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A novel implementation of fixed-point theorems for high-order Hadamard fractional differential equations with multi-point integral boundary conditions

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Abstract. This research inscription gets to grips with a novel type of boundary value problem of nonlinear differential equations encapsuling a fractional derivative known as the Hadamard fractional operator. Our results rely on the standard tools of functional analysis. The existence of the solutions of the aforehand equations is tackled by using Schaefer and Krasnoselskii's fixed point theorems, whereas their uniqueness is handled using the Banach fixed point theorem. Two pertinent examples are presented to point out the applicability of our main results.

Keywords: Fractional differential equation, Hadamard fractional derivative, existence and uniqueness, fixed-point theorem.
AMS Subject Classification 2010: 34A08, 34B15.

1 Introduction

Fractional differential equations (FDEs) are mathematical equations that involve fractional derivatives. Fractional derivatives are generalizations of the integer-order derivatives that appear in classical calculus. They can be defined using various mathematical formulations, such as Riemann-Liouville, Caputo, and Grunwald-Letnikov definitions.

FDEs have been increasingly used in recent years to model phenomena in various fields such as Electrochemistry, material science, physics, engineering, finance, and biology. They are in fact described by differential equations of fractional order [16, 17, 23]. On the other hand, fractional calculus has gained significant attention due to its wide range of applications in various scientific fields. It has been used

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in fields such as biology, blood flow phenomena, image processing, groundwater problems, capacitor theory, viscoelasticity, aerodynamics, geophysics, biophysics, polymer rheology, nonlinear oscillation of earthquakes, and electrical circuits, to name a few. For more applications of FDEs, we refer to sufficient works [19, 21, 22, 24–26, 30] and to the following research papers [1, 3, 5, 8, 15, 31, 33].

Functional analysis is an important branch of mathematics that deals with spaces of functions and their properties. It is used to study a wide range of mathematical objects, including linear and nonlinear differential equations, and has many applications in physics, engineering, and other fields.

In the study of nonlinear fractional differential equations (NFDEs), functional analysis can be used to develop powerful analytical tools for their solution and analysis. Specifically, the theory of fractional calculus and functional analysis can be used to construct a variety of function spaces, such as Sobolev spaces and Besov spaces, that are appropriate for describing the behavior of solutions of NFDEs.

One of the key advantages of using functional analysis in the study of NFDEs is that it provides a rigorous framework for proving the existence and uniqueness of solutions, as well as their regularity properties. Additionally, functional analysis techniques can be used to study the stability and asymptotic behavior of solutions of NFDEs, which are important for understanding the long-term behavior of the system being modeled.

In recent years, many authors have investigated the existence and uniqueness of solutions for nonlinear fractional differential equation boundary value problems. For a small sample of such work, we refer [2, 9–12, 27–29, 32, 34, 35] and references therein. Many latest studies in the existence theory focus on the fractional equations with integral boundary conditions, which improve the classical conditions in the development of mathematical modeling [6, 7, 13].

The present paper which draws inspiration from the aforementioned works and [4, 14, 29], investigates the existence and uniqueness of solutions for the following high-order fractional differential equation:

$$\begin{cases} -D^{\alpha}u(t) = A_1 f_1(t,u(t)) + A_2 I^{\beta} f_2(t,u(t)), \quad n-1<\alpha\le n, \quad n\ge 2, \quad t\in\Pi=[1,e], \\ D^{\gamma+k}u(1)=0,\ 0\le k\le n-2, \quad D^{\gamma}u(e)=\displaystyle\sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} D^{\gamma}u(s)ds, \end{cases} \tag{1}$$

where $0<\gamma<1$, $\alpha-\gamma>n-1$, $0<\beta<1$, $n,k\in\mathbb{N}$ and $0=\eta_0<\eta_1<\cdots<\eta_{m-2}<\eta_{m-1}=1, a_i>0$ for $i\in 1,2,\dots,m-1$, and $1\neq \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1}ds$, D^{ρ} denotes the Hadamard fractional derivative of order $\rho=\{\alpha,\gamma,\gamma+k\}$, I^{β} denotes the Hadamard fractional derivative of order β and f_1, f_2 are given continuous functions, A_1, A_2 are real constants such that A_1 or A_2 is nonzero.

This paper is structured as follows. After introducing, in Section 2, we recall briefly some basic definitions and lemmas and preliminary facts which are required to prove our main results. In Section 3, we shall provide sufficient conditions ensuring the existence of solutions for problem (1) via applications of classical fixed point theorems (Scheafer's and Krasnoselskii's fixed point theorem, and the Banach's Contraction Principle.). Finally in Section 4, we give examples to illustrate the theory presented in the previous sections.

