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Numerical simulation for unsteady Helmholtz problems of anisotropic FGMs

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Abstract. The unsteady Helmholtz type equation of anisotropic functionally graded materials (FGMs) is considered in this study. The study is to find numerical solutions to initial boundary value problems governed by the equation. A combined Laplace and boundary element method is used to solve the problems. The analysis derives a boundary-only integral equation that is used to compute the numerical solutions. The analysis also results in another class of anisotropic FGMs of applications. Some problems are considered. The numerical solutions obtained are accurate and consistent.

Keywords: Anisotropic, FGMs, unsteady state, Helmholtz equation, boundary element method.
AMS Subject Classification 2010: 35K51, 35N10, 44A10, 65M38

1 Introduction

We will investigate numerical solutions to problems governed by an equation with variable coefficients of the form

$$\frac{\partial}{\partial x_i}\left[\beta_{ij}(\mathbf{x})\frac{\partial \phi(\mathbf{x},t)}{\partial x_j}\right]+\gamma^2(\mathbf{x})\,\phi(\mathbf{x},t)=\alpha(\mathbf{x},t)\frac{\partial \phi(\mathbf{x},t)}{\partial t}, \tag{1}$$

where $i,j=1,2$ and for repeated indices the summation holds so that Eq. (1) becomes

$$\frac{\partial}{\partial x_1}\left(\beta_{11}\frac{\partial \phi}{\partial x_1}\right)+\frac{\partial}{\partial x_1}\left(\beta_{12}\frac{\partial \phi}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\beta_{12}\frac{\partial \phi}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\beta_{22}\frac{\partial \phi}{\partial x_2}\right)+\gamma^2\phi=\alpha\frac{\partial c}{\partial t},$$

where $\beta_{12}=\beta_{21}$, $\beta_{11}\beta_{22}-\beta_{12}^2>0$, $\beta_{11},\beta_{12},\beta_{22}>0$. Eq. (1) is usually used to model acoustic problems. Eq. (1) is applicable for FGMs as the coefficients are variable coefficients. Specifically, if $\beta_{11}=\beta_{22}$ and $\beta_{12}=0$ then the material under consideration is an isotropic material, otherwise it is anisotropic.

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Previous studies on the Helmholtz mainly focused on problems of homogeneous materials. In [6] Ma et al. considered a use of the Galerkin boundary element method for exterior problems of 2-D Helmholtz equation with arbitrary wave number. In this paper the authors assumed the internal and the external domain are homogeneous media. Loeffler et al. in [5] investigated numerical solutions for the Helmholtz problems. The Helmholtz equation is treated like a Laplace equation with non-zero right-hand side of the equation. The resulting boundary-domain integral equation is then solved using a direct radial basis function interpolation. In this study, the medium is supposed to be a homogeneous material. In [1] Barucq et al. also considered homogeneous media. The study is focused on the numerical aspect of the BEM used for solving the Helmholtz problems. Similarly, Wu and Alkhalifah in [8] also concerned on the numerical aspect of a finite-difference method for solving the Helmholtz equation of homogeneous media. Li et al. [4] obtained numerical solutions of the Helmholtz equation of homogeneous media using the method of fundamental solutions with Bessel functions, in replacement of Hankel functions, as the fundamental solutions. Some studies for inhomogeneous media have been done, but they are limited to the class of inhomogeneities which take the form of constant-plus-variable functions of inhomogeneity. Later on, Azis and coworkers in [3, 7] use a BEM for solving steady state Helmholtz problems of anisotropic FGMs. The present work is intended to extend the works in [3, 7] for steady-state Helmholtz problems of anisotropic FGMs to unsteady-state Helmholtz problems governed by Eq. (1).

