Journal of Mathematical Modeling
Vol. 11, No. 3, 2023, pp. 587–602. Research Article

JMM

Numerical solution of system of nonlinear Fredholm integro-differential equations using CAS wavelets

Mostafa Akrami Arani, Mehdi Sabzevari*

Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan, Iran
Email(s): akrami57@grad.kashanu.ac.ir, sabzevari@kashanu.ac.ir

Abstract. In this paper, we use the CAS wavelets as basis functions to numerically solve a system of nonlinear Fredholm integro-differential equations. To simplify the problem, we transform the system into a system of algebraic equations using the collocation method and operational matrices. We show the convergence of the presented method and then demonstrate its high accuracy with several illustrative examples. This approach is particularly effective for equations that admit periodic functions because the employed basis CAS functions are inherently periodic. Throughout our numerical examples, we observe that this method provides exact solutions for equations with trigonometric functions at a lower computational cost when compared to other methods.

Keywords: Integro-differential equations, CAS wavelets, collocation method, operational matrices.
AMS Subject Classification 2010: 45J05, 47G20, 65D15, 65T60.

1 Introduction

Integral and differential equations have been one of the principal tools in different areas of applied mathematics. Actually, many problems in various sciences, including physics, control, economics, electrical engineering, mechanics, medicine, etc., can be modeled as initial value or boundary value problems, that can be converted into integral, differential or specially integro-differential equations [1]. On the other hand, it is very difficult or impossible to obtain the analytical solution of such systems in many cases. Therefore, these class of equations are tacked numerically.

In recent years, various types of numerical methods have been presented for solving these equations, for examples [5,10,14–16] and etc. A large class of these research are based on the use of basis functions, especially orthogonal basis functions. The most important advantage of this method is converting the

*Corresponding author
Received: 18 April 2023 / Revised: 11 June 2023 / Accepted: 1 July 2023
DOI: 10.22124/JMM.2023.24341.2179

integral or integro-differential equations into a system of algebraic equations, which greatly simplifies the problem.

Wavelets are an important class of orthogonal basis functions, which have been used since 1991 for numerical solution of such equations [12]. Wavelets are a set of linearly independent functions created by translation and dilation of a single scale function called the mother wavelet. This translation and dilation lead to a very useful property in approximating functions, and hence they can accurately represent the behavior of functions and operators. Therefore, using wavelets for the numerical solution of discussed equations leads to more efficient algorithms than the other methods.

In recent years, different types of wavelets have been used for the numerical solution of integral, differential and integro-differential equations, such as Haar Wavelets [3, 12], Legendre wavelets [11], CAS wavelets [6, 7, 17], Daubechies wavelets [18] and Chebyshev wavelets [2].

In this paper, we consider the system of nonlinear integro-differential equations of the second kind as:

$$\begin{cases} \sum_{j=1}^{l} \lambda_{ij} u_j(s) + \sum_{j=1}^{l} \mu_{ij} u'_j(s) = f_i(s) + \sum_{j=1}^{l} \int_0^1 k_{ij}(s,y)\phi_{ij}(y,u_j(y))\mathrm{d}y, \quad i = 1,2,\dots,l, \\ u_j(0) = u_{j0}, \quad j = 1,2,\dots,l, \end{cases} \tag{1}$$

where the functions $f_i(s)$, $k_{ij}(s,y)$ and $\phi_{ij}(y,u_j(y))$ are known and $u_j(s)$ are unknown for $i, j = 1,2,\dots,l$. We use the CAS wavelets to solve the system of nonlinear integro-differential equations (1). For this purpose, the collocation method as well as operational matrices are used to reduce the integro-differential equations (1) into a system of algebraic equations. This approach significantly simplifies the problem. We also prove the convergence of the presented method and demonstrate the high accuracy of the approach with various numerical examples and compare the results with the other methods.

An important feature of the present method is the periodicity of the basis functions. Therefore, it is reasonable that the method has high efficiency in equations of periodic functions, specially trigonometric functions. It can be seen in Section 7 that our approach leads to the exact solution for the equations with trigonometric functions, while this solution is obtained with less computational cost compared to the other methods. This is the outstanding conclusion of this article. Instead, as can be seen in Example 5 of Section 7, this method is not suitable for equations with non-periodic functions and leads to a solution with low accuracy.

2 Preliminaries of CAS wavelets

Wavelets are a family of linearly independent functions created by translation and dilation of a single scale function $\psi(.)$, called the mother wavelet. When the dilation parameter a and the translation parameter b are continuously varied, a family of continuous wavelets is obtained in the following form:

$$\psi_{a,b}(s) = |a|^{-\frac{1}{2}} \psi(\frac{s-b}{a}), \quad a,b \in \mathbb{R}, \quad a \neq 0.$$

If we restrict the parameters a and b to discrete values as:

$$a = a_0^{-j}, \quad b = kb_0 a_0^{-j},$$

where $a_0 > 1,\ b_0 > 0$ and $j,k \in \mathbb{N}$, the family of discrete wavelets is obtained in the following form:

$$\psi_{j,k}(s) = |a_0|^{\frac{j}{2}} \psi(a_0^j s - kb_0).$$

Here, $\{\psi_{j,k}(s)\}$ is an orthogonal wavelet basis for $L^2(\mathbb{R})$. In particular, if $a_0 = 2$ and $b_0 = 1$, then $\{\psi_{j,k}(s)\}$ will be an orthonormal wavelet basis for $L^2(\mathbb{R})$ [9].

Definition 1. *The set of* CAS *wavelets on the interval* $[0,1)$*, denoted by* $\psi_{nm}(s) = \psi(k,n,m,s)$*, where* k *is an nonnegative integer,* $n = 1,2,\dots,2^k$*,* $m \in \mathbb{Z}$ *and* $s \in [0,1)$*, are defined as follows:*

$$\psi_{nm}(s) = \begin{cases} 2^{\frac{k}{2}}\mathrm{CAS}_m(2^k s - n + 1), & \frac{n-1}{2^k} \le s < \frac{n}{2^k}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathrm{CAS}_m(s) = \cos(2m\pi s) + \sin(2m\pi s)$.