2 Preliminaries

In what follows, we are rendering some results of Hadamard fractional calculus that will be used throughout this paper.

Let $\mathscr{C}(\Pi) = C(\Pi,\mathbb{R})$ be the Banach space of all continuous functions from Π into $\mathbb{R}$ with the norm

$$\|u\| = \sup\left\{|u| : t \in \Pi\right\}.$$

Next, let $\delta := t\frac{d}{dt}$ and define on an interval $[a,b]$, the set

$$AC^n_\delta[a,b] = \left\{g : [a,b] \to \mathbb{R} : \delta^{n-1}g(t) \in AC[a,b]\right\}.$$

Definition 1 ([18,21])**.** The Hadamard fractional integral of order $\alpha > 0$ for a function $\varpi \in L^1([a,b],\mathbb{R})$, is defined as

$$(I^\alpha \varpi)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{\varpi(s)}{s} ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 ([21, 34])**.** *The Hadamard fractional derivative of order $\alpha \geq 0$ of a continuous function $\varpi : (a,b) \to \mathbb{R}$ is given by*

$$D^\alpha \varpi(t) = \delta^n I^{n-\alpha}\varpi(t) = \frac{1}{\Gamma(n-\alpha)}(t\frac{d}{dt})^n \int_a^t (\log\frac{t}{s})^{n-\alpha-1}\varpi(s)\frac{ds}{s}, \tag{2}$$

where $n = [\alpha]+1$, and $[\alpha]$ denotes the integer part of the real number α and $\delta = t\frac{d}{dt}$, provided the right integral converges.

Proposition 1 ([15,20,21])**.** *Let $\alpha,\beta > 0$, $n = [\alpha]+1$, and $a > 0$, then*

$$\begin{aligned} I^\alpha\left(\log\frac{t}{a}\right)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta+\alpha-1}, \\ D^\alpha\left(\log\frac{t}{a}\right)^{\beta-1}(t) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{t}{a}\right)^{\beta-\alpha-1}, \quad \beta > n. \end{aligned} \tag{3}$$

Theorem 1 ([15])**.** *Let $u(t) \in AC^n_\delta[a,b], 0 < a < b < \infty$ and $\alpha \geq 0, \beta \geq 0$, then*

$$\begin{aligned} D^\alpha I^\alpha \varpi(t) &= I^{\beta-\alpha}\varpi(t), \\ D^\alpha D^\beta \varpi(t) &= D^{\alpha+\beta}\varpi(t). \end{aligned} \tag{4}$$

Lemma 1 ([15, 20])**.** *Let $\alpha > 0$ and $n = [\alpha]+1$. If $\varpi(t) \in AC^n_\delta[a,b]$, then the Hadamard fractional differential equation*

$$D^\alpha_{a^+}\varpi(t) = 0,$$

has a solution

$$\varpi(t) = \sum_{k=0}^{n-1} c_k \left(\log\frac{t}{a}\right)^{\alpha-k},$$

and the following formula holds:

$$I^\alpha D^\alpha \varpi(t) = \varpi(t) + \sum_{k=0}^{n-1} c_k\left(\log\frac{t}{a}\right)^{\alpha-k}, \tag{5}$$

where $c_k \in \mathbb{R}$, $k = 1,2,...,n$.

By using the substitution $u(t) = I^{\gamma}y(t) = D^{-\gamma}y(t)$, one can transform the fractional BVP (1) to the following form:

$$\begin{cases} -D^{\alpha-\gamma}y(t) = A_1 f_1(t, I^{\gamma}y(t)) + A_2 I^{\beta} f_2(t, I^{\gamma}y(t)), & t \in \Pi, \\ y^k(1) = 0, 0 \le k \le n-2, y(e) = \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds. \end{cases} \tag{6}$$

To obtain the solution of the fractional BVP (1), the following lemma is essential.

Lemma 2. *For any $h \in \mathscr{C}(\Pi)$, the unique solution of the linear fractional BVP*

$$\begin{cases} -D^{\alpha-\gamma}y(t) = h(t), \quad t \in \Pi, \\ y^k(1) = 0, 0 \le k \le n-2, \quad y(e) = \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds, \end{cases} \tag{7}$$

is

$$y(t) = -I^{\alpha-\gamma}h(t) + \frac{(\log t)^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds} \left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\gamma}h(s)ds \right).$$