2 Statement of problem

Solutions $\phi(\mathbf{x},t)$ to (1) for $t \geq 0$ are sought which are valid in a region Ω in R^2 with boundary $\partial\Omega$, are sought. The boundary $\partial\Omega$ consists of two types, namely $\partial\Omega_1$ on which the unknown variable $\phi(\mathbf{x},t)$ is specified, and $\partial\Omega_2$ on which

$$q(\mathbf{x},t) = \beta_{ij}(\mathbf{x}) \frac{\partial \phi(\mathbf{x},t)}{\partial x_i} n_j, \tag{2}$$

is specified, where $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ and $\mathbf{n} = (n_1, n_2)$ denotes the outward pointing normal to $\partial\Omega$. The zero initial condition is imposed

$$\phi(\mathbf{x},0) = 0. \tag{3}$$

3 Reduction to constant coefficients equation

The coefficients $\beta_{ij}, \gamma^2, \alpha$ are required to take the form

$$\beta_{ij}(\mathbf{x}) = \overline{\beta}_{ij} f(\mathbf{x}), \tag{4}$$

$$\gamma^2(\mathbf{x}) = \overline{\gamma}^2 f(\mathbf{x}), \tag{5}$$

$$\alpha(\mathbf{x},t) = \overline{\alpha}(t) f(\mathbf{x}), \tag{6}$$

where the $\overline{\beta}_{ij}, \overline{\gamma}^2$ are constants, $\overline{\alpha}$ is a function of time t and f is a differentiable function of $\mathbf{x}$. Using (4), (5) and (6) in (1) yields

$$\overline{\beta}_{ij} \frac{\partial}{\partial x_i}\left(f \frac{\partial \phi}{\partial x_j}\right) + \overline{\gamma}^2 f \phi = \overline{\alpha} f \frac{\partial \phi}{\partial t}. \tag{7}$$

Let

$$\phi(\mathbf{x},t) = f^{-1/2}(\mathbf{x})\, \theta(\mathbf{x},t). \tag{8}$$

Therefore substitution of (4) and (8) into (2) gives

$$q(\mathbf{x},t) = -q_f(\mathbf{x})\,\theta(\mathbf{x},t) + f^{1/2}(\mathbf{x})\,q_\theta(\mathbf{x},t), \tag{9}$$

where

$$q_f(\mathbf{x}) = \overline{\beta}_{ij}\frac{\partial f^{1/2}}{\partial x_j}n_i, \qquad q_\theta(\mathbf{x}) = \overline{\beta}_{ij}\frac{\partial \theta}{\partial x_j}n_i.$$

Also, Eq. (7) may be written in the form

$$\begin{aligned}
\overline{\beta}_{ij}\frac{\partial}{\partial x_i}\left[f\frac{\partial\left(f^{-1/2}\theta\right)}{\partial x_j}\right] + \overline{\gamma}^2 f^{1/2}\theta &= \overline{\alpha} f\frac{\partial\left(f^{-1/2}\theta\right)}{\partial t},\\
\overline{\beta}_{ij}\frac{\partial}{\partial x_i}\left[f\left(f^{-1/2}\frac{\partial\theta}{\partial x_j} + \theta\frac{\partial f^{-1/2}}{\partial x_j}\right)\right] + \overline{\gamma}^2 f^{1/2}\theta &= \overline{\alpha} f^{1/2}\frac{\partial\theta}{\partial t},\\
\overline{\beta}_{ij}\frac{\partial}{\partial x_i}\left(f^{1/2}\frac{\partial\theta}{\partial x_j} + f\theta\frac{\partial f^{-1/2}}{\partial x_j}\right) + \overline{\gamma}^2 f^{1/2}\theta &= \overline{\alpha} f^{1/2}\frac{\partial\theta}{\partial t}.
\end{aligned}$$

Using the identity

$$\frac{\partial f^{-1/2}}{\partial x_i} = -f^{-1}\frac{\partial f^{1/2}}{\partial x_i},$$

implies

$$\overline{\beta}_{ij}\frac{\partial}{\partial x_i}\left(f^{1/2}\frac{\partial\theta}{\partial x_j} - \theta\frac{\partial f^{1/2}}{\partial x_j}\right) + \overline{\gamma}^2 f^{1/2}\theta = \overline{\alpha} f^{1/2}\frac{\partial\theta}{\partial t}.$$

