Journal of Mathematical Modeling
Vol. 12, No. 1, 2024, pp. 17–32. Research Article

JMM

Finding a time-dependent reaction coefficient of a nonlinear heat source in an inverse heat conduction problem

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Abstract. In this article, we propose an approximate technique for reconstructing a time-dependent reaction coefficient together with the surface heat flux histories and temperature distribution in a nonlinear inverse heat conduction problem (IHCP). We assume that the initial condition and the transient heat flux on the accessible boundary along with the temperature measured at specified interior locations in the domain of the problem are given as the input data. By applying the given measurements in a transformation, the main problem is reformulated as a certain parabolic problem and later a procedure based upon deploying the Ritz approximation along with the collocation method is applied which converts the problem to a nonlinear system of algebraic equations. Accurate numerical results in dealing with the exact initial and boundary data are obtained and regarding the perturbed boundary data, the regularization method based on cubic spline approximation is used, which results in obtaining stable numerical derivatives.

Keywords: Inverse heat conduction, spectral technique, regularization method, operational matrix.
AMS Subject Classification 2010: 35R30, 65N21, 65N35, 65M30.

1 Introduction

In this work, we discuss the solution of the following parabolic equation for identifying the functions $(p(t), u(x,t))$ [36]

$$u_t(x,t) - u_{xx}(x,t) = p(t)g(u(x,t)) + f(x,t), \quad (x,t) \in Q, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \quad x \in [y_2, 1], \tag{2}$$

and Neumann boundary condition

$$u_x(1,t) = 0, \quad t \in (0,T], \tag{3}$$

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Received: 29 April 2023 / Revised: 28 July 2023 / Accepted: 18 September 2023
DOI: 10.22124/jmm.2023.24413.2186

and subject to the additional temperature measurements given at two sensor positions $0 \le y_2 < y_1 \le 1$ as

$$u(y_1,t) = Y_1(t), \quad u(y_2,t) = Y_2(t), \quad t \in [0,T], \tag{4}$$

where $Q = [y_2, 1] \times [0,T]$ is a bounded domain in $\mathbb{R}^2$ and $u_0(x),\ Y_1(t),\ Y_2(t)$ are considered as the known functions with appropriate conditions describing the initial condition and temperature distribution at the known locations y_1 and y_2. The material source or reaction term $g(u)$ is assumed to be a given function of the state variable and the unknown time-dependent reaction coefficient $p(t)$ characterizes some property of the medium such as the intensity of the source term. We further suppose that the following consistency conditions hold:

$$u_0'(1) = 0, \quad u_0(y_1) = Y_1(0), \quad u_0(y_2) = Y_2(0). \tag{5}$$

Inverse problem given by the system of Eqs. (1)-(4) and other similar problems usually appear in the context of heat conduction or diffusion processes related to ignition and polymerization [15, 36] in chemical or biochemical applications since detecting the intensity of reaction in time can be used for recommendations on the choice of optimal parameters to further applications as monitoring purposes.

The IHCPs concerning the estimation of time-dependent coefficient when $g(u) = u$ has been studied in several articles [6–8, 10, 18, 19, 23, 33, 35]. Despite this fact, the case of the nonlinear function of g has received less attention. In [31], the authors considered a homogeneous IHCP for recovering the time-dependent coefficient $p(t)$ when $g(u) = u^\gamma,\ \gamma \in (0,1)$ with homogeneous Neumann boundary conditions and established the conditions under which the existence and uniqueness of the solution is guaranteed. In [4, 11], the authors proved the uniqueness of the solution of inverse problems for recovering unknown state dependent reaction term in some reaction-diffusion equation and integral identities based on associated adjoint problem derived to show the relationship between the changes in the source term and measured output data. In [12], the authors discussed the stable identification of a nonlinear source term in parabolic equations from a single set of boundary measurements via Carleman estimates. In [24], the authors employed a meshless method based on the application of the radial basis functions (RBFs) technique and the Tikhonov regularization method for retrieving a state dependent source term in a one dimensional heat equation. In [9], the author proved the unique solvability of a nonlinear inverse problem for recovering the right-hand side of a quasi-linear parabolic equation and presented a reconstruction technique based on parametric representation of the sought coefficient. In [13], the author proposed a polygonal approximation technique to find the nonlinear source term in a one-dimensional heat equation.

In [36], the authors utilized the following transformation

$$y = \frac{x - y_2}{1 - y_2}, \quad v(y,t) = u(x,t) - Y_2(t), \tag{6}$$

to obtain the equivalent problem

$$v_t - \gamma v_{yy} = p(t) g(v + Y_2(t)) + f\left(y_2 + (1 - y_2) y, t\right) - Y_2'(t), \quad (y,t) \in Q', \tag{7}$$

$$v(y,0) = u_1(y), \quad y \in [0,1], \tag{8}$$

$$v_y(1,t) = 0, \quad t \in (0,T], \tag{9}$$

$$v(0,t) = 0, \quad v(y^*,t) = Y_1(t) - Y_2(t), \quad t \in [0,T], \tag{10}$$

where

$$\gamma = \frac{1}{(1-y_2)^2}, \quad u_1(y) = u_0\Big(y_2 + (1-y_2)y\Big) - Y_2(0), \quad y^* = \frac{y_1 - y_2}{1 - y_2}, \quad Q' = [0,1]\times[0,T],$$

and showed that the problem presented by Eqs. (1)-(4) has a unique classical solution for small values of T and proposed a numerical scheme based on the finite difference method (FDM).