Proof. By Lemma 1, the solutions of Eq. (7) are

$$y(t) = -I^{\alpha-\gamma}h(t) - c_1(\log t)^{\alpha-\gamma-1} - c_2(\log t)^{\alpha-\gamma-2} - \cdots - c_n(\log t)^{\alpha-\gamma-n},$$

where $c_i (i = 1, 2, \ldots, n) \in \mathbb{R}$ are arbitrary constants. By the conditions $y^{(k)}(1) = 0,\ 0 \le k \le n-2$, we obtain $c_2 = \cdots = c_n = 0$. Then, we conclude that

$$y(t) = -I^{\alpha-\gamma}h(t) - c_1(\log t)^{\alpha-\gamma-1}. \tag{8}$$

Now, by the condition $y(e) = \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} y(s)ds$, we can get

$$c_1 = \frac{1}{1 - \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds} \left(-I^{\alpha-\gamma}h(e) + \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\gamma}h(s)ds \right).$$

Combining this value with Eq. (8), we obtain

$$y(t) = -I^{\alpha-\gamma}h(t) + \frac{(\log t)^{\alpha-\gamma-1}}{1 - \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds} \left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} I^{\alpha-\gamma}h(s)ds \right).$$

The proof is complete. □

Notice that, the solution of the equation $-D^{\alpha}u(t) = h(t)$ depends on the boundary conditions given by (1) can be expressed as

$$\begin{aligned}
u(t) &= I^{\gamma}y(t)\\
&= I^{\gamma}\left[-I^{\alpha-\gamma}h(t) + \frac{(\log t)^{\alpha-\gamma-1}}{1-\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}(\log s)^{\alpha-\gamma-1}ds}\left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\gamma}h(s)ds\right)\right]\\
&= -I^{\alpha}h(t) + \frac{1}{1-\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}(\log s)^{\alpha-\gamma-1}ds}\left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\gamma}h(s)ds\right)\\
&\quad\times\frac{1}{\Gamma(\gamma)}\int_1^e(\log\frac{t}{s})^{\gamma-1}(\log s)^{\alpha-\gamma-1}ds\\
&= -I^{\alpha}h(t) + \frac{(\log t)^{\alpha-1}\Gamma(\alpha-\gamma)}{\Gamma(\alpha)(1-\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}(\log s)^{\alpha-\gamma-1}ds)}\left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\gamma}h(s)ds\right)\\
&= -I^{\alpha}h(t) + (\log t)^{\alpha-1}\Delta\left(I^{\alpha-\gamma}h(e) - \sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}I^{\alpha-\gamma}h(s)ds\right).
\end{aligned}$$

Next, we introduce an operator $\mathscr{T}:\mathscr{C}(\Pi)\to\mathscr{C}(\Pi)$ as

$$\begin{aligned}
(\mathscr{T}u(t) = &-\frac{A_1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}f_1(s,u(s))\frac{ds}{s} - \frac{A_2}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}f_2(s,u(s))\frac{ds}{s}\\
&+(\log t)^{\alpha-1}\Delta\left[\frac{A_1}{\Gamma(\alpha-\gamma)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-\gamma-1}f_1(s,u(s))\frac{ds}{s}\right.\\
&+\frac{A_2}{\Gamma(\alpha-\gamma+\beta)}\int_1^t\left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1}f_2(s,u(s))\frac{ds}{s}\\
&-\frac{A_1}{\Gamma(\alpha-\gamma)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1}f_1(\eta,u(\eta))\frac{d\eta}{\eta}ds\\
&\left.-\frac{A_2}{\Gamma(\alpha-\gamma+\beta)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1}f_2(\eta,u(\eta))\frac{d\eta}{\eta}ds\right], \qquad (9)
\end{aligned}$$

where

$$\Delta = \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)(1-\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}(\log s)^{\alpha-\gamma-1}ds)}.$$

It can be said that u is a solution of the fractional BVP (1) if and only if u is a fixed point of the operator $\mathscr{T}$ on $\mathscr{C}(\Pi)$.

For easy statement, denote

$$\Lambda_1 = |A_1|\left[\frac{1}{\Gamma(\alpha+1)} + \Delta\left(\frac{1}{\Gamma(\alpha-\gamma+1)} + \frac{1}{\Gamma(\alpha-\gamma)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1}\frac{d\eta}{\eta}ds\right)\right],$$

$$\Lambda_2 = |A_2|\left[\frac{1}{\Gamma(\alpha+\beta+1)} + \Delta\left(\frac{1}{\Gamma(\alpha-\gamma+\beta+1)} + \sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\frac{\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1}}{\Gamma(\alpha-\gamma+\beta)}\frac{d\eta}{\eta}ds\right)\right],$$

$$
\begin{aligned}
K :=& \left[\frac{|A_1|}{\Gamma(\alpha-\gamma+1)} + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta+1)} + \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} \frac{d\eta}{\eta} ds \right.\\
&\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} \frac{d\eta}{\eta} ds \right],\\
\omega :=& \left[\frac{|A_1|}{\Gamma(\alpha+1)} + \frac{|A_2|}{\Gamma(\alpha+\beta+1)}\right],\\
\Delta :=& \frac{\Gamma(\alpha-\gamma)}{\Gamma(\alpha)(1-\sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds)}.
\end{aligned}
$$