The set of CAS wavelets forms an orthonormal basis in $L^2[0,1)$. For example, the set of CAS wavelets for $k = 1$ and $m = -1,\ 0,\ 1$ on the interval $[0,1)$ are given by:

$$\begin{cases} \psi_{1,-1}(s) &= \sqrt{2}(\cos(4\pi s) - \sin(4\pi s)), \\ \psi_{1,0}(s) &= \sqrt{2}, \\ \psi_{1,1}(s) &= \sqrt{2}(\cos(4\pi s) + \sin(4\pi s)), \end{cases} \quad \text{if } 0 \le s < \frac{1}{2},$$

$$\begin{cases} \psi_{2,-1}(s) &= \sqrt{2}(\cos(4\pi s) - \sin(4\pi s)), \\ \psi_{2,0}(s) &= \sqrt{2}, \\ \psi_{2,1}(s) &= \sqrt{2}(\cos(4\pi s) + \sin(4\pi s)), \end{cases} \quad \text{if } \frac{1}{2} \le s < 1.$$

3 Functions approximation

Any function $u(s)$ which is square integrable in the interval $[0,1)$ can be expanded by CAS wavelets as:

$$u(s) = \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(s), \tag{2}$$

where $c_{nm} = \langle u(s), \psi_{nm}(s)\rangle$, in which $\langle \cdot,\cdot\rangle$ is the inner product in $L^2[0,1)$ that is defined as follows:

$$c_{nm} = \langle u(s), \psi_{nm}(s)\rangle = \int_0^1 u(s)\psi_{nm}(s)\mathrm{d}s.$$

The series expansion appearing in Eq. (2) usually contains infinite number of terms for a smooth function $u(s)$. If this infinite series is truncated, an approximation of the function $u(s)$ is obtained as follows:

$$u(s) \approx \sum_{n=1}^{2^k} \sum_{m=-M}^{M} c_{nm}\psi_{nm}(s) = C^T\Psi(s), \tag{3}$$

where C and Ψ are $2^k(2M+1) \times 1$ vectors given by:

$$C = [c_{1(-M)}, c_{1(-M+1)}, \dots,\ c_{1(M)}, c_{2(-M)}, \dots, c_{2(M)}, \dots, c_{2^k(-M)}, \dots, c_{2^k(M)}]^T, \tag{4}$$

$$\begin{aligned} \Psi(s) = [&\psi_{1(-M)}(s), \psi_{1(-M+1)}(s), \dots,\ \psi_{1(M)}(s), \psi_{2(-M)}(s), \dots, \psi_{2(M)}(s), \\ &\dots, \psi_{2^k(-M)}(s), \dots, \psi_{2^k(M)}(s)]^T. \end{aligned} \tag{5}$$

Similarly, the bivariate function $k(s,y) \in L^2([0,1)\times[0,1))$ can be approximated based on CAS wavelets as

$$k(s,y) \approx \Psi^T(s) K \Psi(y), \tag{6}$$

where K is a $2^k(2M+1)\times 2^k(2M+1)$ matrix and its elements are calculated for $i,j = 1,2,\ldots,2^k(2M+1)$ as below

$$K_{ij} = \langle \Psi_i(s), \langle k(s,y), \Psi_j(y)\rangle\rangle = \int_0^1\int_0^1 \Psi_i(s) k(s,y) \Psi_j(y) \mathrm{d}y\mathrm{d}s.$$

Here, $\Psi_i(s)$ and $\Psi_j(y)$ are the i-th and j-th elements of $\Psi(s)$ and $\Psi(y)$ vectors, respectively.

4 CAS wavelet operational matrix of integration

The CAS wavelets operational matrix of integration is an important tool in converting the system of nonlinear Fredholm integro-differential equations (1) into the system of algebraic equations. Consider the $2^k(2M+1)\times 1$ vector $\Psi(s)$ defined in (5), which its elements are CAS wavelets. The integration of the vector is given by

$$\int_0^s \Psi(y)\mathrm{d}y = P\Psi(s), \tag{7}$$

where P is a square matrix of order $2^k(2M+1)$ and we call it operational matrix of integration for CAS wavelets.

Now, in order to calculate the matrix P, we consider the case $k=0$ and $M=1$. In this case we have

$$\begin{cases} \psi_{1,-1}(s) &= \cos(2\pi s) - \sin(2\pi s), \\ \psi_{1,0}(s) &= 1, \\ \psi_{1,1}(s) &= \cos(2\pi s) + \sin(2\pi s), \end{cases} \quad \text{if } 0 \le s < 1.$$

Hence

$$\begin{aligned} \int_0^s \Psi(y)\mathrm{d}y &= \int_0^s \left[\psi_{1,-1}(y),\ \psi_{1,0}(y),\ \psi_{1,1}(y)\right]^T \mathrm{d}y \\ &= \frac{1}{2\pi}\left[\psi_{1,1}(s) - \psi_{1,0}(s),\ 2\pi s,\ \psi_{1,0}(s) - \psi_{1,-1}(s)\right]^T. \end{aligned} \tag{8}$$

Expanding the second element of the above vector in terms of the CAS basis functions, we have

$$2\pi s = a_{1,-1}\psi_{1,-1}(s) + a_{1,0}\psi_{1,0}(s) + a_{1,1}\psi_{1,1}(s), \tag{9}$$

where, $a_{1,m} = \langle 2\pi s, \psi_{1,m}(s)\rangle_{L^2[0,1)}$, $m = -1,0,1$. Substituting the above equations in (8), we get

$$\int_0^s \Psi(y)\mathrm{d}y = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ -\frac{1}{\pi} & \frac{1}{\pi} & 0 \end{pmatrix}\begin{pmatrix} \psi_{1,-1}(s) \\ \psi_{1,0}(s) \\ \psi_{1,1}(s) \end{pmatrix}.$$

