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A robust optimization approach for multi-objective linear programming under uncertainty

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Abstract. This paper proposes a new robust optimization approach for solving multi-objective linear programming problems under uncertainty. The uncertainty is assumed to be in the objective function coefficients and the constraint parameters. The proposed approach is based on an alternative model for obtaining robust efficient solutions to the original problem. A numerical example is given to test and illustrate the effectiveness of the proposed approach, and a comparison with a method given in the literature is discussed based on certain performance metrics.

Keywords: Multi-objective linear programming, uncertainty, robust optimization, weighted sum scalarization.
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1 Introduction

In many real-life problems, decision-makers are looking for solutions that satisfy their constraints and simultaneously optimize several conflicting criteria. Examples are the optimization of cost and time in transportation problems [3], cost and profit in production planning [5], cost of therapy and patients satisfaction in healthcare management problems [26], and revenue and risk in portfolio optimization problems [29]. Theoretical issues, applicability and numerous approaches for solving such problems have been widely considered. Some reviews in this regard can be found in [10, 13, 18]. In deterministic multi-objective optimization problems, objective coefficients and constraint parameters are nominal and precisely determined. They are known before solving the problem under consideration, which makes it possible to obtain nominal solutions to the problem.

However, uncertainty is an important characteristic of real-world problems. When solving multi-objective optimization problems, the precise input data are often unknown beforehand. The input data

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may be unknown and uncertain for several reasons, such as measurement errors, fluctuations, lack of precise information, disturbances and perturbation of nominal data because of the interaction of the problem with other related problems. For example, in import and export problems, transportation costs can be uncertain because of changes in fuel prices, availability of transportation modes, and global circumstances [1]. Customer demand may not be known by those delivering goods before the goods arrive at the destination [6, 17]. Transportation times may also be uncertain because of weather conditions, accumulated delays and road congestion [22]. Moreover, in the case of transport of dangerous goods, the number of persons exposed to the danger should be taken into account, but is usually unknown. Another example is uncertainty of returns from financial markets in portfolio selection problem [25]. This list of examples is not exhaustive, but the main common challenge is to consider uncertainty when addressing these multi-objective optimization problems.

There have been two approaches to deal with uncertainty in optimization problems: stochastic optimization and robust optimization [19]. In the first approach, the aim is to find feasible solutions such that the uncertain parameters are assumed to be random variables and to obey a probability distribution. On the other hand, the second approach robust optimization consists of looking for solutions that still apply in different situations and protect against uncertainty regardless of the parameter values within an uncertain set, without any probability distribution assumptions.

Multi-objective optimization and robust optimization have both been thoroughly considered in many problems. However they have rarely been combined in the same problem, and this research area deserves further investigation.

Robustness has been carefully considered in the literature for single-objective optimization; see [4, 16, 27]. Topics related to robust optimization, formulations and solution approaches are reviewed in [21]. Because of the multiple and conflicting objectives of real-world problems, it is crucial to deal with uncertainties in multi-objective optimization problems.

Generally speaking, in multi-objective optimization problems uncertainty is taken into consideration using various concepts of robustness such as minmax robustness [8], component-wise worst case robustness [12, 20], robustness based on set order relations [9], and robustness in an optimization problem such as a bi-level vector optimization problem [28].

In particular, multi-objective linear programming with uncertainties has been investigated by various authors using different research viewpoints. Rivaz and Yaghoobi [24] introduced a minmax regret approach to obtain solutions to a multi-objective linear programming problem with interval uncertainties in the objective function coefficients. These authors modelled the uncertain coefficients of the problem using closed intervals. In addition, to take into account all the possibilities for the coefficients of the objective functions, they presented an approach which consists of using a minmax regret method based on an assumed reference point given by the decision-maker. Rivaz and Saeidi [23] introduced an interval-based approach for solving multi-objective linear problems. They proposed an order relation for interval numbers to deal with the uncertainties that affect the objective coefficients and the constraint parameters. Goberna et al. [15] investigated a multi-objective linear programming problem with uncertainties in both the objective functions and the constraints. They introduced a radius concept of robustness for checking the non-emptiness of the robust feasible set. Their approach involves considering all the possible scenarios across all parameters within the uncertainty set; uncertainty is considered by using ellipsoidal, norm and box data uncertainty sets, which leads to an optimization problem with second-order constraints. Georgiev et al. [14] studied robustness in terms of post-optimal analysis instead of looking for a solution to a multi-objective linear programming problem that remains feasible even if the data for the problem vary. The idea presented by the authors is to solve the deterministic problem and explore the conditions under which its efficient solution remains efficient when the coefficients of the objective functions are subject to change.