Although we know that the method used by [36] is effective in solving the above problem and even a wide class of PDEs [17, 32], but the computational cost of the FDM is high and the accuracy of the method is low. For this reason, we seek to provide a spectral method [1–3, 5, 20–22] that can be easily implemented to obtain numerical accuracy beyond two or three digits. Moreover, we will provide appropriate instructions based on regularization methods, using which, the proposed technique can calculate robust and reliable solutions [16] when dealing with noisy boundary conditions.

The organization of this article is as follows. In Section 2, we present the approximate solution of the considered inverse problem. In Section 3, some simulations are presented to demonstrate the effectiveness of the proposed method. Finally, in Section 4, we present the concluding remarks.

2 Computational Scheme

By defining the following functions

$$b(y,t) := y\frac{Y_1(t) - Y_2(t)}{y^*}, \tag{11}$$

$$s(y,t) := b(y,t) + u_1(y) - b(y,0), \quad w(y,t) := v(y,t) - s(y,t), \tag{12}$$

and taking the compatibility conditions (5) into account, we get

$$w(y,0) = 0, \quad 0 \le y \le 1, \quad w(0,t) = w(y^*,t) = 0, \quad t \in [0,T]. \tag{13}$$

Then, by applying the assumptions (11)-(12) into the problem (7)-(10) the following is derived

$$\begin{aligned} w_t - \gamma w_{yy} = {} & p(t)g\Big(w + s(y,t) + Y_2(t)\Big) + f\Big(y_2 + (1-y_2)y, t\Big) \\ & - Y_2'(t) + \gamma u_1''(y) - s_t(y,t), \quad (y,t) \in Q', \end{aligned} \tag{14}$$

$$w_y(1,t) = \frac{1}{y^*}\Big(Y_1(t) - Y_2(t) - Y_1(0) + Y_2(0)\Big) + u_1'(1), \quad t \in (0,T], \tag{15}$$

where

$$s_t(y,t) = y\frac{Y_1'(t) - Y_2'(t)}{y^*}. \tag{16}$$

Accordingly, if the functions of boundary and initial conditions are continuous in corners, i.e. the conditions (5) are satisfied, then problems (7)-(10) and (13)-(16) are equivalent if

$$u_1(y) \in C^2([0,1]), \quad Y_1(t) \;\&\; Y(t) \in C^1([0,T]), \quad y_1 \neq y_2.$$

Thus in the sequel, solving the system of Eqs. (13)-(16) is considered instead of problem (1)-(4). In this regard, we employ the orthonormal Bernstein basis functions (OBBFs) $\{\phi_i(y)\}_{i=0}^{\infty}$ and $\{\psi_j(t)\}_{j=0}^{\infty}$ [25, 26] which are complete in the Hilbert spaces $L^2[0,1]$ and $L^2[0,T]$ respectively, that are defined as follows

$$\phi_i(y) = \frac{\phi_i^*(y)}{\sqrt{\int_0^1 (\phi_i^*(y))^2 dy}},\ y \in [0,1], \quad \psi_i(t) = \frac{\psi_i^*(t)}{\sqrt{\int_0^T (\psi_i^*(t))^2 dt}},\ t \in [0,T], \tag{17}$$

such that for any fixed natural numbers $1 \leq n \leq m$, functions $\phi_i^*(y)$ and $\psi_i^*(t)$ are constructed via the following Gram-Schmidt formulas

$$\phi_0^*(y) := (1-y)^m, \quad \phi_n^*(y) := \frac{m! y^n (1-y)^{m-n}}{(m-n)!n!} - \sum_{k=0}^{n-1} \frac{\int_0^1 m! y^n (1-y)^{m-n} \phi_k^*(y) dy}{(m-n)!n! \int_0^1 \left(\phi_k^*(y)\right)^2 dy} \phi_k^*(y), \tag{18}$$

$$\psi_0^*(t) := \frac{(T-t)^m}{T^m}, \quad \psi_n^*(t) := \frac{m! t^n (T-t)^{m-n}}{T^m (m-n)!n!} - \sum_{k=0}^{n-1} \frac{\int_0^T m! t^n (T-t)^{m-n} \psi_k^*(t) dt}{T^m (m-n)!n! \int_0^T \left(\psi_k^*(t)\right)^2 dt} \psi_k^*(t). \tag{19}$$

By taking the following column vectors of the OBBFs

$$\phi(y) = [\phi_0(y), \phi_1(y), \ldots, \phi_N(y)]^\top, \quad \psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_{N'}(t)]^\top, \tag{20}$$

we introduce the differentiation of the vectors $\phi(y)$ and $\psi(t)$ as

$$\frac{d}{dy}\phi(y) := D_N \phi(y), \quad 0 \leq y \leq 1, \quad \frac{d}{dt}\psi(t) := D_{N'} \psi(t), \quad 0 \leq t \leq T, \tag{21}$$

where the notations D_N and $D_{N'}$ stand for the $(N+1)\times(N+1)$ and $(N'+1)\times(N'+1)$ operational matrices of differentiation [25, 26] corresponding to the basis functions $\phi_i(y)$ and $\psi_i(t)$, respectively. By denoting

$$r_i^\phi(y) := \frac{d\phi_i(y)}{dy}, \quad r_i^\psi(t) := \frac{d\psi_i(t)}{dt},$$

and considering the entries of the matrices D_N and $D_{N'}$ by d_{ij}^N and $d_{ij}^{N'}$, respectively, we find

$$d_{ij}^N = \int_0^1 r_i^\phi(y) \phi_j(y) dy, \quad i,\ j = \overline{0,N}, \quad d_{ij}^{N'} = \int_0^T r_i^\psi(t) \psi_j(t) dt, \quad i,\ j = \overline{0,N'}. \tag{22}$$