Therefore, the CAS wavelet operational matrix of integration in the case $k=0$ and $M=1$ is a square matrix of order 3 as follows:

$$P_{3\times 3}=\frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{\pi} & \frac{1}{\pi} \\ \frac{1}{\pi} & 1 & -\frac{1}{\pi} \\ -\frac{1}{\pi} & \frac{1}{\pi} & 0 \end{pmatrix}.$$

In general, for arbitrary parameters k and M, it is proved by induction that the operational matrix of integration for CAS wavelets is a square matrix of order $2^k(2M+1)$ as follows

$$P=\frac{1}{2^{k+1}}\begin{pmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & F \\ O & O & L & \cdots & F \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & \cdots & \cdots & O & L \end{pmatrix},$$

where L and F are the square matrices of order $2M+1$ given by

$$L=\begin{pmatrix} 0 & 0 & \cdots & 0 & -\frac{1}{M\pi} & 0 & \cdots & 0 & \frac{1}{M\pi} \\ 0 & 0 & \cdots & 0 & -\frac{1}{(M-1)\pi} & 0 & \cdots & \frac{1}{(M-1)\pi} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\frac{1}{\pi} & \frac{1}{\pi} & \cdots & 0 & 0 \\ \frac{1}{\pi} & \frac{1}{\pi} & \cdots & \frac{1}{\pi} & 1 & -\frac{1}{\pi} & \cdots & -\frac{1}{\pi} & -\frac{1}{\pi} \\ 0 & 0 & \cdots & -\frac{1}{\pi} & \frac{1}{\pi} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -\frac{1}{(M-1)\pi} & \cdots & 0 & \frac{1}{(M-1)\pi} & 0 & \cdots & 0 & 0 \\ -\frac{1}{M\pi} & 0 & \cdots & 0 & \frac{1}{M\pi} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$F=\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 2 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

and O is a zero matrix of order $(2M+1)\times(2M+1)$.

Also, the integration of the product of two CAS wavelet function vectors is obtained as:

$$\int_0^1 \Psi(s)\Psi^T(s)\mathrm{d}s=I, \tag{10}$$

where I is an identity matrix of order $2^k(2M+1)\times 2^k(2M+1)$.

5 Implementation of the method

In this section, according to the properties and relations stated in the previous sections for the CAS wavelets and operational matrices, we deal with the numerical solution of the system of nonlinear Fredholm integro-differential equations (1).

At first, suppose that

$$\phi_{ij}(y,u_j(y)) = \rho_{ij}(y), \quad 0 \le y < 1. \tag{11}$$

Now, considering Eqs. (3) and (6), each of the functions appearing in the system of nonlinear Fredholm integro-differential equations (1) can be approximated as follows:

$$\begin{aligned} u_j(s) &\approx \Psi^T(s)U_j, \\ u'_j(s) &\approx \Psi^T(s)U'_j, \\ f_i(s) &\approx \Psi^T(s)F_i, \\ k_{ij}(s,y) &\approx \Psi^T(s)K_{ij}\Psi(y), \\ \phi_{ij}(y,u_j(y)) = \rho_{ij}(y) &\approx \Psi^T(y)\Phi_{ij}, \end{aligned} \tag{12}$$

where U_j, U'_j and Φ_{ij} are $2^k(2M+1)\times 1$ vectors with unknown elements, F_i are $2^k(2M+1)\times 1$ vectors and K_{ij} are $2^k(2M+1)\times 2^k(2M+1)$ matrices with known elements for $i,j = 1,2,\ldots,l$.

Also, $u_j(0)$ can be considered as constant functions which are approximated by CAS wavelets as follows:

$$u_j(0) \approx \Psi^T(s)U_{j0}, \qquad j = 1,2,\ldots,l, \tag{13}$$

where U_{j0} are $2^k(2M+1)\times 1$ vectors with known elements and can be obtained as the following:

$$U_{j0} = [\underbrace{\overbrace{0,\ldots,\underbrace{v_{1j}(0)}_{(M+1)-th},\ldots,0}^{2M+1},\overbrace{0,\ldots,\underbrace{v_{2j}(0)}_{(M+1)-th},\ldots,0}^{2M+1},\ldots,\overbrace{0,\ldots,\underbrace{v_{2^kj}(0)}_{(M+1)-th},\ldots,0}^{2M+1}}_{2^k(2M+1)}]^T,$$

where $v_{rj}(0)$ are calculated as follows:

$$v_{rj}(0) = \int_0^1 u_j(0)\psi_{r0}(s)\mathrm{d}s, \qquad r = 1,2,\ldots,2^k.$$

In order to reduce the number of unknowns in Eq. (12), we use the initial conditions of the system of nonlinear Fredholm integro-differential equations (1). Actually, the CAS wavelet operational matrix of integration (7) leads to the following result

$$u_j(s) - u_j(0) = \int_0^s u'_j(y)\mathrm{d}y \approx \int_0^s U_j'^T\Psi(y)\mathrm{d}y = U_j'^T P\Psi(s) = \Psi^T(s)P^TU'_j.$$

On the other hand, according to the Eqs. (12) and (13), we have

$$u_j(s) - u_j(0) \approx \Psi^T(s)U_j - \Psi^T(s)U_{j0}.$$

Therefore, comparing the two previous equations, the following result is obtained:

$$U_j - U_{j0} = P^TU'_j. \tag{14}$$

Now, by substituting each of approximate functions (12) into the system of integro-differential equations (1), we have

$$\sum_{j=1}^{l} \lambda_{ij} \Psi^T(s) U_j + \sum_{j=1}^{l} \mu_{ij} \Psi^T(s) U_j' = \Psi^T(s) F_i + \sum_{j=1}^{l} \int_0^1 \Psi^T(s) K_{ij} \Psi(y) \Psi^T(y) \Phi_{ij} \mathrm{d}y, \quad i = 1,2,\ldots,l.$$