3 Main results

In this section, we prove some existence and uniqueness results to the nonlinear fractional differential equation (1). For the sake of convenience, we impose the following hypotheses:

(H1) $f_1, f_2 : \Pi \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

(H2) There exists a constant $L_1, L_2 > 0$ such that

$$
|f_1(t,x) - f_1(t,y)| \le L_1 |x-y|,
$$
$$
|f_2(t,x) - f_2(t,y)| \le L_2 |x-y|,
$$

for $t \in \Pi$, and each $x, y \in \mathbb{R}$, where $L = \max\{L_1, L_2\}$.

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 2. *Suppose that* $1 \neq \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds$ *and assume that the hypothesis (H2) holds. If the inequality*

$$
L(\Lambda_1 + \Lambda_2) < 1,
$$

is valid, then BVP (1) has a unique solution on Π.

Proof. Transform problem (1) into a fixed point problem for the operator $\mathscr{T}$ given by

$$
\begin{aligned}
\mathscr{T}u(t) =& -\frac{A_1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f_1(s,u(s)) \frac{ds}{s} - \frac{A_2}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} f_2(s,u(s)) \frac{ds}{s}\\
&+ (\log t)^{\alpha-1} \Delta \left[\frac{A_1}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} f_1(s,u(s)) \frac{ds}{s} \right.\\
&+ \frac{A_2}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} f_2(s,u(s)) \frac{ds}{s}\\
&- \frac{A_1}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} f_1(\eta,u(\eta)) \frac{d\eta}{\eta} ds\\
&\left. - \frac{A_2}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} f_2(\eta,u(\eta)) \frac{d\eta}{\eta} ds \right].
\end{aligned}
$$

Applying the Banach contraction mapping principle, we shall show that $\mathscr{T}$ is a contraction. Let $u, v \in AC^n_\delta(\Pi, \mathbb{R})$. Then for each $t \in \Pi$, we have

$$\begin{aligned}
|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \leq & \frac{|A_1|}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f_1(s,u(s)) - f_1(s,v(s))| \frac{ds}{s} \\
& + \frac{|A_2|}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} |f_2(s,u(s)) - f_2(s,v(s))| \frac{ds}{s} \\
& + (\log t)^{\alpha-1} \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s)) - f_1(s,v(s))| \frac{ds}{s} \right. \\
& + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log \frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s)) - f_2(s,v(s))| \frac{ds}{s} \\
& + \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(\eta,u(\eta)) - f_1(\eta,v(\eta))| \frac{d\eta}{\eta} ds \\
& \left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta)) - f_2(\eta,v(\eta))| \frac{d\eta}{\eta} ds \right].
\end{aligned}$$

Hence,

$$\begin{aligned}
|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \leq L_1 \|u-v\| & \left[|A_1| \left[\frac{1}{\Gamma(\alpha+1)} + \Delta \left(\frac{1}{\Gamma(\alpha-\gamma+1)} \right.\right.\right. \\
& \left.\left.\left. + \frac{1}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} \frac{d\eta}{\eta} ds \right) \right] \right] \\
& + L_2 \|u-v\| \left[|A_1| \left[\frac{1}{\Gamma(\alpha+\beta+1)} + \Delta \left(\frac{1}{\Gamma(\alpha-\gamma+\beta+1)} \right.\right.\right. \\
& \left.\left.\left. + \frac{1}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} \frac{d\eta}{\eta} ds \right) \right] \right].
\end{aligned}$$

Thus, $|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)| \leq (L_1\Lambda_1 + L_2\Lambda_2)\|u-v\|$. Consequently,

$$\|(\mathscr{T}u)(t) - (\mathscr{T}v)(t)\| \leq L(\Lambda_1 + \Lambda_2)\|u-v\|.$$

As $L < 1/(\Lambda_1 + \Lambda_2)$, $\mathscr{T}$ is a contraction. Hence, by the Banach's fixed point theorem, the fractional BVP (1) has a unique solution. The proof is completed. □

3.2 Existence result via Scheafer's fixed point theorem

Theorem 3. *Suppose that* $1 \neq \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds$ *and assume that assumptions (H1)-(H2) hold. Then, problem (1) has at least one solution on* Π.

Proof. We use Scheafer's fixed point theorem to prove that $\mathscr{T}$ has at least a fixed point. The proof will be given in several steps.

- **Step 1** $\mathscr{T}$ is continuous.

  In view of the continuity of f_1, f_2, we conclude that the operator $\mathscr{T}$ is continuous.