We generally note that, in most research studies that are interested in multi-objective linear optimization under uncertainties, the robustness is considered for certain objective functions of the problem or the constraints. However, uncertainty in both the constraints and all the objective functions simultaneously has not had enough attention in the research community.

In most of these studies, robustness is introduced using interval-based approaches, minmax regret, and all possible results from the data. Considering all possible scenarios is obviously expensive in terms of computation and time, especially with a very large number of scenarios. In other studies, uncertainty is taken into account by using a norm-based approach, which leads to the loss of linearity of the problem under study. Taking into account all the scenarios and the loss of linearity of the problem represent drawbacks that will be avoided in the approach proposed in our paper.

This article attempts to suggest an approach to solve a multi-objective linear programming problem with uncertainties in the constraints and also in the objective functions. This paper extends the approach in [7] for single-objective linear programming problems to the multi-objective version of the problem. Moreover, a scalarization approach is proposed to obtain efficient solutions to the problem with uncertainties. Indeed, instead of looking directly for robust and efficient solutions to a multi-objective linear programming problem with uncertainties, the core idea is to solve an alternative model that is a deterministic and single-objective problem.

The remainder of this paper is organized as follows. Section 2 introduces the necessary preliminaries. Section 3 discusses the proposed formulation of multi-objective linear programming problems with uncertainties, and the proposed alternative model. The scalarization approach is demonstrated in Section 4. A numerical example, computational results and comparison with another method are provided in Section 5. Finally, the paper is concluded in Section 6.

2 Preliminaries

Consider the following multi-objective linear programming problem

$$
\begin{aligned}
\min \quad & Z_k(x) = \sum_{j=1}^{m} c_{kj} x_j, && k = 1,\dots,p, \\
\text{s.t.} \quad & \sum_{j=1}^{m} a_{ij} x_j \le b_i, && i = 1,\dots,n, \\
& x_j \ge 0, && j = 1,\dots,m.
\end{aligned} \tag{1}
$$

We introduce the order relation used in this paper as defined by Ehrgott [11].

Definition 1. *Let $v = (v_1, v_2, \dots, v_p)$ and $u = (u_1, u_2, \dots, u_p)$ be two vectors in $\mathbb{R}^p$. Then, $v \preceq u$ if $v_i \le u_i\ \forall i \in \{1,2,\dots,p\}$ and $\exists\, i_0 \in \{1,\dots,p\}$ such that $v_{i_0} < u_{i_0}$. In other words, $v \preceq u$ if v is smaller than or equal to u in every component and strictly smaller in at least one component.*

Definition 2. *Let $v = (v_1, v_2, \dots, v_p)$ and $u = (u_1, u_2, \dots, u_p)$ be two vectors in $\mathbb{R}^p$. We write $v \prec u$ if $v_i < u_i\ \forall i \in \{1,2,\dots,p\}$, which means v is smaller than u in every component.*

By using such order relations, we can compare the feasible solutions of problem (1) from a multi-objective point of view. The following definitions describe the characterization of the feasible solutions that we look for when solving problem (1).

Definition 3. *A feasible solution x^* is a Pareto efficient solution (or simply an efficient solution) to problem (1) if there is no other feasible solution x such that $Z(x) \preceq Z(x^*)$.*

Definition 4. *A feasible solution x^* is called a weakly efficient solution of problem (1) if there is no other feasible solution x such that $Z(x) \prec Z(x^*)$.*

3 Robust multi-objective linear program

Consider the multi-objective linear programming problem (1). Assume that the objective coefficient c_{kj} and the constraint parameter a_{ij} are affected by data uncertainty. The coefficient $c_{kj}, k \in K = \{1,2,\ldots,p\}, j \in J = \{1,2,\ldots,m\}$ is modelled as a parameter $\tilde{c}_{kj}$ that takes values in the interval $[c_{kj} - \hat{c}_{kj}, c_{kj} + \hat{c}_{kj}]$ where $\hat{c}_{kj}$ is the deviation from the nominal value c_{kj}. Let J_k define the set of coefficients of the k^{th} objective function that are subject to uncertainty. Thus, $J_k = \{j \in J : \hat{c}_{kj} > 0\}$. We introduce an integer parameter $\nabla_k \in [0, |J_k|]$. This parameter adjusts the robustness of the k^{th} objective. The coefficient $a_{ij}, i \in I = \{1,2,\ldots,n\}, j \in J = \{1,2,\ldots,m\}$ is modelled as a parameter $\tilde{a}_{ij}$ that takes values in the interval $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$, where $\hat{a}_{ij}$ is the deviation from the nominal value a_{ij}. Let I_i define the set of parameters of the i^{th} constraint that are subject to uncertainty (we assume that a_{ij} and not b_i may be affected by uncertainty). In other words, $I_i = \{j \in J : \hat{a}_{ij} > 0\}$. We introduce another integer parameter $\Gamma_i \in [0, |I_i|]$ which adjusts the robustness of the data in the i^{th} constraint.