Using Eq. (10) it becomes

$$\Psi^T(s) \sum_{j=1}^{l} \lambda_{ij} U_j + \Psi^T(s) \sum_{j=1}^{l} \mu_{ij} U_j' = \Psi^T(s) F_i + \Psi^T(s) \sum_{j=1}^{l} K_{ij} \Phi_{ij}, \quad i = 1,2,\ldots,l.$$

Therefore

$$\sum_{j=1}^{l} \lambda_{ij} U_j + \sum_{j=1}^{l} \mu_{ij} U_j' = F_i + \sum_{j=1}^{l} K_{ij} \Phi_{ij}, \quad i = 1,2,\ldots,l.$$

If we multiply both sides of the above equation by P^T from the left and use Eq. (14), the following system of equations is obtained:

$$\sum_{j=1}^{l} \lambda_{ij} P^T U_j + \sum_{j=1}^{l} \mu_{ij} (U_j - U_{j0}) = P^T F_i + \sum_{j=1}^{l} P^T K_{ij} \Phi_{ij}, \tag{15}$$

which $i = 1,2,\ldots,l$. The above equation is a system of $2^k(2M+1)l$ equations and $2^k(2M+1)(l^2+l)$ unknowns, including the elements of U_j and Φ_{ij} vectors. So we need to find $2^k(2M+1)l^2$ more equations to reach a system of equations with a unique solution. For this purpose, by substituting $2^k(2M+1)$ collocation points

$$s_t = \frac{t-0.5}{2^k(2M+1)}, \quad t = 1,2,\ldots,2^k(2M+1),$$

in Eq. (11), the following $2^k(2M+1)l^2$ equations are gained:

$$\phi_{ij}(s_t, U_j^T \Psi(s_t)) = \Phi_{ij}^T \Psi(s_t), \quad i,j = 1,2,\ldots,l. \tag{16}$$

Finally, combining Eqs. (15) and (16), a system of nonlinear algebraic equations with $2^k(2M+1)(l^2+l)$ equations and the same number of unknowns is obtained as follows:

$$\begin{cases} \sum_{k=1}^{l} \lambda_{ik} P^T U_k + \sum_{k=1}^{l} \mu_{ik} (U_k - U_{k0}) = P^T F_i + \sum_{k=1}^{l} P^T K_{ik} \Phi_{ik}, \\ \phi_{ij}(s_t, U_j^T \Psi(s_t)) = \Phi_{ij}^T \Psi(s_t), \end{cases} \tag{17}$$

where $i,j = 1,2,\ldots,l$. By solving the above system of equations, the unknown U_j vectors are calculated and consequently the solution of the system of integral equations (1) will be obtained by $u_j(s) = \Psi^T(s) U_j$, for $j = 1,2,\ldots,l$.

6 Convergence and error analysis

We dedicate this section to convergence and error analysis of the presented method. First, let us state the following theorem on the existence and uniqueness of the solution for a general system of Fredholm integral equations [8].

Theorem 1. *Consider the following system of Fredholm integral equations*

$$\begin{cases} x_1(t) = g_1(t) + \int_a^b K_1(x,s,x_1(s))\mathrm{d}s + \displaystyle\int_a^b H_1(x,s,x_2(s))\mathrm{d}s, \\ x_2(t) = g_2(t) + \int_a^b K_2(x,s,x_2(s))\mathrm{d}s + \displaystyle\int_a^b H_2(x,s,x_1(s))\mathrm{d}s. \end{cases}$$

The above system has a unique solution $x^* = (x_1^*, x_2^*) \in (C[a,b])^2$, if the following two conditions hold

(i) *there exist $L_{K_j}, L_{H_j} > 0$, $j \in \{1,2\}$ such that*

$$|K_j(t,s,u) - K_j(t,s,v)| \le L_{K_j}|u-v|,$$
$$|H_j(t,s,u) - H_j(t,s,v)| \le L_{H_j}|u-v|.$$

(ii) $\frac{b-a}{2}\left[(L_{K_1}+L_{K_2}) \pm \sqrt{(L_{K_1}-L_{K_2})^2 + 4L_{H_1}L_{H_2}}\right] \in (-1,1)$.

Now, suppose $u(s) \in L^2([0,1))$ is a continuous and bounded function on the interval $[0,1)$ and $\tilde{u}$ is an approximation of u from Eq. (3) as

$$\tilde{u}(s) = \sum_{i=1}^{2^k(2M+1)} c_i \psi_i(s), \tag{18}$$

where

$$c_i = \int_0^1 u(s)\psi_i(s)\mathrm{d}s, \qquad i = 1,2,\dots 2^k(2M+1). \tag{19}$$

Here, we want to consider the behavior of $\|u(s) - \tilde{u}(s)\|_2$, when $k \to \infty$.

Theorem 2. *Suppose $u(s) \in L^2([0,1))$ is a continuous and bounded function on the interval $[0,1)$, which for every $s \in [0,1)$, $|u(s)| \le R$, where R is a constant. Also, let $\tilde{u}(s)$ be the approximation of $u(s)$ by CAS wavelets in form of (18). Then*

$$\|E(s)\|_2^2 = \|u(s) - \tilde{u}(s)\|_2^2 \le \sum_{i=2^k+1}^{\infty} 2^{-k+1}R^2, \tag{20}$$

and therefore $\tilde{u}$ converges to u when $k \to \infty$.