- **Step 2** The operator $\mathscr{T}$ maps bounded sets into bounded sets in $\mathscr{C}(\Pi)$:

  For $r > 0$, we take $u \in B_r = \{u \in \mathscr{C}(\Pi), \| u \| \leq r\}$. Let $\sup_{t\in\Pi} f_1(t,1) = M_1$ and $\sup_{t\in\Pi} f_2(t,1) = M_2$, and assume that $M = \max\{M_1, M_2\}$.
  Choosing $r > \frac{M(\Lambda_1+\Lambda_2)}{1-L(\Lambda_1+\Lambda_2)}$, and from (H2), we obtain

$$|f_1(s,x(s))| \leq |f_1(s,x(s)) - f_1(s,1)| + |f_1(s,1)| \leq L_1 r + M_1,$$

$$|f_2(s,x(s))| \leq |f_2(s,u(s)) - f_2(s,1)| + |f_2(s,1)| \leq L_2 r + M_2.$$

  For $u \in B_r$, and for each $t \in \Pi$, we get

$$\begin{aligned}
|\mathscr{T}u(t)| \leq{}& \frac{|A_1|}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f_1(s,u(s))| \frac{ds}{s} - \frac{|A_2|}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
&+ (\log t)^{\alpha-1} \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \right. \\
&+ \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(\eta,u(\eta))| \frac{d\eta}{\eta} ds \\
&\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta))| \frac{d\eta}{\eta} ds \right].
\end{aligned}$$

  We obtain

$$\begin{aligned}
|\mathscr{T}u(t)| \leq (Lr+M) \Bigg(\Bigg[|A_1| \Bigg[& \frac{1}{\Gamma(\alpha+1)} + \Delta \Bigg(\frac{1}{\Gamma(\alpha-\gamma+1)} \\
&+ \frac{1}{\Gamma(\alpha-\gamma+1)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} \frac{d\eta}{\eta} ds \Bigg) \Bigg] \Bigg] \\
&+ \Bigg[|A_1| \Bigg[\frac{1}{\Gamma(\alpha+\beta+1)} + \Delta \Bigg(\frac{1}{\Gamma(\alpha-\gamma+\beta+1)} \\
&+ \frac{1}{\Gamma(\alpha-\gamma+\beta+1)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} d\eta ds \Bigg) \Bigg] \Bigg] \Bigg).
\end{aligned}$$

  Consequently,

$$\|\mathscr{T}u(t)\| \leq (Lr+M)(\Lambda_1+\Lambda_2) < r.$$

  Hence, $\mathscr{T}$ is uniformly bounded.

- **Step 3** $\mathscr{T}$ is equicontinuous on Π.

Let us take $u \in B_r$, $t_1, t_2 \in \Pi$, $t_1 < t_2$, we get

$$
\begin{aligned}
|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)| \le{}& \frac{|A_1|}{\Gamma(\alpha)} \int_1^t ((\log\frac{t_2}{s})^{\alpha-1} - (\log\frac{t_1}{s})^{\alpha-1}) |f_1(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_1|}{\Gamma(\alpha)} \int_1^t (\log\frac{t_2}{s})^{\alpha-1} |f_1(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_2|}{\Gamma(\alpha+\beta)} \int_1^t ((\log t_2 - \log s)^{\alpha+\beta-1} (\log\frac{t_1}{s})^{\alpha+\beta-1}) |f_2(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_2|}{\Gamma(\alpha)} \int_1^t (\log\frac{t_2}{s})^{\alpha+\beta-1} |f_1(s,u(s))| \frac{ds}{s} \\
&+ ((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}) \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \right. \\
&+ \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s (\log\frac{s}{\eta})^{\alpha-\gamma-1} |f_1(\eta,u(\eta))| \frac{d\eta}{\eta} ds \\
&\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta))| \frac{d\eta}{\eta} ds \right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)| \le{}& \frac{|A_1|Lr}{\Gamma(\alpha+1)} ((\log t_1)^{\alpha} - (\log t_2)^{\alpha}) + \frac{|A_2|Lr}{\Gamma(\alpha+\beta+1)} ((\log t_1)^{\alpha} - (\log t_2)^{\alpha}) \\
&+ ((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}) \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \right. \\
&+ \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(\eta,u(\eta))| \frac{d\eta}{\eta} ds \\
&\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta))| \frac{d\eta}{\eta} ds \right].
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)| \le{}& Lr \left(\frac{|A_1|}{\Gamma(\alpha+1)} + \frac{|A_2|}{\Gamma(\alpha+\beta+1)} \right) ((\log t_1)^{\alpha} - (\log t_2)^{\alpha}) \\
&+ ((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}) \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \right. \\
&+ \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
&+ \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(\eta,u(\eta))| \frac{d\eta}{\eta} ds \\
&\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta))| \frac{d\eta}{\eta} ds \right],
\end{aligned}
$$

which implies $\|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)\| \to 0$ as $t_2 \to t_1$, By Arzela-Ascoli theorem, we conclude that $\mathscr{T}$ is completely continuous operator.