Rearranging and neglecting some zero terms gives

$$f^{1/2}\overline{\beta}_{ij}\frac{\partial^2\theta}{\partial x_i\partial x_j} - \theta\overline{\beta}_{ij}\frac{\partial^2 f^{1/2}}{\partial x_i\partial x_j} + \overline{\gamma}^2 f^{1/2}\theta = \overline{\alpha} f^{1/2}\frac{\partial\theta}{\partial t}.$$

It follows that if f is such that

$$\overline{\beta}_{ij}\frac{\partial^2 f^{1/2}}{\partial x_i\partial x_j} - \lambda f^{1/2} = 0, \tag{10}$$

where λ is a constant, then equation (8) brings the variable coefficients equation (7) to a constant coefficients equation

$$\overline{\beta}_{ij}\frac{\partial^2\theta}{\partial x_i\partial x_j} + \left(\overline{\gamma}^2 - \lambda\right)\theta = \overline{\alpha}\frac{\partial\theta}{\partial t}. \tag{11}$$

Laplace transform of (8), (9), (11) subjected to equation (3) are respectively

$$\theta^*(\mathbf{x},s) = f^{1/2}(\mathbf{x})\,\phi^*(\mathbf{x},s), \tag{12}$$

$$q_{\theta^*}(\mathbf{x},s) = \left[q^*(\mathbf{x},s) + q_f(\mathbf{x})\,\theta^*(\mathbf{x},s)\right]f^{-1/2}(\mathbf{x}), \tag{13}$$

$$\overline{\beta}_{ij}\frac{\partial^2\theta^*}{\partial x_i\partial x_j} + \left(\overline{\gamma}^2 - s\overline{\alpha}^* - \lambda\right)\theta^* = 0, \tag{14}$$

where s is the variable of the Laplace-transformed domain.

A boundary integral equation for the solution of (14) is given in the form

$$\eta(\boldsymbol{\xi})\,\theta^*(\boldsymbol{\xi},s) = \int_{\partial\Omega} \left[G(\mathbf{x},\boldsymbol{\xi})\,\theta^*(\mathbf{x},s) - F(\mathbf{x},\boldsymbol{\xi})\,q_{\theta^*}(\mathbf{x},s)\right] dS(\mathbf{x}), \tag{15}$$

where $\boldsymbol{\xi} = (\xi_1,\xi_2)$, $\eta = 0$ if $(\xi_1,\xi_2) \notin \Omega \cup \partial\Omega$, $\eta = 1$ if $(\xi_1,\xi_2) \in \Omega$, $\eta = \frac{1}{2}$ if $(\xi_1,\xi_2) \in \partial\Omega$ and $\partial\Omega$ has a continuously turning tangent at (ξ_1,ξ_2). The so called fundamental solution F in (15) is any solution of the equation

$$\overline{\beta}_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \left(\overline{\gamma}^2 - s\overline{\alpha}^* - \lambda\right)F = \delta(\mathbf{x}-\boldsymbol{\xi}),$$

and the G is given by

$$G(\mathbf{x},\boldsymbol{\xi}) = \overline{\beta}_{ij}\frac{\partial F(\mathbf{x},\boldsymbol{\xi})}{\partial x_j} n_i,$$

where δ is the Dirac delta function. For two-dimensional problems F and G are given by