The Ritz approximate solutions $p_{N'}(t)$ and $w_{N,N'}(y,t)$ for the unknown functions $p(t)$ and $w(y,t)$, based on the OBBFs are suggested as follows:

$$p_{N'}(t) := \theta^\top \psi(t) = \sum_{j=0}^{N'} \theta_j \psi_j(t), \tag{23}$$

$$w_{N,N'}(y,t) := ty(y-y^*)\phi^\top(y) C \psi(t) = \sum_{i=0}^{N} \sum_{j=0}^{N'} c_{ij} t y (y-y^*) \phi_i(y) \psi_j(t), \tag{24}$$

In this work, we represent the transpose operator by notation $\top$

such that the unknown matrix C and vector $\theta^\top$ are given by

$$C = \begin{pmatrix} c_{00} & \cdots & c_{0N'} \\ \vdots & & \vdots \\ c_{N0} & \cdots & c_{NN'} \end{pmatrix}, \quad \theta^\top = [\theta_0, \theta_1, ..., \theta_{N'}]. \tag{25}$$

It is worthy to point out that the approximation $w_{N,N'}(y,t)$ proposed by Eq. (24) precisely fulfills the boundary and initial conditions (13). We call (24) as the approximate solution of the system of Eqs. (13)-(16) if the following conditions are also included

$$\begin{aligned} R_1(y,t,p(t),w(y,t)) := w_t - \gamma w_{yy} - p(t)g\Big(w + s(y,t) + Y_2(t)\Big) - f\Big(y_2 + (1-y_2)y, t\Big) \\ + Y_2'(t) - \gamma u_1''(y) + s_t(y,t) = 0, \end{aligned} \tag{26}$$

$$R_2(t, w_y(1,t)) := w_y(1,t) - \frac{1}{y^*}\Big(Y_1(t) - Y_2(t) - Y_1(0) + Y_2(0)\Big) - u_1'(1) = 0. \tag{27}$$

In this respect, by taking (23) and (24) into account and using the operational matrices of differentiation D_N and $D_{N'}$, we compute the following approximations of $w_t(y,t)$, $w_y(y,t)$ and $w_{yy}(y,t)$

$$w_t(y,t) \simeq y(y-y^*)\phi^\top(y)C\Big(\psi(t) + tD_{N'}\psi(t)\Big), \tag{28}$$

$$w_y(y,t) \simeq t\Big((2y-y^*)\phi^\top(y) + y(y-y^*)\phi^\top(y)D_N^\top\Big)C\psi(t), \tag{29}$$

$$w_{yy}(y,t) \simeq t\Big(2\phi^\top(y) + (4y-2y^*)\phi^\top(y)D_N^\top + y(y-y^*)\phi^\top(y)(D_N^\top)^2\Big)C\psi(t). \tag{30}$$

By substituting the approximations (28)-(30) in Eqs. (26)-(27) we get

$$\begin{aligned} R_1(y,t,p_{N'}(t), w_{N,N'}(y,t)) \simeq\ & y(y-y^*)\phi^\top(y)C\Big(\psi(t) + tD_{N'}\psi(t)\Big) - f\Big(y_2 + (1-y_2)y, t\Big) - \gamma u_1''(y) \\ & - \gamma t\Big(2\phi^\top(y) + (4y-2y^*)\phi^\top(y)D_N^\top + y(y-y^*)\phi^\top(y)(D_N^\top)^2\Big)C\psi(t) \\ & - \theta^\top\psi(t)g\Big(ty(y-y^*)\phi^\top(y)C\psi(t) + s(y,t) + Y_2(t)\Big) \\ & + Y_2'(t) + y\frac{Y_1'(t) - Y_2'(t)}{y^*} = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} R_2(t, w_{y_{N,N'}}(1,t)) \simeq\ & t\Big((2-y^*)\phi^\top(1) + (1-y^*)\phi^\top(1)D_N^\top\Big)C\psi(t) \\ & - \frac{1}{y^*}\Big(Y_1(t) - Y_2(t) - Y_1(0) + Y_2(0)\Big) - u_1'(1) = 0. \end{aligned} \tag{32}$$

Next, by collocating [27–30] the residual functions (31) and (32) at the points

$$x_i = \frac{i}{N+2}, \quad t_j = \frac{jT}{N'+2}, \quad i = \overline{1, N+1}, \quad j = \overline{1, N'+1}, \tag{33}$$

we achieve to the nonlinear system of algebraic equations as follows

$$\begin{cases} R_1(x_i,t_j,p_{N'}(t_j),w_{N,N'}(x_i,t_j))=0, \quad i=\overline{1,N+1}, \quad j=\overline{1,N'+1}, \\ R_2(t_j,w_{y_{N,N'}}(1,t_j))=0, \qquad j=\overline{1,N'+1}. \end{cases} \tag{34}$$

By solving the system of Eqs. (34) for the elements

$$c_{ij},\ i=\overline{0,N},\ j=\overline{0,N'}, \quad \theta_j,\ j=\overline{0,N'},$$

utilizing the Newton's iterative method [34], the unknown matrix C and vector θ are specified. It is worth noting that as the parameters N and N' increase, if the functions $R_2(t_j,w_{y_{N,N'}}(1,t_j))$ and $R_1(x_i,t_j,p_{N'}(t_j),w_{N,N'}(x_i,t_j))$ converge to zero, then Eqs. (26) and (27) are satisfied. On the other hand, the initial and boundary conditions (13) are also accurately fulfilled by the approximation (24), thus $p_{N'}(t)$ and $w_{N,N'}(y,t)$ as the approximations to the system of Eqs. (13)-(16) tend to the analytical solutions $p(t)$ and $w(y,t)$, respectively. If so, then from relations (6) and (12), the approximation of $u(x,t)$ is obtained as

$$u_{N,N'}(x,t)=w_{N,N'}(\frac{x-y_2}{1-y_2},t)+s(\frac{x-y_2}{1-y_2},t)+Y_2(t). \tag{35}$$