Proof. Using Eqs. (2), (18) and according to orthonormality of CAS wavelets we get

$$\begin{aligned} \|E(s)\|_2^2 = \|u(s) - \tilde{u}(s)\|_2^2 &= \int_0^1 (u(s) - \tilde{u}(s))^2 \mathrm{d}s \\ &= \sum_{i_1=2^k(2M+1)+1}^{\infty} \sum_{i_2=2^k(2M+1)+1}^{\infty} c_{i_1}c_{i_2} \int_0^1 \psi_{i_1}(s)\psi_{i_2}(s)\mathrm{d}s \\ &= \sum_{i=2^k(2M+1)+1}^{\infty} c_i^2. \end{aligned} \tag{21}$$

From Definition 1 and Eq. (19) we have

$$c_i = \int_{\frac{n-1}{2^k}}^{\frac{n}{2^k}} 2^{\frac{k}{2}} u(s)\mathrm{CAS}_m(2^k s - n + 1)\mathrm{d}s.$$

Now by using mean value theorem for the above integral, we can find $s_1 \in [(n-1)2^{-k}, n2^{-k})$ such that

$$c_i = 2^{\frac{k}{2}} \left(n2^{-k} - (n-1)2^{-k}\right) u(s_1)\mathrm{CAS}_m(2^k s_1 - n + 1) = 2^{\frac{-k}{2}} u(s_1)\mathrm{CAS}_m(2^k s_1 - n + 1). \tag{22}$$

Since $|u(s)| \leq R$, we have

$$c_i^2 = 2^{-k}u^2(s_1)\mathrm{CAS}_m^2(2^k s_1 - n + 1) = 2^{-k}u^2(s_1)\left(1 + \sin\left(2(2^k s_1 - n + 1)\right)\right) \leq 2^{-k+1}R^2. \tag{23}$$

Substituting Eq. (23) into Eq. (21), we obtain

$$\begin{aligned} \|E(s)\|_2^2 &= \|u(s) - \tilde{u}(s)\|_2^2 \\ &\leq \sum_{i=2^k(2M+1)+1}^{\infty} 2^{-k+1}R^2 \leq \sum_{i=2^k+1}^{\infty} 2^{-k+1}R^2 \\ &= 2R^2 \lim_{N\to\infty} \sum_{i=2^k+1}^{N} 2^{-k} = 2R^2 \lim_{N\to\infty} (N2^{-k} - 1). \end{aligned} \tag{24}$$

Since $N2^{-k} - 1 < N2^{-k}$ and $\lim_{N,k\to\infty}(N2^{-k}) = 0$, from Eq. (24) we get $\lim\limits_{k\to\infty} \|E(s)\|_2^2 = 0$, i.e., $\tilde{u}$ converges to u when $k \to \infty$. □

7 Numerical results

In this section, we implement our method on five different examples. In each example, we use the method for appropriate values of k and M and present the corresponding values of absolute errors in the relevant tables. In Examples 1 and 2, we compare our results with the results obtained from other methods. It may be worth to notice that the functions in the Examples 1-4 are all periodic, in view of the fact the basis functions are inherently periodic. The outputs of these examples show the high accuracy of the method in particular in the case of integro-differential equations of periodic functions. It even results surprisingly to exact solutions in many cases with less computational cost compared to the other methods. Instead, the functions used in Example 5 are not periodic and therefore the results of the presented method are not accurate enough. It is noteworthy that all computations are done by *Maple* 17.

Example 1. Consider the system of Fredholm integro-differential equations as follows [13]

$$\begin{cases} 2\pi u_1(s) = 2\pi\cos(2\pi s)(1 + \sin(2\pi s)) \\ \qquad\qquad - 2\pi \displaystyle\int_0^1 \sin(4\pi s)\cos(2\pi y)u_2(y)\mathrm{d}y - 2\pi \int_0^1 \sin(4\pi s + 2\pi y)u_1(y)\mathrm{d}y, \\ u_2'(s) - 2\pi u_1(s) = -\frac{1}{2}\sin(4\pi s) \\ \qquad\qquad - \displaystyle\int_0^1 \cos(4\pi s)\sin(2\pi y)u_1(y)\mathrm{d}y - \int_0^1 \cos(4\pi s + 2\pi y)u_2(y)\mathrm{d}y, \\ u_1(0) = 1, \quad u_2(0) = 0, \end{cases} \tag{25}$$

where the exact solutions are $u_1(s) = \cos(2\pi s)$ and $u_2(s) = \sin(2\pi s)$.

We use the CAS wavelet method for $k = 0$ and $M = 1$, to solve the system of integro-differential Eqs. (25). Therefore, the F_1, F_2, U_{10} and U_{20} vectors are obtained as follows:

$$F_1 = \begin{pmatrix} \pi \\ 0 \\ \pi \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad U_{10} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad U_{20} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Also, the K_{11}, K_{12}, K_{21} and K_{22} are the 3×3 zero matrices.

Substituting the above vectors and matrices in the system of nonlinear algebraic equations (17), the unknown vectors U_1 and U_2 are obtained as follows:

$$U_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \qquad U_2 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}.$$

Finally, the approximate solutions of Example 1 are obtained by Eq. (12) as $u_1(s) = \cos(2\pi s)$, $u_2(s) = \sin(2\pi s)$, which are the exact solutions.

The system of Fredholm integro-differential equations (25) has already been solved using Haar wavelets [13]. The results obtained by our method for $k = 0$ and $M = 1$ have been compared with the results obtained by Haar wavelet method for $k = 128$ and this comparison has been reported in Table 1. As is visible from this table, the present method leads to an exact solution with very low computational cost that includes solving a system of algebraic equations with 3 equations and 3 unknowns. However, in the method presented in [13], after solving an algebraic system with 128 equations and the same number of unknowns, an error of order 10^{-2} has been obtained.

Table 1: The corresponding values of absolute errors of $(u_1(s), u_2(s))$ for Example 1.

s	Presented method for $k = 0$ and $M = 1$	Method in [13] for $k = 128$
0.1	$(0,0)$	$(8\times10^{-3}, 1\times10^{-2})$
0.2	$(0,0)$	$(5\times10^{-3}, 2\times10^{-3})$
0.3	$(0,0)$	$(5\times10^{-3}, 2\times10^{-3})$
0.4	$(0,0)$	$(9\times10^{-3}, 1\times10^{-2})$
0.5	$(0,0)$	$(3\times10^{-4}, 2\times10^{-2})$
0.6	$(0,0)$	$(9\times10^{-3}, 1\times10^{-2})$
0.7	$(0,0)$	$(5\times10^{-3}, 2\times10^{-3})$
0.8	$(0,0)$	$(5\times10^{-3}, 2\times10^{-3})$
0.9	$(0,0)$	$(8\times10^{-3}, 1\times10^{-2})$

Example 2. Consider the following initial value Fredholm integro-differential equation [4]:

$$\begin{cases} u'(s) = -2\pi\sin(2\pi s) - \frac{1}{2}\sin(4\pi s) + \int_0^1 \sin(4\pi s + 2\pi y)u(y)\mathrm{d}y, \\ u(0) = 1, \end{cases} \tag{26}$$

where the exact solution is $u(s) = \cos(2\pi s)$.