- **Step 4** We show that the set Ω defined by

$$\Omega = \{u \in \mathscr{C}(\Pi), u = \rho \mathscr{T}(u), 0 < \rho < 1\},$$

is bounded.
Let $u \in \Omega$, then $u = \rho \mathscr{T}(u)$, for $0 < \rho < 1$. Thus, for each $t \in \Pi$, we get

$$\begin{aligned}\frac{1}{\rho}|\mathscr{T}u(t)| \leq & \frac{|A_1|}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f_1(s,u(s))| \frac{ds}{s} + \frac{|A_2|}{\Gamma(\alpha+\beta)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\ & + (\log t)^{\alpha-1} \Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| ds \right. \\ & + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log \frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\ & + \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(\eta,u(\eta))| \frac{d\eta}{\eta} ds \\ & \left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(\eta,u(\eta))| \frac{d\eta}{\eta} ds \right].\end{aligned}$$

So, we can write

$$\frac{1}{\rho}|\mathscr{T}u(t)| \leq (Lr+M)(\Lambda_1+\Lambda_2).$$

Therefore,

$$\|\mathscr{T}u(t)\| \leq \rho[(Lr+M)(\Lambda_1+\Lambda_2)].$$

This shows that $\mathscr{T}$ is bounded.
As consequence of Schaefer's fixed point theorem, problem (1) has at least one solution on Π.

□

3.3 Existence result via Krasnoselskii's fixed point theorem

Theorem 4. *Suppose that* $1 \neq \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} (\log s)^{\alpha-\gamma-1} ds$ *and assume that the hypotheses (H1)-(H2) hold, such that*

$$L\omega < 1. \tag{10}$$

If there exist $\bar{r} \in \mathbb{R}$ *such that*

$$\bar{r} \geq (L\bar{r}+M)(\Lambda_1+\Lambda_2). \tag{11}$$

then, problem (1) has at least one solution on Π.

Proof. Suppose that (11) holds, where $B_{\bar{r}} = \{u \in \mathscr{C}(\Pi), \| u \| \leq \bar{r}\}$. Let us split the operator $\mathscr{T} : \mathscr{C}(\Pi) \to \mathscr{C}(\Pi)$ defined by Eq (9) as

$$\mathscr{T}u(t) := \mathscr{R}u(t) + \mathscr{Q}u(t),$$

where $\mathscr{R}$ and $\mathscr{Q}$ are given by

$$\mathscr{R}u(t) := \frac{A_1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f_1(s,u(s))\frac{ds}{s} - \frac{A_2}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1} f_2(s,u(s))\frac{ds}{s},$$

and

$$\begin{aligned}\mathscr{Q}u(t) := (\log t)^{\alpha-1}\Delta\Bigg[&\frac{A_1}{\Gamma(\alpha-\gamma)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} f_1(s,u(s))\frac{ds}{s}\\ &+\frac{A_2}{\Gamma(\alpha-\gamma+\beta)}\int_1^t\left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} f_2(s,u(s))\frac{ds}{s}\\ &+\frac{A_1}{\Gamma(\alpha-\gamma)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} f_1(\eta,u(\eta))\frac{d\eta}{\eta}ds\\ &+\frac{A_2}{\Gamma(\alpha-\gamma+\beta)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} f_2(\eta,u(\eta))\frac{d\eta}{\eta}ds\Bigg].\end{aligned}$$

Claim 1: $\mathscr{R}(u)+\mathscr{Q}(v)\in B_{\bar{r}}$.

For $u,v\in B_{\bar{r}}$ and for each $t\in\Pi$, we get

$$\begin{aligned}|\mathscr{R}u(t)+\mathscr{Q}v(t)| \le\ &\frac{|A_1|}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}|f_1(s,u(s))|\frac{ds}{s}\\ &+\frac{|A_2|}{\Gamma(\alpha+\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha+\beta-1}|f_2(s,u(s))|\frac{ds}{s}\\ &+(\log t)^{\alpha-1}\Delta\Bigg[\frac{|A_1|}{\Gamma(\alpha-\gamma)}\int_1^e\left(\log\frac{e}{s}\right)^{\alpha-\gamma-1}|f_1(s,u(s))|\frac{ds}{s}\\ &+\frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)}\int_1^t\left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1}|f_2(s,u(s))|\frac{ds}{s}\\ &+\frac{|A_1|}{\Gamma(\alpha-\gamma)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1}|f_1(\eta,u(\eta))|\frac{d\eta}{\eta}ds\\ &+\frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1}|f_2(\eta,u(\eta))|\frac{d\eta}{\eta}ds\Bigg].\end{aligned}$$