The proposed extension to a robust multi-objective linear programming problem (RMLP) can be formulated as follows

$$
\begin{aligned}
\text{Min} \quad & \tilde{Z}_k(x) = \sum_{j=1}^{m} c_{kj} x_j + \max_{\{U_k : U_k \subset J_k,\, |U_k| \le \nabla_k\}} \sum_{j \in U_k} \hat{c}_{kj}.x_j, && k = 1,\ldots,p, \\
\text{s.t.} \quad & \sum_{j=1}^{m} a_{ij} x_j + \max_{\{S_i : S_i \subset I_i,\, |S_i| \le \Gamma_i\}} \sum_{j \in S_i} \hat{a}_{ij}.x_j \le b_i, && i = 1,\ldots,n, \\
& x_j \ge 0, && j = 1,\ldots,m.
\end{aligned}
\tag{2}
$$

The k^{th} objective function is protected by

$$\beta_k(x, \nabla_k) = \max_{\{U_k : U_k \subset J_k,\, |U_k| \le \nabla_k\}} \sum_{j \in U_k} \hat{c}_{kj}.x_j.$$

The i^{th} constraint is protected by

$$\varphi_i(x, \Gamma_i) = \max_{\{S_i : S_i \subset I_i,\, |S_i| \le \Gamma_i\}} \sum_{j \in S_i} \hat{a}_{ij}.x_j.$$

Note that an efficient solution of problem (2) is called a robust efficient solution. Next, we establish the following theorem.

Theorem 1. *The robust multi-objective linear programming problem (2) can be equivalently formulated as the following deterministic multi-objective linear programming problem (3), which is an alternative model (ARMLP)*

$$\begin{array}{lll} \min & \theta_k(x) = \sum_{j=1}^{m} c_{kj}.x_j + \sum_{j \in J_k} p_{kj} + \nabla_k.g_k, & k = 1, \dots, p, \\ s.t. & \sum_{j=1}^{m} a_{ij}.x_j + \sum_{j \in I_i} \omega_{ij} + \Gamma_i.\eta_i \le b_i, & i = 1, \dots, n, \\ & p_{kj} + g_k \ge \hat{c}_{kj}.x_j, & j \in J_k, \ k = 1, \dots, p, \\ & \omega_{ij} + \eta_i \ge \hat{a}_{ij}.x_j, & j \in I_i, \ i = 1, \dots, n, \\ & p_{kj} \ge 0, & j \in J_k, \ k = 1, \dots, p, \\ & \omega_{ij} \ge 0, & j \in I_i, \ i = 1, \dots, n, \\ & g_k \ge 0, & k = 1, \dots, p, \\ & \eta_i \ge 0, & i = 1, \dots, n, \\ & x_j \ge 0, & j = 1, \dots, m. \end{array} \tag{3}$$

Proof. Given a fixed vector x, the protection function of the k^{th} objective function is:

$$\beta_k(x, \nabla_k) = \max_{\{U_k : U_k \subset J_k, \, |U_k| \le \nabla_k\}} \sum_{j \in U_k} \hat{c}_{kj}.x_j$$

This is equivalent to the following optimization problem

$$\begin{array}{rll} \beta_k(x, \nabla_k) = \max & \sum_{j \in J_k} \hat{c}_{kj}.x_j.t_{kj}, & \\ \text{s.t.} & \sum_{j \in J_k} t_{kj} \le \nabla_k, & \\ & t_{kj} \in \{0,1\}, \quad j \in J_k, & \end{array} \tag{4}$$

$\beta_k(x, \nabla_k)$ is equivalent to the selection of a subset $U_k \subset J_k$ with a cardinality of at most ∇_k, that is, the selection of no more than ∇_k elements with a value of 1, which constitutes the optimum solution for problem (4).