Furthermore, it should be noted that the approximations recast by Eqs. (24)-(35) are valid as long as the input initial and boundary data of the problem are free of errors. Otherwise, appropriate procedure should be adopted so that the errors in the input data are controlled. For the case of inaccurate boundary conditions, we suppose that $Y_i^{\eta}(t),\ i\in\{1,2\}$ be perturbations subject to $\|Y_i(t)-Y_i^{\eta}(t)\|_\infty\leq\eta$. Then, we fix the constant M and consider the approximations of $Y_i^{\eta}(t)$ and $(Y_i^{\eta})'(t)$ based on the following spline functions

$$Y_i^{\eta}(t)\simeq\overline{S}_i(t)=\begin{cases} S_{1i}(t), & 0\leq t<\frac{T}{M}, \\ S_{2i}(t), & \frac{T}{M}\leq t<\frac{2T}{M}, \\ \vdots & \\ S_{Mi}(t), & \frac{(M-1)T}{M}\leq t<T, \end{cases} \qquad (Y_i^{\eta})'(t)\simeq\overline{S}'_i(t)=\begin{cases} S'_{1i}(t), & 0\leq t<\frac{T}{M}, \\ S'_{2i}(t), & \frac{T}{M}\leq t<\frac{2T}{M}, \\ \vdots & \\ S'_{Mi}(t), & \frac{(M-1)T}{M}\leq t<T, \end{cases} \tag{36}$$

where $i\in\{1,2,...,M\},\ j\in\{1,2\}$, and

$$S_{ij}(t)=\alpha_{i,1}^{(j)}t^3+\alpha_{i,2}^{(j)}t^2+\alpha_{i,3}^{(j)}t+\alpha_{i4}^{(j)}, \quad S'_{ij}(t)=3\alpha_{i,1}^{(j)}t^2+2\alpha_{i,2}^{(j)}t+\alpha_{i,3}^{(j)}. \tag{37}$$

We wish to find the unknown coefficients $\alpha_{i,j}^{(k)},\ i=\overline{1,M},\ j=\overline{1,4},\ k\in\{1,2\}$ such that the functions $\overline{S}_i(t)$ be the natural spline approximations of $Y_i^{\eta}(t)$. In this respect, for each $k\in\{1,2\}$ we form the following equations

$$\begin{cases} \alpha_{i,1}^{(k)}(\frac{jT}{M})^3+\alpha_{i,2}^{(k)}(\frac{jT}{M})^2+\alpha_{i,3}^{(k)}(\frac{jT}{M})+\alpha_{i,4}^{(k)}=Y_k^{\eta}(\frac{jT}{M}), \quad j\in\{i-1,i\}, \quad i=\overline{1,M}, \\ \alpha_{1,2}^{(k)}=0, \quad 6\alpha_{M,1}^{(k)}T+2\alpha_{M,2}^{(k)}=0, \\ 3\alpha_{i,1}^{(k)}(\frac{iT}{M})^2+2\alpha_{i,2}^{(k)}(\frac{iT}{M})+\alpha_{i,3}^{(k)}-3\alpha_{i+1,1}^{(k)}(\frac{iT}{M})^2-2\alpha_{i+1,2}^{(k)}(\frac{iT}{M})-\alpha_{i,3}^{(k)}=0, \quad i=\overline{1,M-1}, \\ 6\alpha_{i,1}^{(k)}(\frac{iT}{M})+2\alpha_{i,2}^{(k)}-6\alpha_{i+1,1}^{(k)}(\frac{iT}{M})-2\alpha_{i+1,2}^{(k)}=0, \quad i=\overline{1,M-1}, \end{cases} \tag{38}$$

and the result of which will be the systems represented generically by

$$B_k\alpha^{(k)}=e_k,$$

where the vectors $\alpha^{(k)}$ contain the unknowns $\alpha_{ij}^{(k)},\ i=\overline{1,M},\ j=\overline{1,4},\ k\in\{1,2\}$. The Tikhonov regularization method solves the modified system

$$(B_k^\top B_k+\lambda_k I)\alpha^{(k)}=B_k^\top e_k,\quad \lambda_k>0, \tag{39}$$

to get the unknown vectors $\alpha_k^{(k)}$ and then the functions $\overline{S_i'}(t)$ are specified which are used in Eqs. (16) and (31) instead of $Y_i'(t)$.

Remark 1. *Although the instruction given by Eqs. (36)-(39) provides a suitable method to find stable numerical derivative, but as a new approach and by writing Eq. (26) in the following integral form*

$$\begin{aligned}R_3(y,t,p(t),w(y,t)):=\ & w(y,t)-\int_0^t\left\{\gamma w_{yy}(y,r)+p(r)g\Big(w(y,r)+s(y,r)+Y_2(r)\Big)\right\}dr\\&-\int_0^t f\Big(y_2+(1-y_2)y,r\Big)dr+Y_2(t)+s(y,t)\\&-Y_2(0)-u_1(y)-t\gamma u_1''(y)=0,\end{aligned} \tag{40}$$

we can avoid differentiating with respect to the time variable t regarding the noisy boundary conditions. By substituting the approximations of $w(y,t)$, $w_{yy}(y,t)$ and $p(t)$ in Eq. (40), we have