Using the proposed method for $k = 0$ and $M = 1$, the F and U_0 vectors are computed as:

$$F = \begin{pmatrix} \pi \\ 0 \\ -\pi \end{pmatrix}, \qquad U_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Also, the matrix K is

$$K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Substituting the above vectors and matrix in the system of nonlinear algebraic equations (17), the unknown vector U is obtained as follows:

$$U = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix},$$

and consequently by Eq. (12), $u(s) = \cos(2\pi s)$, which is the exact solution of the Fredholm integro-differential equation (26).

The absolute values of error obtained by CAS wavelets for $k = 0$ and $M = 1$ have been compared with the results obtained by an iterative scheme [4] for $J = 33$ and reported in Table 2. As is visible from this table, the present method has reached much better results than [4] with less computational cost.

Table 2: The corresponding values of absolute errors of $u(s)$ for Example 2.

s	Presented method for $k = 0$ and $M = 1$	Method in [4] for $J = 33$
0.1	0	4.93×10^{-4}
0.2	0	8.90×10^{-4}
0.3	0	4.00×10^{-3}
0.4	0	5.70×10^{-3}
0.5	0	6.40×10^{-3}
0.6	0	3.72×10^{-3}
0.7	0	7.78×10^{-4}
0.8	0	8.96×10^{-4}
0.9	0	9.40×10^{-4}

Example 3. Consider the system of nonlinear Fredholm integro-differential equations as follows

$$\begin{cases} u_1(s)+4\pi u_2(s)-u_1'(s)=\sin(2\pi s)+2\pi\cos(2\pi s)+\frac{1}{2}\cos^2(\pi s)-\frac{1}{4} \\ \qquad\qquad +\int_0^1 \cos(2\pi s+4\pi y)u_1^2(y)\mathrm{d}y+\int_0^1 \sin(4\pi s+2\pi y)u_2^2(y)\mathrm{d}y, \\ 2\pi u_1(s)-u_2'(s)=4\pi\sin(2\pi s) \\ \qquad\qquad +\int_0^1 \cos(2\pi y)u_1^2(y)\mathrm{d}y+\int_0^1 \sin(4\pi s)\cos(2\pi y)u_2^2(y)\mathrm{d}y, \\ u_1(0)=0, \qquad u_2(0)=1, \end{cases} \tag{27}$$

where the exact solutions are $u_1(s)=\sin(2\pi s)$ and $u_2(s)=\cos(2\pi s)$.

To solve this example, using the method presented in this paper, we consider the case of $k=0$ and $M=2$, which leads to a system of algebraic equations including 5 equations and 5 unknowns in the form (17). In this case, F_1, F_2, U_{10} and U_{20} vectors and K_{11}, K_{12}, K_{21} and K_{22} matrices are computed as follows:

$$F_1=\begin{pmatrix}0\\-\frac{3}{8}+\pi\\0\\\frac{5}{8}+\pi\\0\end{pmatrix},\quad F_2=\begin{pmatrix}0\\-2\pi\\0\\2\pi\\0\end{pmatrix},\quad U_{10}=\begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix},\quad U_{20}=\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix},$$

$$K_{11}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&\frac{1}{2}\\0&0&0&0&0\\\frac{1}{2}&0&0&0&0\\0&0&0&0&0\end{pmatrix},\qquad K_{12}=\begin{pmatrix}0&-\frac{1}{2}&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&\frac{1}{2}&0\end{pmatrix},$$

$$K_{21}=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\qquad K_{22}=\begin{pmatrix}0&-\frac{1}{4}&0&-\frac{1}{4}&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&\frac{1}{4}&0&\frac{1}{4}&0\end{pmatrix}.$$

Substituting the above vectors and matrices in the system of nonlinear algebraic equations (17), the unknown vectors U_1 and U_2 are obtained as follows:

$$U_1=\begin{pmatrix}0\\-\frac{1}{2}\\0\\\frac{1}{2}\\0\end{pmatrix},\qquad U_2=\begin{pmatrix}0\\\frac{1}{2}\\0\\\frac{1}{2}\\0\end{pmatrix},$$

and consequently by Eq. (12), $u_1(s)=\sin(2\pi s)$, $u_2(s)=\cos(2\pi s)$, which are clearly the exact solutions of system (27).

Example 4. Consider the system of nonlinear Fredholm integro-differential equations as follows:

$$
\begin{cases}
u_1(s) + 4\pi u_2(s) - u_1'(s) = \sin(4\pi s) \\
\qquad\qquad\qquad + \int_0^1 \cos(2\pi s + 4\pi y) u_1^2(y) \mathrm{d}y + \int_0^1 \sin(4\pi s + 2\pi y) u_2^2(y) \mathrm{d}y, \\
2\pi u_1(s) - u_2'(s) = 6\pi \sin(4\pi s) \\
\qquad\qquad\qquad + \int_0^1 \cos(2\pi y) u_1^2(y) \mathrm{d}y + \int_0^1 \sin(4\pi s)\cos(2\pi y) u_2^2(y) \mathrm{d}y, \\
u_1(0) = 0, \qquad u_2(0) = 1,
\end{cases}
\tag{28}
$$

where the exact solutions are $u_1(s) = \sin(4\pi s)$ and $u_2(s) = \cos(4\pi s)$.