The dual version of problem (4) is the following problem

$$\begin{array}{rll} \beta_k(x, \nabla_k) = \min & \nabla_k.g_k + \sum_{j \in J_k} p_{kj}, & \\ \text{s.t.} & p_{kj} + g_k \ge \hat{c}_{kj}.x_j, \quad j \in J_k, & \\ & p_{kj} \ge 0, \quad j \in J_k, & \\ & g_k \ge 0. & \end{array} \tag{5}$$

In the same manner, the protection function of the i^{th} constraint is

$$\varphi_i(x, \Gamma_i) = \max_{\{S_i : S_i \subset I_i, \, |S_i| \le \Gamma_i\}} \sum_{j \in S_i} \hat{a}_{ij}.x_j.$$

This is equivalent to the following optimization problem

$$\begin{aligned} \varphi_i(x,\Gamma_i) &= \max \quad \sum_{j\in I_i} \hat{a}_{ij}.x_j.v_{ij}, \\ \text{s.t.} \quad & \sum_{j\in I_i} v_{ij} \leq \Gamma_i, \\ & v_{ij} \in \{0,1\}, \qquad j \in I_i, \end{aligned} \tag{6}$$

$\varphi_i(x,\Gamma_i)$ consists of the selection of a subset $S_i \subset I_i$ with a cardinality of at most Γ_i, that is, the selection of no more than Γ_i elements with a value of 1, which constitutes the optimal solution of problem (6).

The dual version of problem (6) is the following problem

$$\begin{aligned} \varphi_i(x,\Gamma_i) &= \min \quad \Gamma_i.\eta_i + \sum_{j\in I_i} \omega_{ij}, \\ \text{s.t.} \quad & \omega_{ij} + \eta_i \geq \hat{a}_{ij}.x_j, \qquad j \in I_i, \\ & \omega_{ij} \geq 0, \qquad j \in I_i. \\ & \eta_i \geq 0 \end{aligned} \tag{7}$$

As a remark, when $\nabla_k = 0$ and $\Gamma_i = 0$ then $\beta_k(x,\nabla_k) = 0$ and $\varphi_i(x,\Gamma_i) = 0$. Thus, the problem is equivalent to the deterministic problem with nominal data (problem (1)).

For all $\nabla_k \in [0,|J_k|]$, problem (4) is bounded and feasible. By strong duality, the dual problem (5) is also bounded and feasible. Moreover, the objective function values of the primal problem (4) and the dual problem (5) are equal. Similarly, the primal problem (6) and its dual problem (7) are feasible and bounded for all $\Gamma_i \in [0,|I_i|]$, and their objective function values are equal.

Let us return to problem (2). By integrating the new formulations of $\beta_k(x,\nabla_k)$ and $\varphi_i(x,\Gamma_i)$, we find that problem (2) is equivalent to the optimization problem (3). Next, we propose the following theorem. □

Theorem 2. *Any efficient solution of problem (3) is a robust efficient solution of problem (2).*

Proof. Let x^* be an efficient solution of problem (3). In fact, x^* is a component of a solution (x^*,p^*,g^*) which efficiently optimizes the objective functions of problem (3).

Suppose x^* is not an efficient solution of problem (2). Then there exists a feasible point $x, x \neq x^*$, such that $\tilde{z}(x) \preceq \tilde{z}(x^*)$. In other words, $\exists k_0 \in \{1,\ldots,p\}$ at least, such that $\tilde{z}_{k_0}(x) < \tilde{z}_{k_0}(x^*)$. This means

$$\sum_{j=1}^{m} c_{k_0 j}.x_j + \max_{\{U_{k_0}:U_{k_0}\subset J_{k_0},|U_{k_0}|\leq\nabla_{k_0}\}} \sum_{j\in U_{k_0}} \hat{c}_{k_0 j}.x_j < \sum_{j=1}^{m} c_{k_0 j}.x_j^* + \max_{\{U_{k_0}:U_{k_0}\subset J_{k_0},|U_{k_0}|\leq\nabla_{k_0}\}} \sum_{j\in U_{k_0}} \hat{c}_{k_0 j}.x_j^*,$$

which implies

$$\sum_{j=1}^{m} c_{k_0 j}.x_j + \nabla_{k_0}.g_{k_0} + \sum_{j\in J_{k_0}} p_{k_0 j} < \sum_{j=1}^{m} c_{k_0 j}.x_j^* + \nabla_{k_0}.g_{k_0}^* + \sum_{j\in J_{k_0}} p_{k_0 j}^*.$$

Using the results of the primaldual conversion of problems (4) and (5), we have $\theta_{k_0}(x) < \theta_{k_0}(x^*)$. This means that x^* is not an efficient solution of problem (3), and this is a contradiction. □

In the next section, the weighted sum method is used as a scalarization approach to find robust efficient solutions to the multi-objective linear programming problem with uncertainties.