Hence,

$$\begin{aligned}|\mathscr{R}u(t)+\mathscr{Q}v(t)| \le (L\bar{r}+M)\Bigg(&|A_1|\Bigg[\frac{1}{\Gamma(\alpha+1)}+\Delta\Bigg(\frac{1}{\Gamma(\alpha-\gamma+1)}\\ &+\frac{1}{\Gamma(\alpha-\gamma)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1}\frac{d\eta}{\eta}ds\Bigg)\Bigg]\\ &+|A_1|\Bigg[\frac{1}{\Gamma(\alpha+\beta+1)}+\Delta\Bigg(\frac{1}{\Gamma(\alpha-\gamma+\beta+1)}\\ &+\frac{1}{\Gamma(\alpha-\gamma+\beta)}\sum_{i=1}^{m-1}a_i\int_{\eta_{i-1}}^{\eta_i}\int_1^s\left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1}\frac{d\eta}{\eta}ds\Bigg)\Bigg]\Bigg).\end{aligned}$$

Consequently,

$$\|\mathscr{R}u(t)+\mathscr{Q}v(t)\| \leq (L\bar{r}+M)(\Lambda_1+\Lambda_2) < \bar{r}.$$

Now, using condition (11), we conclude that $\mathscr{R}(u)+\mathscr{Q}(v) \in B_{\bar{r}}$.

Claim 2: We shall prove that $\mathscr{Q}$ is continuous and compact.

i) The continuity of f_1 and f_2 imply that the operator $\mathscr{Q}$ is continuous.

ii) Now, we prove that $\mathscr{Q}$ maps bounded sets into bounded sets of $\mathscr{C}(\Pi)$. For $u \in B_{\bar{r}}$ and for each $t \in \Pi$, we have

$$\begin{aligned}
|\mathscr{Q}u(t)| \leq (\log t)^{\alpha-1}\Delta \Bigg[& \frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \\
& + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log \frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s} \\
& + \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{d\eta}{\eta} ds \\
& + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{d\eta}{\eta} ds \Bigg] \\
\leq \Delta \Bigg[& \frac{|A_1|(L_1\bar{r}+M_1)}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-\gamma-1} \frac{ds}{s} \\
& + \frac{|A_2|(L_2\bar{r}+M_2)}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log \frac{e}{s}\right)^{\alpha-\gamma+\beta-1} \frac{ds}{s} \\
& + \frac{|A_1|(L_1\bar{r}+M_1)}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} \frac{d\eta}{\eta} ds \\
& + \frac{|A_2|(L_2\bar{r}+M_2)}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} \frac{d\eta}{\eta} ds \Bigg].
\end{aligned}$$

We obtain

$$\begin{aligned}
|\mathscr{Q}u(t)| \leq (L\bar{r}+M)\Delta \times \Bigg[& \frac{|A_1|}{\Gamma(\alpha-\gamma+1)} + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta+1)} \\
& + \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma-1} \frac{d\eta}{\eta} ds \\
& + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log \frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} \frac{d\eta}{\eta} ds \Bigg].
\end{aligned}$$

So,

$$|\mathscr{Q}u(t)| \leq (L\bar{r}+M)\Delta.K.$$

Consequently,

$$\|\mathscr{Q}u(t)\| \leq \infty.$$

Thus, it follows from the above inequalities that the operator $\mathscr{Q}$ is uniformly bounded.

iii) The operator $\mathscr{Q}$ maps bounded sets into equicontinuous sets of $\mathscr{C}(\Pi)$.

Let $t_1, t_2 \in \Pi; t_2 < t_1$, $u \in B_{\bar{r}}$. Then, we have

$$|\mathscr{T}u(t_2) - \mathscr{T}u(t_1)| \leq ((\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1})\Delta \left[\frac{|A_1|}{\Gamma(\alpha-\gamma)} \int_1^e \left(\log\frac{e}{s}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{ds}{s} \right.$$
$$+ \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \int_1^t \left(\log\frac{e}{s}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{ds}{s}$$
$$+ \frac{|A_1|}{\Gamma(\alpha-\gamma)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma-1} |f_1(s,u(s))| \frac{d\eta}{\eta} ds$$
$$\left. + \frac{|A_2|}{\Gamma(\alpha-\gamma+\beta)} \sum_{i=1}^{m-1} a_i \int_{\eta_{i-1}}^{\eta_i} \int_1^s \left(\log\frac{s}{\eta}\right)^{\alpha-\gamma+\beta-1} |f_2(s,u(s))| \frac{d\eta}{\eta} ds \right].$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As consequence of **[i), ii), iii)]**, together with the Arzela-Ascoli theorem, we can conclude that $\mathscr{Q}$ is continuous and completely continuous.