$$\begin{aligned}R_3(y,t,p_{N'}(t),w_{N,N'}(y,t))\ \simeq\ & y(y-y^*)\phi^\top(y)C\psi(t)+Y_2(t)+s(y,t)-u_0\Big(y_2+(1-y_2)y\Big)-t\gamma u_1''(y)\\&-\gamma\Big(2\phi^\top(y)+(4y-2y^*)\phi^\top(y)D_N^\top+y(y-y^*)\phi^\top(y)(D_N^\top)^2\Big)C\int_0^t z_1(r)dr\\&-\int_0^t z_2(y,r)dr,\end{aligned}$$

where $z_1(r)=[r\psi_0(r),\ldots,r\psi_{N'}(r)]^\top$ and

$$z_2(y,r)=f\Big(y_2+(1-y_2)y,r\Big)+p(r)g\Big(w(y,r)+s(y,r)+Y_2(r)\Big).$$

Finally, the following system of nonlinear algebraic equations is solved to find the unknowns C and θ:

$$\begin{cases}R_3(x_i,t_j,p_{N'}(t_j),w_{N,N'}(x_i,t_j))=0, & i=\overline{1,N+1},\quad j=\overline{1,N'+1},\\ R_2(t_j,w_{y_{N,N'}}(1,t_j))=0, & j=\overline{1,N'+1}.\end{cases}$$

Meanwhile, for computing the values $\int_0^{t_j}z_1(y)dy$ and $\int_0^{t_j}z_2(x_i,y)dy$ we apply the Simpson's rule as follows

$$\int_0^{t_j}z_1(y)dy\ \simeq\ \frac{H_j}{3}\left\{z_1(0)+z_1(t_j)+4\sum_{k=1}^{\frac{M'}{2}}z_1\Big((2k-1)H_j\Big)+2\sum_{k=1}^{\frac{M'}{2}-1}z_1\Big(2kH_j\Big)\right\},$$

$$\int_0^{t_j}z_2(x_i,y)dy\ \simeq\ \frac{H_j}{3}\left\{z_2(x_i,0)+z_2(x_i,t_j)+4\sum_{k=1}^{\frac{M'}{2}}z_2\Big(x_i,(2k-1)H_j\Big)+2\sum_{k=1}^{\frac{M'}{2}-1}z_2\Big(x_i,2kH_j\Big)\right\},$$

where $H_j=t_j/M'$.

3 Illustrative tests

Three examples are solved in this section. The functions

$$E(u(x,t)) = |u(x,t) - u_{N,N'}(x,t)|, \quad E(p(t)) = |p(t) - p_{N'}(t)|,$$

are used to represent the absolute error corresponding to the functions $u(x,t)$ and $p(t)$. Meanwhile, by denoting the unknown function $u_x(0,t) = q(t)$, we define the absolute error of function $q(t)$ by $E(q(t)) = |u_x(0,t) - u_{x_{N,N'}}(0,t)|$. Throughout this work, numerical calculations are performed in Mathematica software. The utilized commands are as follows: FindRoot (to solve the nonlinear systems of algebraic equations), LinearSolve (to solve the linear systems of algebraic equations) and RandomReal$[\{-1,1\}]$ (to generate random real numbers belonging to the interval $[-1,1]$). The L-Curve criterion [14] is employed for selecting the regularization parameters $\lambda_j,\ j \in \{1,2\}$.

Example 1. The inverse problem with the following properties is considered

$$\begin{aligned} u_t - u_{xx} &= p(t)e^{-u} + (\frac{x^2}{2} - x - 2)e^{\frac{t}{2}} - (\frac{t+0.8}{1+(t+0.1)^2})e^{(2x-x^2)e^{\frac{t}{2}}}, \quad \text{in} \quad [0.1,1]\times[0,1], \\ u_0(x) &= x^2 - 2x, \quad 0.1 \le x \le 1, \quad u_x(1,t) = 0, \quad 0 < t \le 1, \\ u(0.5,t) &= -0.75e^{\frac{t}{2}}, \quad u(0.1,t) = -0.19e^{\frac{t}{2}}, \quad t \in [0,1], \end{aligned}$$

to retrieve the following analytical solutions

$$p(t) = \frac{t+0.8}{1+(t+0.1)^2},\ u(x,t) = e^{\frac{t}{2}}(x^2 - 2x).$$

The problem is solved by employing the presented scheme in Section 2 with $N = N' = 2$, where the exact initial and boundary conditions are applied and the result of which is depicted in Figures 1-3. In addition, we gradually increase the number of basis functions in the computations and tabulate the outcomes of this experiment in Tables 1-2 to show the numerical convergence of the presented technique. The obtained numerical results confirm the reduction of error and increase of accuracy in numerical approximations. Next, we want to investigate the sensitivity of our solution method to small changes in the boundary conditions. In this regard we utilize the following

$$Y_j^{\eta}(t_i) = Y_j(t_i) + \eta \times \text{RandomRea}[\{-1,1\}], \quad t_i \in [0,1],\ j \in \{1,2\}, \tag{41}$$

$$u_x(1,t) = 0 - \eta \sin(\frac{t}{\eta^2}), \quad t \in (0,1], \tag{42}$$

where $\eta \in \{10^{-2}, 3\times10^{-2}, 5\times10^{-2}\}$ is the percentage of noise. By solving the problem via the presented technique in Section 2 and utilizing the regularization method given by Eqs. (36)-(39) with the parameters $N_1 = N_2 = 4$ and $M = 10$, we get the results of approximating $p(t)$ and $q(t)$ shown in Figures 4-5 and Table 3. It is observed that by applying the regularization method, we control the impact of the introduced errors to the boundary conditions and acceptable approximations are derived.

