6}$.

Figure 3: Solution profile of Example 1 in Case (2).

equations. It is proved both theoretically and computationally that the developed method is robust, layer resolving and parameter independent convergent.

Figures presented in this article reveal the fact that the boundary layers changes rapidly near both the boundaries of the domain of the problem. From the tables in this article, it is evident that the maximum pointwise errors decreases monotonically through the diagonal. Further, the computed rate of convergence increases monotonically whereas the computed error constant decreases monotonically when the number of mesh points N is increased; this shows the consistency of the proposed computational technique.

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