4 Weighted sum scalarization

The weighted sum scalarization approach is one of the best-known and most common methods for solving deterministic multi-objective optimization problems like (1) that is formulated as follows:

$$\begin{aligned} \min \quad & \sum_{k=1}^{p} \lambda_k . Z_k(x), & \\ \text{s.t.} \quad & \sum_{j=1}^{m} a_{ij}.x_j \leq b_i, & i = 1,\dots,n, \\ & x_j \geq 0, & j = 1,\dots,m, \end{aligned}$$

such that $\lambda_k \geq 0 \ \ \forall k = 1,\dots,p$ and $\sum_{k=1}^{p} \lambda_k = 1$. The main advantage of weighted sum scalarization is that it allows us to find efficient solutions of a multi-objective optimization problem by reducing it to a single-objective optimization problem. Next, we introduce the weighted sum scalarization formulation of problem (3) with the objective function $\sum_{k=1}^{p} \lambda_k.\theta_k(x)$ as follows:

$$\begin{aligned} \min \quad & \psi(x) = \sum_{k=1}^{p}\sum_{j=1}^{m} \lambda_k.c_{kj}.x_j + \sum_{k=1}^{p}\sum_{j\in J_k} \lambda_k.p_{kj} + \sum_{k=1}^{p} \lambda_k.\nabla_k.g_k, \\ \text{s.t.} \quad & \\ & \textit{All the constraints of problem (3)} \end{aligned} \tag{8}$$

Theorem 3. *If x^* is a unique optimal solution of problem (8), then x^* is an efficient solution of problem (3).*

Proof. Let x^* be a component of the unique optimal solution of problem (8). Suppose that x^* is not an efficient solution for problem (3). So, there exists x such that $\theta(x) \preceq \theta(x^*)$. Then, $\theta_k(x) \leq \theta_k(x^*) \ \ \forall k = 1,\dots,p$ and $\theta_{k_0}(x) < \theta_{k_0}(x^*)$ for an element $k_0 \in \{1,\dots,p\}$, which means $\forall k = 1,\dots,p$:

$$\sum_{j=1}^{m} c_{kj}.x_j + \sum_{j\in J_k} p_{kj} + \nabla_k.g_k \leq \sum_{j=1}^{m} c_{kj}.x_j^* + \sum_{j\in J_k} p_{kj}^* + \nabla_k.g_k^*$$

and for an element $k_0 \in \{1,\dots,p\}$

$$\sum_{j=1}^{m} c_{k_0 j}.x_j + \sum_{j\in J_{k_0}} p_{k_0 j} + \nabla_{k_0}.g_{k_0} < \sum_{j=1}^{m} c_{k_0 j}.x_j^* + \sum_{j\in J_{k_0}} p_{k_0 j}^* + \nabla_{k_0}.g_{k_0}^*.$$

We arbitrarily choose $\lambda_k \geq 0$ such that $\sum_{k=1}^{p} \lambda_k = 1$. Then we have

$$\sum_{k=1}^{p}\sum_{j=1}^{m} \lambda_k c_{kj}.x_j + \sum_{k=1}^{p}\sum_{j\in J_k} \lambda_k p_{kj} + \sum_{k=1}^{p} \lambda_k \nabla_k.g_k \leq \sum_{k=1}^{p}\sum_{j=1}^{m} \lambda_k c_{kj}.x_j^* + \sum_{k=1}^{p}\sum_{j\in J_k} \lambda_k p_{kj}^* + \sum_{k=1}^{p} \lambda_k \nabla_k.g_k^* \,.$$

This means that $\psi(x) \leq \psi(x^*)$, and this is in contradiction with the optimality and the uniqueness of the solution x^* of problem (8). □

Remark 1. *The result of Theorem 3 remains true with* $\lambda_k > 0$ *strictly positive* $\forall k = 1,\ldots,p$ *such that* $\sum_{k=1}^{p} \lambda_k = 1$*, even if the uniqueness of the optimal solution is not guaranteed. The proof is similar to the previous one.*

To summarize this section, if (x^*, p^*, g^*) is an optimal solution to problem (8), then x^* is a robust efficient solution to problem (2). Indeed, if (x^*, p^*, g^*) is an optimal solution to problem (8), then by Theorem 1 (x^*, p^*, g^*) is an efficient solution to problem (3). Thus, by Theorem 2, x^* is an efficient solution of the robust multi-objective linear programming problem (2), which is the main optimization problem that we are trying to solve. Overall, to find the robust efficient solutions for a multi-objective linear programming problem with uncertain data, it is sufficient to solve the deterministic single-objective problem (8).