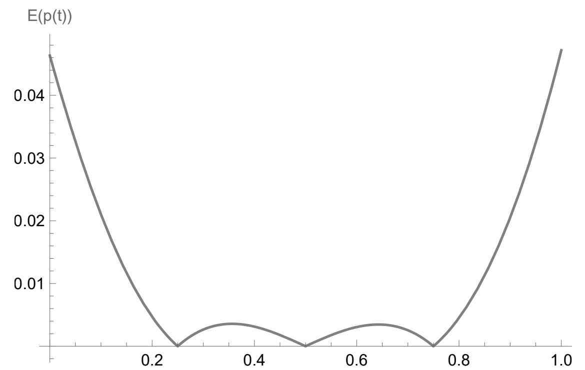

Figure 1: Graph of $E(p(t))$ derived by the proposed method with $N = N' = 2$ when exact input data are used in Example 1.

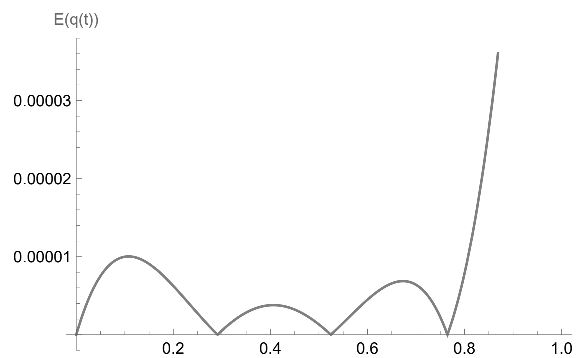

Figure 2: Graph of $E(q(t))$ derived by the proposed method with $N = N' = 2$ when exact input data are used in Example 1.

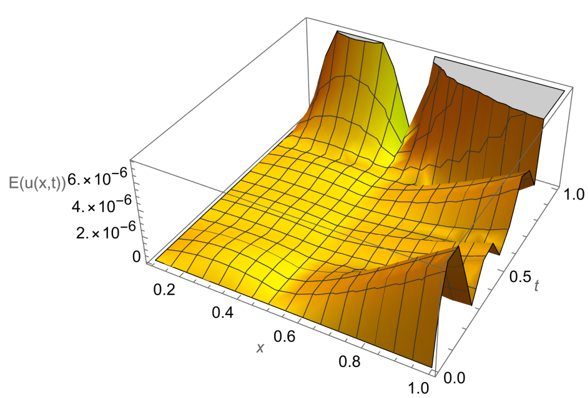

Figure 3: Graph of $E(u(x,t))$ derived by the proposed method with $N = N' = 2$ when exact input data are used in Example 1.

Table 1: Computational results for Example 1 with the accurate boundary conditions.

(N,N')	$\|E(p(t))\|_\infty$	$\|E(q(t))\|_\infty$	$\|E(u(x,t))\|_\infty$	$\|R_2(t,w_y(1,t))\|_\infty$	$\|R_1(y,t,p(t),w(y,t))\|_\infty$
$(3,3)$	8×10^{-3}	5.5×10^{-6}	4.6×10^{-6}	1.6×10^{-5}	3.3×10^{-2}
$(5,5)$	1.7×10^{-3}	4.96×10^{-9}	3.2×10^{-9}	1.53×10^{-8}	9.6×10^{-3}
$(7,7)$	2.9×10^{-4}	4.2×10^{-12}	1.45×10^{-12}	8.15×10^{-12}	6.7×10^{-4}

Table 2: Computational results for Example 1 with the accurate boundary conditions.

(N,N')	$\|E(p(t))\|_2$	$\|E(q(t))\|_2$	$\|E(u(x,t))\|_2$	$\|R_2(t,w_y(1,t))\|_2$	$\|R_1(y,t,p(t),w(y,t))\|_2$
$(3,3)$	2×10^{-3}	1.08×10^{-6}	2.9×10^{-7}	3.5×10^{-6}	5.4×10^{-3}
$(5,5)$	4.2×10^{-4}	7.49×10^{-10}	1.8×10^{-10}	2.4×10^{-9}	1.2×10^{-3}
$(7,7)$	4.5×10^{-5}	5.7×10^{-13}	1.04×10^{-14}	1.1×10^{-12}	1×10^{-4}

Example 2. In this example [36], we assume that the problem given by Eqs. (1)-(4) possesses the following conditions:

$$f(x,t)=\left(\frac{\pi^2}{4(1-y_2)^2}-1\right)\sin\left(\frac{\pi}{2}.\frac{x-y_2}{1-y_2}\right)e^{-t}+(1+t^2)\sin^3\left(\sin\left(\frac{\pi}{2}.\frac{x-y_2}{1-y_2}\right)e^{-t}\right), \tag{43}$$

$$g(u)=\sin^3(u),\quad u_0(x)=\sin\left(\frac{\pi}{2}.\frac{x-y_2}{1-y_2}\right),\quad Y_1(t)=\sin\left(\frac{\pi}{2}.\frac{y_1-y_2}{1-y_2}\right)e^{-t},\quad Y_2(t)=0, \tag{44}$$

and its exact solutions are as follows

$$p(t)=-1-t^2,\quad u(x,t)=\sin\left(\frac{\pi}{2}.\frac{x-y_2}{1-y_2}\right)e^{-t},\quad Q=[0.1,1]\times[0,1].$$

By taking $y_1=0.9$, $y_2=0.1$ and applying the approximations presented in Section 2 with different values of $N=N'\in\{1,2,3,4,5,6\}$, we derive the findings tabulated in Tables 4-5. Following the numerical solutions, it can be observed that very good numerical approximations to the analytical solutions are provided and the results further improve with increasing the parameters N and N'. As a conclusion, when no noise is introduced to the initial and boundary data the proposed method produces excellent and convergent approximations. Moreover, compared to the results presented in [36], it can be observed that the algorithm proposed in the present paper yields better results because of providing higher accuracy with lower computational cost.