Here, we use the presented method for $k = 1$ and $M = 2$, which leads to a system in the form (17) with 10 equations and 10 unknowns. Then, F_1, F_2, U_{10} and U_{20} vectors are as follows:

$$
\begin{aligned}
F_1 &= \begin{bmatrix} 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 & 0 & -\frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 \end{bmatrix}^T, \\
F_2 &= \begin{bmatrix} 0 & -\frac{3\sqrt{2}}{2}\pi & 0 & \frac{3\sqrt{2}}{2}\pi & 0 & 0 & -\frac{3\sqrt{2}}{2}\pi & 0 & \frac{3\sqrt{2}}{2}\pi & 0 \end{bmatrix}^T, \\
U_{10} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
U_{20} &= \begin{bmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \end{bmatrix}^T.
\end{aligned}
$$

Also, K_{11}, K_{12}, K_{21} and K_{22} are 10×10 matrices.

By solving the system of nonlinear algebraic equations (17), which is obtained after replacing the above vectors and matrices, the approximate solutions $u_1(s)$ and $u_2(s)$ are obtained. Although the exact solution is not obtained in this example, the accuracy increases in the same proportion as the "Digits" parameter increases in Maple 17. The absolute error functions with $k = 1$ and $M = 2$ are plotted in Figures 1 and 2, when *Digits* $= 10$ and 30 respectively.

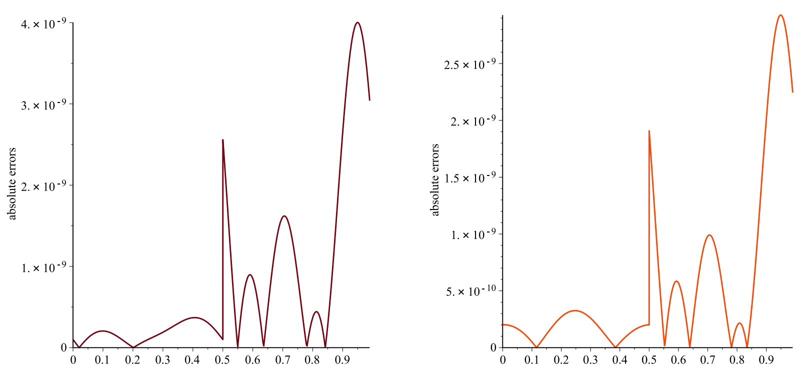

Figure 1: The absolute error functions of $u_1(s)$ (left figure) and $u_2(s)$ (right figure) for Example 4 with $k = 1$, $M = 2$ and Digits=10.

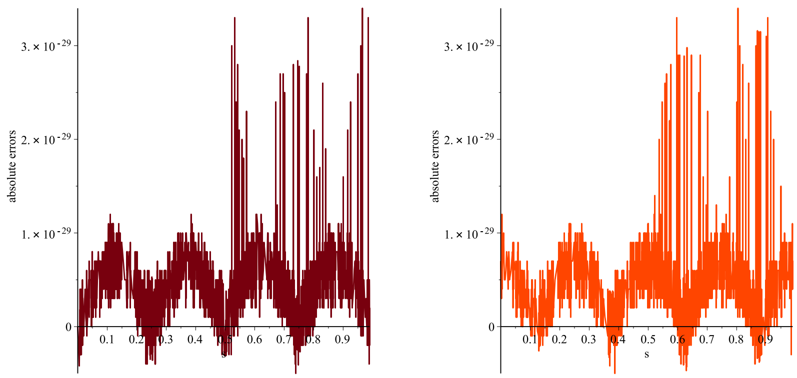

Figure 2: The absolute error functions of $u_1(s)$ (left figure) and $u_2(s)$ (right figure) for Example 4 with $k=1$, $M=2$ and Digits=30.

Example 5. Consider the following system of nonlinear Fredholm integro-differential equation [16]:

$$\begin{cases} u_1(s)+3u_2(s)-u_2'(s) = -\dfrac{1}{9}s(1+2e^3)+\displaystyle\int_0^1 e^{s-2y}u_1^2(y)\mathrm{d}y+\int_0^1 syu_2(y)\mathrm{d}y, \\ u_2(s)+2u_1'(s)+u_2'(s) = 4e^{3s}+\dfrac{15}{7}e^s-s-\dfrac{1}{7}e^{s+7}+\displaystyle\int_0^1 se^{-3y}u_1^3(y)\mathrm{d}y+\int_0^1 e^{s+y}u_2^2(y)\mathrm{d}y, \\ u_1(0)=u_2(0)=1, \end{cases} \tag{29}$$

where the exact solutions are $u_1(s)=e^s$ and $u_2(s)=e^{3s}$.

We use the CAS wavelet method for $k=2$ and $M=2$. The approximate solution obtained from the proposed method is displayed in Fig. 3 together with the exact solution. Also, our results have been compared with the results obtained by the hybrid of Legendre polynomials and Block-Pulse functions method [16] for $N=M=3$ and this comparison has been reported in Table 3. The results show that the presented method is not efficient in equations with non-periodic functions.

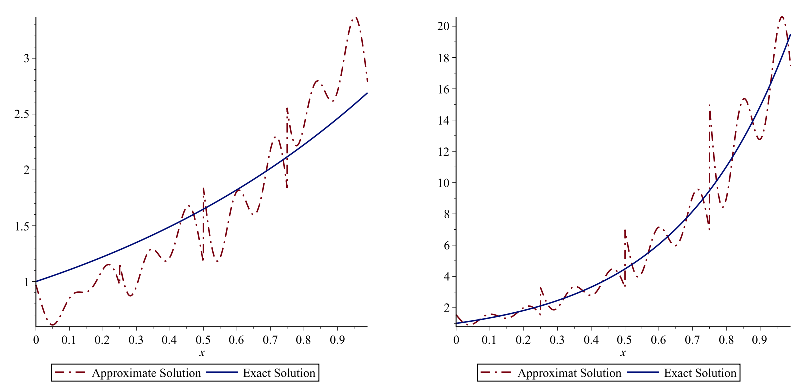

Figure 3: Comparison of exact solution and approximate solution of $u_1(s)$ (left figure) and $u_2(s)$ (right figure) for Example 5 with $k=2$ and $M=2$.