The next section provides a numerical example to demonstrate the effectiveness of the proposed approach.

5 Numerical example

Suppose a company produces five products P_1, P_2, P_3, P_4 and P_5. The aim is to simultaneously minimize the total cost and the total time of manufacturing, in order to make a profit that exceeds 5000\$. The profits per unit of P_1, P_2, P_3, P_4 and P_5 are, respectively, $60\$, 30\$, 60\$, 50\$$, and 40\$.

The manufacturing cost and time of one unit of each product is summarized in Table 1. The company uses four raw materials M_1, M_2, M_3 and M_4 for manufacturing the products. Table 2 summarizes the number of units of each raw material M_j required to manufacture one unit of each product P_i.

Table 1: Manufacturing cost and time for producing one unit of each product.

	P_1	P_2	P_3	P_4	P_5
Manufacturing cost of one unit (in \$)	250	200	300	500	300
Manufacturing time of one unit (in min)	100	400	300	100	60

Table 2: Number of units of each raw material used for producing one unit of each product.

	M_1	M_2	M_3	M_4
One unit of product P_1	2	1	1	2
One unit of product P_2	1	1	2	3
One unit of product P_3	1	2	1	2
One unit of product P_4	2	1	3	1
One unit of product P_5	2	3	1	1

Furthermore, raw materials M_1, M_2, M_3 and M_4 are assumed to be available with limited quantities of $150, 175, 201$ and 400 units respectively. The aim is to minimize the total cost and the total time of manufacturing. The formulation of this problem is given as the following deterministic multi-objective linear model:

$$\begin{aligned}
\min \quad & 250x_1+200x_2+300x_3+500x_4+300x_5,\\
\min \quad & 100x_1+400x_2+300x_3+100x_4+60x_5,\\
\text{s.t.} \quad & \\
& 2x_1+x_2+x_3+2x_4+2x_5 \le 150,\\
& x_1+x_2+2x_3+x_4+3x_5 \le 175,\\
& x_1+2x_2+x_3+3x_4+x_5 \le 201,\\
& 2x_1+3x_2+2x_3+x_4+x_5 \le 400,\\
& 60x_1+30x_2+60x_3+50x_4+40x_5 \ge 5000,\\
& x_1,x_2,x_3,x_4,x_5 \ge 0.
\end{aligned}$$

Let us assume that the manufacturing cost and time of some products, as well as the quantities of some raw materials, are uncertain. For instance, let us suppose the costs of producing one unit of products P_3 and P_5 are uncertain. Instead of the nominal values for the costs $c_{13}=300$ and $c_{15}=500$, we have $\tilde{c}_{13}\in[200,400]$ and $\tilde{c}_{14}\in[400,600]$. Similarly, the production times of P_2 and P_3 are assumed to be uncertain. Instead of the nominal values $c_{22}=400$ *min* and $c_{24}=300$ *min*, these values vary in intervals as follows: $\tilde{c}_{22}\in[370,430]$ and $\tilde{c}_{23}\in[270,330]$.

For the constraint parameters, it is assumed that, instead of the nominal values $a_{11}=2, a_{15}=2, a_{23}=2, a_{25}=3, a_{32}=2, a_{41}=2$ and $a_{42}=3$, these parameters are uncertain and vary in intervals as follows: $\tilde{a}_{11}\in[1,3], \tilde{a}_{15}\in[1,3], \tilde{a}_{23}\in[1,3], \tilde{a}_{25}\in[2,4], \tilde{a}_{32}\in[1,3], \tilde{a}_{41}\in[1,3]$ and $\tilde{a}_{42}\in[1,5]$.

To obtain a robust efficient solution to the problem with these uncertain data, the mathematical model in the form of the alternative problem (3) is used. The formulation is given by the following problem

$$\begin{aligned}
\min \quad & 250x_1+200x_2+300x_3+500x_4+300x_5+p_{13}+p_{14}+\nabla_1.g_1,\\
\min \quad & 100x_1+400x_2+300x_3+100x_4+60x_5+p_{22}+p_{23}+\nabla_2.g_2,\\
\text{s.t.} \quad & \\
& 2x_1+x_2+x_3+2x_4+2x_5+\omega_{11}+\omega_{15}+\Gamma_2.v_2 \le 150,\\
& x_1+x_2+2x_3+x_4+3x_5+\omega_{23}+\omega_{25}+\Gamma_1.v_1 \le 175,\\
& x_1+2x_2+x_3+3x_4+x_5+\omega_{32}+\Gamma_3.v_3 \le 201,\\
& 2x_1+3x_2+2x_3+x_4+x_5+\omega_{41}+\omega_{42}+\Gamma_4.v_4 \le 400,\\
& 60x_1+30x_2+60x_3+50x_4+40x_5 \ge 5000,\\
& p_{kj}+g_k \ge \hat{c}_{kj}.x_j, \quad j\in J_k, k=1,\dots,p,\\
& \omega_{ij}+v_i \ge \hat{a}_{ij}.x_j, \quad j\in I_i, i=1,\dots,n,\\
& p_{kj},g_k \ge 0, \quad j\in J_k,\ k=1,\dots,p,\\
& \omega_{ij},v_i \ge 0, \quad j\in I_i,\ i=1,\dots,n,\\
& x_j \ge 0, \quad j=1,\dots,m.
\end{aligned} \tag{9}$$