Furthermore, to realize that how the proposed approximate method deals with inaccurate boundary conditions, we consider the problem involving Eqs. (41)-(42) and solve it by utilizing the suggested technique of Remark 2.1 along with the following parameters

$$N=N'=3,\quad M'=50,\quad \eta\in\{0.001,0.002,0.003\}.$$

The results are illustrated in Figures 6-7 and Table 6. As it can be seen, for small amounts of perturbations in the boundary conditions, the obtained approximations are close to the analytical solutions. Nevertheless, as time increases the errors increase and it results in some drawbacks for recovering the value of $p(t)$ at $t=T$, accurately. In addition, we should mention that we could not get acceptable solutions for $\eta>0.003$.

Table 3: The L^2-norm of errors corresponding to the unknown functions $p(t)$ and $q(t)$ for Example 1, while the boundary data are contaminated with errors.

η	$\|E(p(t))\|_2$	$\|E(q(t))\|_2$
0.01	0.009	0.003
0.03	0.04	0.02
0.05	0.09	0.03

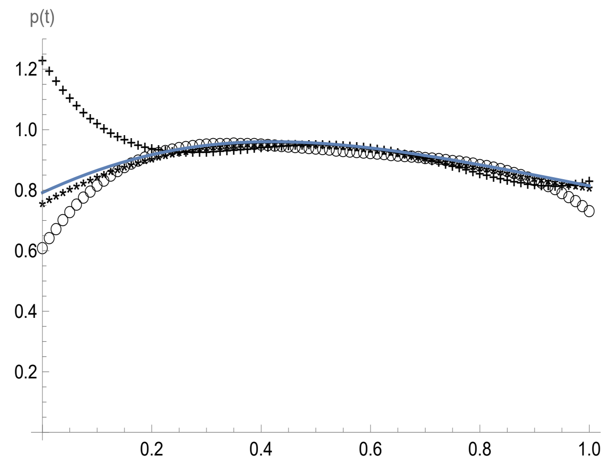

Figure 4: Graph of the exact solution of $p(t)$ (blue curve), and approximate solutions found by the proposed technique with the perturbed boundary data and appropriate regularization parameters λ_1, λ_2 for Example 1 ($***$: when $\eta = 0.01$, $\lambda_1 = 10^{-6}$, $\lambda_2 = 10^{-2}$; $\circ\circ\circ$: when $\eta = 0.03$, $\lambda_1 = \lambda_2 = 10^{-2}$; $+++$: when $\eta = 0.05$, $\lambda_1 = \lambda_2 = 10^{-2}$).

Table 4: Computational results for Example 2 with the accurate boundary conditions.

(N,N')	$\|E(p(t))\|_\infty$	$\|E(q(t))\|_\infty$	$\|E(u(x,t))\|_\infty$	$\|R_2(t,w_y(1,t))\|_\infty$	$\|R_3(y,t,p(t),w(y,t))\|_\infty$
$(2,2)$	0.24	0.02	0.0004	0.002	0.063
$(3,3)$	0.058	0.0022	0.000064	0.00021	0.015
$(4,4)$	0.0045	0.00049	7.86×10^{-6}	0.000014	0.0013
$(5,5)$	0.00053	4.2×10^{-5}	5.59×10^{-7}	8.2×10^{-7}	0.00018
$(6,6)$	3×10^{-5}	5.8×10^{-6}	3.9×10^{-8}	4×10^{-8}	1.3×10^{-5}

Example 3. Consider the equation

$$u_t - u_{xx} = p(t)e^u + \frac{4(x-1)^2 - (1+t+(x-1)^2)}{(1+t+(x-1)^2)^2} - (1+t+(x-1)^2)e^t, \quad in \quad [0.1,1]\times[0,T], \quad (45)$$

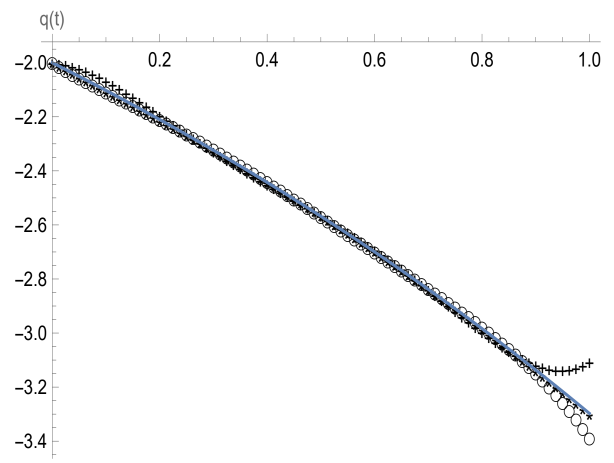

Figure 5: Graph of the exact solution of $q(t)$ (blue curve), and approximate solutions found by the proposed technique with the perturbed boundary data and appropriate regularization parameters λ_1, λ_2 for Example 1 ($***$: when $\eta = 0.01$, $\lambda_1 = 10^{-6}$, $\lambda_2 = 10^{-2}$; $\circ\circ\circ$: when $\eta = 0.03$, $\lambda_1 = \lambda_2 = 10^{-2}$; $+++$: when $\eta = 0.05$, $\lambda_1 = \lambda_2 = 10^{-2}$).

Table 5: Computational results for Example 2 with the accurate boundary conditions.