Table 3: The corresponding values of absolute errors of $(u_1(s), u_2(s))$ for Example 5.

s	Presented method for $k = M = 2$	Method in [16] for $M = N = 3$
0.1	$(2.93 \times 10^{-1}, 8.33 \times 10^{-2})$	$(5.28 \times 10^{-4}, 5.79 \times 10^{-3})$
0.2	$(2.43 \times 10^{-2}, 7.80 \times 10^{-3})$	$(5.61 \times 10^{-4}, 4.17 \times 10^{-3})$
0.3	$(3.18 \times 10^{-1}, 3.46 \times 10^{-2})$	$(1.08 \times 10^{-3}, 1.17 \times 10^{-3})$
0.4	$(1.99 \times 10^{-3}, 1.85 \times 10^{-1})$	$(1.97 \times 10^{-3}, 9.72 \times 10^{-3})$
0.5	$(4.13 \times 10^{-1}, 2.47 \times 10^{-1})$	$(2.82 \times 10^{-3}, 1.37 \times 10^{-3})$
0.6	$(6.05 \times 10^{-2}, 3.86 \times 10^{-1})$	$(3.87 \times 10^{-3}, 1.98 \times 10^{-2})$
0.7	$(3.43 \times 10^{-1}, 1.65 \times 10^{-1})$	$(6.12 \times 10^{-3}, 2.07 \times 10^{-2})$
0.8	$(3.01 \times 10^{-1}, 4.15 \times 10^{-1})$	$(7.73 \times 10^{-3}, 2.42 \times 10^{-2})$
0.9	$(6.85 \times 10^{-1}, 1.42 \times 10^{-1})$	$(8.97 \times 10^{-3}, 5.60 \times 10^{-2})$

8 Conclusions

In this article, CAS wavelets method is proposed to solve the system of nonlinear fredholm integro-differential equations. Using operational matrices of CAS wavelets as well as collocation method, the system of integro-differential equations is transformed into a system of algebraic equations, which greatly simplified the problem.

The high accuracy of the presented method was shown with different numerical Examples 1-4. In these examples, we used equations with periodic functions, specially trigonometric functions, in view of the fact the basis functions are inherently periodic. The outputs of these examples surprisingly led to exact solutions in many cases with less computational cost compared to the other methods. Therefore, using the proposed method in the numerical solution of system of nonlinear fredholm integro-differential equations with periodic functions is highly recommended. Instead, Example 5 showed that this method is not suitable for equations with non-periodic functions and led to a solution with low accuracy.

References

[1] J. Abdul, *Introduction to Integral Equations with Applications*, Wiley, New York, 1999.

[2] H. Adibi, P. Assari, *Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind*, Math. Prob. Engin. **2010** (2010) 1–17.

[3] I. Aziz, *New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets*, J. Comput. Appl. Math. **239** (2013) 333–345.

[4] M.I. Berenguer, D. Gámez, A.J. López Linares, *An iterative scheme for solving systems of nonlinear Fredholm integrodifferential equations*, Abstr. Appl. Anal. **2014** (2014) 1–10.

[5] M. Bisheh-Niasar, A. Saadatmandi, M. Akrami-Arani, *A new family of high-order difference schemes for the solution of second order boundary value problems*, Iranian J. Math. Chem. **9** (2018) 187–199.

[6] H. Danfu, S. Xufeng, *Numerical solution of integro-differential equations by using* CAS *wavelet operational matrix of integration*, Appl. Math. Comput. **194** (2007) 460–466.

[7] R. Ezzati, S. Najafalizadeh, *Numerical methods for solving linear and nonlinear Volterra-Fredholm integral equations by using* CAS *wavelets*, World Appl. Sci. J. **18** (2012) 1847–1854.

[8] A.D. Filip, A. Petruel, *Existence and uniqueness of the solution for a general system of Fredholm integral equations*, Math. Meth. Appl. Sci. **2020** (2020) 1–10.

[9] J.S. Guf, W.S. Jiang, *The Haar wavelets operational matrix of integration*, Int. J. Sys. Sci. **27** (1996) 623–628.

[10] S.M. Hashemiparast, M. Sabzevari, H. Fallahgoul, *Improving the solution of nonlinear Volterra integral equations using rationalized Haar s-functions*, Vietnam J. Math. **39** (2011) 145–157.

[11] M.H. Heydari, M.R. Hooshmandasl, A. Shakiba, C. Cattani, *Legendre wavelets Galerkin method for solving nonlinear stochastic integral equations*, Nonlin. Dyn. **85** (2016) 1185–1202.

[12] Ü. Lepik, *Numerical solution of differential equations using Haar wavelets*, Math. Comput. Simul. **68** (2005) 127–143.

[13] K. Maleknejad, F. Mirzaee and S. Abbasbandy, *Solving linear integro-differential equations system by using rationalized Haar functions method*, Appl. Math. comput. **155** (2004) 317–328.

[14] M. Sabzevari, *Erratum to "Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions" [Appl. Math. Comput. 149 (2004) 799–806]*, Appl. Math. Comput. **339** (2018) 302–307.

[15] M. Sabzevari, *A review on "Numerical solution of nonlinear Volterra-Fredholm integral equations using hybrid of block-pulse functions and Taylor series" [Alexandria Eng. J. 52 (2013) 551–555]*, Alexandria Eng. J. **58** (2019) 1099–1102.

[16] M. Sabzevari, F. Molaei, *Numerical solution of system of nonlinear integro-differential equations using hybrid of Legendre polynomials and block-pulse functions*, Math. Interdisc. Res. **5** (2020) 225–238.

[17] S.C. Shiralashetti, L. Lamani, CAS *wavelets stochastic operational matrix of integration and its application for solving stochastic ito-volterra integral equations*, Jordan J. Math. Statis. **14** (2021) 555–580.

[18] J. Xiao, L. Wen, D. Zhang, *Solving second kind Fredholm integral equations by periodic wavelet Galerkin method*, Appl. Math. Comput. **175** (2000) 508–518.