Claim 3: Now, we prove that $\mathscr{R}$ is a contraction mapping.

Let $u, v \in \mathscr{C}(\Pi)$. Then, for each $t \in \Pi$, we have

$$|(\mathscr{R}u)(t) - (\mathscr{R}v)(t)| \leq \frac{|A_1|}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f_1(s,u(s)) - f_1(s,v(s))| \frac{ds}{s}$$
$$+ \frac{|A_2|}{\Gamma(\alpha+\beta)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha+\beta-1} |f_2(s,u(s)) - f_2(s,v(s))| \frac{ds}{s}.$$

By (H1), we obtain

$$|(\mathscr{R}u)(t) - (\mathscr{R}v)(t)| \leq L\|u-v\| \left[\frac{|A_1|}{\Gamma(\alpha+1)} + \frac{|A_2|}{\Gamma(\alpha+\beta+1)}\right].$$

Consequently,

$$|(\mathscr{R}u)(t) - (\mathscr{R}v)(t)| \leq L\omega\|u-v\|.$$

Using condition (10) we conclude that $\mathscr{R}$ is a contraction mapping.

As a consequence of Krasnoselskii's fixed point theorem, we deduce that $\mathscr{T}$ has a fixed point which is a solution of (1).

□

4 Applications

To elaborate, our results constructed in the previous two subsections, here we provide two examples.

Example 1. Consider the following fractional boundary value problem

$$\begin{cases} -D^{\frac{3}{4}}u(t) = f_1(t,u(t)) + I^{\frac{1}{3}} f_2(t,u(t)), \quad 1 < \alpha \leq 2, \quad t \in \Pi, \\ D^{\frac{1}{4}}u(1) = 0, \quad D^{\frac{1}{4}}u(e) = \frac{1}{2}\int_0^{\frac{1}{4}} D^{\frac{1}{4}}u(s)ds + \frac{1}{2}\int_{\frac{1}{3}}^1 D^{\frac{1}{4}}u(s)ds. \end{cases} \tag{12}$$

Here $n=2$, $\alpha=3/2$, $\gamma=1/4$, $\beta=1/3$, $a_1=1/2$, $a_2=0$, $a_3=1/2$, $\eta_0=0$, $\eta_1=1/4$, $\eta_2=1/3$, $\eta_3=1$ and

$$f_1(t,u)=\frac{1}{t^2+10}\cos(u),\quad f_2(t,u)=\frac{1}{t^2+12}\sin(u).$$

As

$$|f_1(t,u)-f_1(t,v)|\leq\frac{1}{10}|u-v|,$$

and

$$|f_2(t,u)-f_2(t,v)|\leq\frac{1}{12}|u-v|,$$

(H2) is satisfied with $L=\max\{L_1,L_2\}=\frac{1}{10}$. and $L(\Lambda_1+\Lambda_2)<1$. Therefore, by the conclusion of Theorem 2, the fractional BVP (12) has a unique solution on Π.

Example 2. Consider the problem

$$\begin{cases}-D^{\frac{5}{2}}u(t)=f_1(t,u(t))+I^{\frac{1}{2}}f_2(t,u(t)),\ \ 1<\alpha\leq 3,\ \ t\in\Pi,\\ D^{\frac{1}{4}}u(1)=D^{\frac{5}{4}}u(1)=0,\ \ D^{\frac{1}{4}}u(e)=\frac{1}{2}\int_0^{\frac{1}{8}}D^{\frac{1}{4}}u(s)ds+\frac{1}{2}\int_{\frac{1}{8}}^1 D^{\frac{1}{4}}u(s)ds.\end{cases}\tag{13}$$

Here $n=3$, $\alpha=5/2$, $\gamma=1/4$, $\beta=1/2$, $a_1=1/2$, $a_2=0$, $a_3=1/2$, $\eta_0=0$, $\eta_1=1/8$, $\eta_2=1/8$, $\eta_3=1/4$ and

$$f_1(t,u)=\frac{1}{t^2+10}\cos(u),\quad f_2(t,u)=\frac{1}{t^2+12}\sin(u).$$

As $|f_1(t,u)-f_1(t,v)|\leq\frac{1}{10}|u-v|$, and $|f_2(t,u)-f_2(t,v)|\leq\frac{1}{12}|u-v|$, (H2) is satisfied with $L=\max\{L_1,L_2\}=\frac{1}{10}$. and $L\omega<1$. Hence by Theorem 4, the boundary value problem (13) has at least one solution on Π.

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