(N,N')	$\|E(p(t))\|_2$	$\|E(q(t))\|_2$	$\|E(u(x,t))\|_2$	$\|R_2(t,w_y(1,t))\|_2$	$\|R_3(y,t,p(t),w(y,t))\|_2$
$(2,2)$	0.1	0.013	0.00022	0.0006	0.012
$(3,3)$	0.022	0.0014	0.00003	0.000042	0.002
$(4,4)$	0.002	0.0003	3×10^{-6}	2.5×10^{-6}	1.7×10^{-4}
$(5,5)$	0.0002	0.00003	2.4×10^{-7}	1.3×10^{-7}	2×10^{-5}
$(6,6)$	1.3×10^{-5}	3.8×10^{-6}	1.84×10^{-8}	5.8×10^{-9}	1.3×10^{-6}

with the following initial and boundary conditions

$$u(x,0) = Ln((x-1)^2+1),\ x\in[0.1,1],\ u(\frac{j}{10},t) = Ln((\frac{j-10}{10})^2+t+1),\ j\in\{1,4\},\ t\in[0,T]. \quad (46)$$

The problem is solved via the presented method with exact initial and boundary data per two different time intervals, i.e. $T\in\{1,2\}$. The results are tabulated in Table 7. It is seen that the approximations of unknown functions are improved by increasing the number of basis functions. However, for larger time-span T, we need to increase the number of basis functions to get more accurate approximate solutions and this increases the computational cost.

4 Conclusions

The content of this paper concerns an inverse problem originated from the heat conduction with a nonlinear source term. The main goal is solving numerically the considered nonlinear IHCP for simultaneously

Table 6: The L^2-norm of errors corresponding to the unknown functions $p(t)$ and $q(t)$ for Example 2, while the boundary data are contaminated with errors.

η	$\|E(p(t))\|_2$	$\|E(q(t))\|_2$
0.001	0.03	0.003
0.002	0.42	0.008
0.003	0.5	0.01

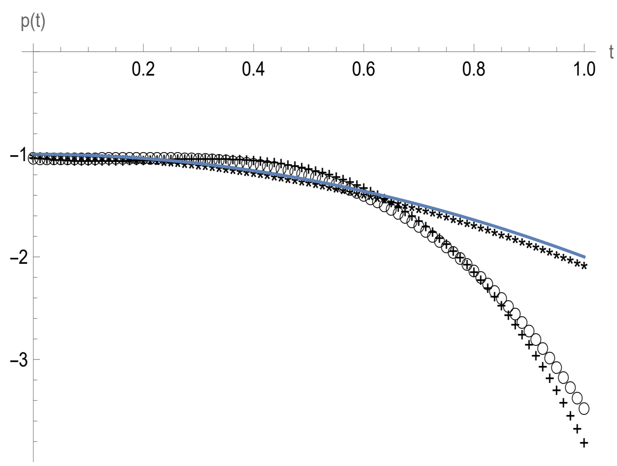

Figure 6: Graph of the exact solution of $p(t)$ (blue curve), and approximate solutions found by the proposed technique with the perturbed boundary for Example 2 ($***$: when $\eta = 0.001$; $\circ\circ\circ$: when $\eta = 0.002$; $+++$: when $\eta = 0.003$).

retrieving a time-dependent reaction coefficient and the surface heat flux histories and temperature distribution. First, the problem is recast as a certain PDE and the Ritz approximations based on the OBBFs are employed to detect the unknown functions. Then, the collocation method is applied to reduce the inverse problem to the solution of algebraic equations. The method employs the natural cubic spline technique in order to approximate the perturbed boundary data as well as its derivative and additionally Tikhonov regularization method is utilized for achieving stable solutions. We present some numerical tests and the issues of numerical convergence and stability are discussed. It can be seen that by employing the proposed method satisfactory results are obtained such that in the presence of the exact initial and boundary data the unknown functions are excellently retrieved and regarding the noisy boundary data the obtained approximations deviate from the analytical solution almost proportional to the amount of introduced noise.

Acknowledgements

The author is very grateful to the referees for their valuable comments and helpful suggestions to improve the earlier version of this article.

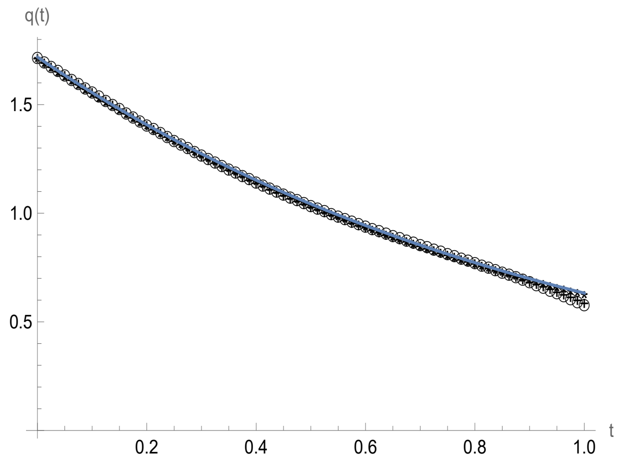

Figure 7: Graph of the exact solution of $q(t)$ (blue curve), and approximate solutions found by the proposed technique with the perturbed boundary for Example 2 ($***$: when $\eta = 0.001$; $\circ\circ\circ$: when $\eta = 0.002$; $+++$: when $\eta = 0.003$).

Table 7: Computational results for Example 3 with the accurate boundary conditions.

(N, N', T)	$\|E(p(t))\|_2$	$\|E(q(t))\|_2$	$\|E(u(x,t))\|_2$
$(2,2,1)$	0.025	0.016	0.004
$(2,2,2)$	0.22	0.15	0.03
$(4,4,1)$	0.001	0.0006	0.0002
$(4,4,2)$	0.005	0.003	0.001

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