Using the weighted sum scalarization formulation given by problem (8), we can solve the robust multi-objective programming problem with data uncertainties. Note that the models are implemented in Cplex12.5 using a Personal Computer core i3, 2.2GHz with 4GB of RAM.

5.1 Comparison and performance metrics

For comparison, we test by solving the problem in the numerical example using the interval-based approach IBA proposed in [23]. In summary, the IBA method suggested in [23] consists of solving problem (1):

$$\begin{array}{lll} \min & Z_k(x) = \sum_{j=1}^{m} c_{kj}.x_j, & k = 1,\ldots,p, \\ \text{s.t.} & \sum_{j=1}^{m} a_{ij}.x_j \leq b_i, & i = 1,\ldots,n, \\ & x_j \geq 0, & j = 1,\ldots,m, \end{array}$$

such that $c_{kj} \in [c_{kj}^L, c_{kj}^R], a_{ij} \in [a_{ij}^L, a_{ij}^R]$ and $b_i \in [b_i^L, b_i^R]$. Note that when a coefficient is not uncertain, the left and the right bounds coincide. Then the interval problem is given as

$$\begin{array}{lll} \min & Z_k^{'}(x) = \sum_{j=1}^{m} [c_{kj}^L, c_{kj}^R].x_j, & k = 1,\ldots,p, \\ \text{s.t.} & \sum_{j=1}^{m} [a_{ij}^L, a_{ij}^R].x_j \leq [b_i^L, b_i^R], & i = 1,\ldots,n, \\ & x_j \geq 0, & j = 1,\ldots,m. \end{array} \tag{10}$$

The authors of [23] supposed that an interval A can be represented by its left and right bounds $[A^L, A^R]$ or by its centre point A^c and the half-width length A^w, where

$$A^c = \frac{A^L + A^R}{2} \quad \text{and} \quad A^w = \frac{A^R - A^L}{2}.$$

They also proposed an order relation between any pair of intervals A and B defined as:
$A \preceq B$ if and only if $A^R \leq B^R$ and $A^c \leq B^c$, and $A < B$ if and only if $A \preceq B$ and $A \neq B$. In order to find robust and efficient solutions, the authors suggested solving the following problem:

$$\begin{array}{lll} \min & z_1(x) = \sum_{k=1}^{p} \sum_{j=1}^{m} \lambda_k.c_{kj}^R.x_j, & \\ \min & z_2(x) = \sum_{k=1}^{p} \sum_{j=1}^{m} \lambda_k.\frac{c_{kj}^L + c_{kj}^R}{2}.x_j, & \\ \text{s.t.} & \sum_{j=1}^{m} (a_{ij}^L + \alpha_i.(a_{ij}^R - a_{ij}^L)).x_j \leq b_i^R - \alpha_i.(b_i^R - b_i^L), & i = 1,\ldots,n, \\ & x_j \geq 0, & j = 1,\ldots,m, \end{array} \tag{11}$$

where $0 \leq \alpha_i \leq 1,\ i = 1,\ldots,n$. Note that if $\alpha_i = 0\ \forall = 1,\ldots,n$ then the feasible set of solutions is larger. The solutions of the previous problem (11) are robust efficient solutions for the multi-objective linear problem with uncertain data. Indeed, suppose that $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is an efficient solution of problem (11) and assume that x^* is not an efficient solution of problem (10). Therefore, there is some feasible solution $x = (x_1, x_2, \ldots, x_n)$ to problem (10) such that $(Z_1^{'}(x), Z_2^{'}(x), \ldots, Z_p^{'}(x)) \preceq (Z_1^{'}(x^*), Z_2^{'}(x^*), \ldots, Z_p^{'}(x^*))$, which means that: $Z_k^{'}(x) \leq Z_k^{'}(x^*)$ for $k = 1,\ldots,p$ and $Z_{k_0}^{'}(x) < Z_{k_0}^{'}(x^*)$ for some $1 \leq k_0 \leq p$. This implies that

$$\sum_{k=1}^{p}\sum_{j=1}^{m}[\lambda_k.c_{kj}^L, \lambda_k.c_{kj}^R].x_j \leq \sum_{k=1}^{p}\sum_{j=1}^{m}[\lambda_k.c_{kj}^L, \lambda_k.c_{kj}^R].x_j^*$$

which is a contradiction to the efficiency of x^* for problem (11) according to the interval order relation. More details about this solution approach are available in [23].

To test the effectiveness of our approach in finding robust efficient solutions, and to compare it with the previous *IBA* method, some performance metrics are used. Performance metrics are a well-known tool to validate the reliability of a solution approach and to evaluate the accuracy of the solutions obtained. According to [1], in order to measure the quality of the solutions obtained, comparison metrics are needed and three aspects are measured: the distribution and position of the solutions with respect to an ideal point, the closeness of the solutions obtained to a theoretical set of solutions, and the number of solutions obtained.

The first performance metric is the mean ideal distance *Mid*. This calculates the mean distance between the set of efficient solutions and the ideal point, and can be defined as follows:

$$Mid = \sum_{i=1}^{n}\frac{\|f_i - f_{ideal}\|}{n},$$

where $f_{ideal} = (\min f_1(x), \min f_2(x))$, f_i are the objective function values for each approach, and n is the number of robust efficient solutions obtained. Note that the approach with a lower value of *Mid* is more efficient [2].

The second metric is the inverted generational distance *IGD* [30], which is the average distance from each reference point to the nearest obtained solution. When the value of the *IGD* is small, the good convergence of solutions and their good distribution can be deduced. This metric can be calculated using the following expression:

$$IGD = \frac{1}{n}.\sum_{p\in P} d(p,X),$$

where P is the set of efficient solutions of the deterministic problem, X is the set of robust efficient solutions obtained by the approach, and $d(p,X)$ is the minimum distance between the point p and the set X. The method is more efficient if the *IGD* value is lower.

5.2 Computational results

For the deterministic multi-objective linear programming problem, the efficient solutions obtained are $(64,0,18,0,2)$, $(66,0,18,0,0)$ and $(66,1,17,0,0)$ with the objective function values $(22000,11920)$, $(21900,12000)$ and $(21800,12100)$, respectively. These efficient solutions are feasible for the deterministic problem with nominal data. However, they are not feasible for the problem with uncertain data, which means that they are not robust efficient. The following Table 3 provides the objective function values of the robust efficient solutions found using our approach and the IBA method, in addition to the performance metrics of each one.

Table 3: Objective funtions values of robust efficient solutions and performance metrics.

IBA method			Our approach		
(f_1, f_2)	*Mid*	*IGD*	(f_1, f_2)	*Mid*	*IGD*
$(35450, 18690)$	15237	15109	$(37850, 16600); (31150, 17400); (30500, 18400)$	12801	10706

For the robust multi-objective linear programming problem with uncertainties in both the objective function coefficients and the constraint parameters, the robust efficient solutions obtained are with the objective function values $(37850, 16600)$, $(31150, 17400)$ and $(30500, 18400)$, while the IBA approach provides one robust efficient solution with an objective function value of up to $(35450, 18690)$. Note that these robust efficient solutions obtained by both methods are feasible for each potential value of the uncertain data. However, we can note that the set of solutions found using the method proposed in this paper contains more robust efficient solutions than the set using the IBA method. It is interesting for decision-makers if a method comes up with a finite set of varied solutions from which they can choose an appropriate compromise solution, rather than a single solution.

We can conclude that, according to the results obtained from our approach and the performance metrics, the robust efficient solutions obtained for the multi-objective linear programming problem with uncertainties are various, well positioned in relation to the ideal points and close to the set of efficient solutions of the deterministic problem of the numerical example.

6 Conclusions

In this paper, a new robust optimization approach is proposed for solving multi-objective linear programming problems with uncertainties in both the objective function coefficients and the constraint parameters. The robust multi-objective problem is converted to an alternative deterministic multi-objective problem, and then a scalarization approach is applied to obtain an efficient solution that is robust against data uncertainties. The necessary properties and proofs, as well as a numerical example, are provided to explain and illustrate the effectiveness of the proposed approach. A comparison to an interval-based method from the literature is discussed in order to illustrate the principle of each and to introduce some performance metrics.

Proposing new approaches and addressing large-scale problems with uncertain data and real-life case studies could be the aim of further research.

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