$$\begin{aligned}
F(\mathbf{x},\boldsymbol{\xi}) &= \begin{cases} \frac{\sigma}{2\pi}\ln R, & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda = 0, \\ \frac{\iota\sigma}{4}H_0^{(2)}(\omega R), & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda > 0, \\ \frac{-\sigma}{2\pi}K_0(\omega R), & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda < 0, \end{cases} \\
G(\mathbf{x},\boldsymbol{\xi}) &= \begin{cases} \frac{\sigma}{2\pi}\frac{1}{R}\overline{\beta}_{ij}\frac{\partial R}{\partial x_j}n_i, & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda = 0, \\ \frac{-\iota\sigma\omega}{4}H_1^{(2)}(\omega R)\,\overline{\beta}_{ij}\frac{\partial R}{\partial x_j}n_i, & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda > 0, \\ \frac{\sigma\omega}{2\pi}K_1(\omega R)\,\overline{\beta}_{ij}\frac{\partial R}{\partial x_j}n_i, & \text{if } \overline{\gamma}^2 - s\overline{\alpha}^* - \lambda < 0, \end{cases}
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
\sigma &= \ddot{\rho}/D, \\
\omega &= \sqrt{|\overline{\gamma}^2 - s\overline{\alpha}^* - \lambda|/D}, \\
D &= \left[\overline{\beta}_{11} + 2\overline{\beta}_{12}\dot{\rho} + \overline{\beta}_{22}\left(\dot{\rho}^2 + \ddot{\rho}^2\right)\right]/2, \\
R &= \sqrt{(\dot{x}_1 - \dot{\xi}_1)^2 + (\dot{x}_2 - \dot{\xi}_2)^2}, \\
\dot{x}_1 &= x_1 + \dot{\rho}x_2, \\
\dot{\xi}_1 &= \xi_1 + \dot{\rho}\xi_2, \\
\dot{x}_2 &= \ddot{\rho}x_2, \\
\dot{\xi}_2 &= \ddot{\rho}\xi_2,
\end{aligned}$$

where $\dot{\rho}$ and $\ddot{\rho}$ are respectively the real and the positive imaginary parts of the complex root ρ of the quadratic

$$\overline{\beta}_{11} + 2\overline{\beta}_{12}\rho + \overline{\beta}_{22}\rho^2 = 0,$$

and $H_0^{(2)}$, $H_1^{(2)}$ are the Hankel function of second kind and order zero and order one respectively. K_0, K_1 represent the modified Bessel function of order zero and order one respectively, $\iota = \sqrt{-1}$. The derivatives

$\partial R/\partial x_j$ needed for the calculation of the G in (16) are given by

$$\begin{aligned}\frac{\partial R}{\partial x_1} &= \frac{1}{R}\left(\dot{x}_1 - \dot{a}\right),\\ \frac{\partial R}{\partial x_2} &= \dot{\rho}\left[\frac{1}{R}\left(\dot{x}_1 - \dot{a}\right)\right] + \ddot{\rho}\left[\frac{1}{R}\left(\dot{x}_2 - \dot{b}\right)\right].\end{aligned}$$

Using Eqs. (12) and (13) in (15) yields

$$\eta f^{1/2}\phi^* = \int_{\partial\Omega}\left[\left(f^{1/2}G - q_f F\right)\phi^* - \left(f^{-1/2}F\right)q^*\right]dS. \tag{17}$$

Eq. (17) provides a boundary integral equation for determining $\phi^*, \partial\phi^*/\partial x_1, \partial\phi^*/\partial x_2$ at all points of Ω.

After obtaining the solutions $\phi^*, \partial\phi^*/\partial x_1, \partial\phi^*/\partial x_2$, the Stehfest formula can be used to find the values of $\phi, \partial\phi/\partial x_1, \partial\phi/\partial x_2$. The Stehfest formula is

$$\begin{aligned}\phi(\mathbf{x},t) &\approx \frac{\ln 2}{t}\sum_{m=1}^{N} W_m \phi^*(\mathbf{x},s_m),\\ \frac{\partial\phi(\mathbf{x},t)}{\partial x_1} &\approx \frac{\ln 2}{t}\sum_{m=1}^{N} W_m \frac{\partial\phi^*(\mathbf{x},s_m)}{\partial x_1},\\ \frac{\partial\phi(\mathbf{x},t)}{\partial x_2} &\approx \frac{\ln 2}{t}\sum_{m=1}^{N} W_m \frac{\partial\phi^*(\mathbf{x},s_m)}{\partial x_2},\end{aligned} \tag{18}$$

where

$$s_m = \frac{\ln 2}{t}m, \quad W_m = (-1)^{\frac{N}{2}+m}\sum_{k=\left[\frac{m+1}{2}\right]}^{\min\left(m,\frac{N}{2}\right)} \frac{k^{N/2}(2k)!}{\left(\frac{N}{2}-k\right)!k!(k-1)!(m-k)!(2k-m)!}.$$

Possible multi-parameter solution $f^{1/2}(\mathbf{x})$ to (10)

$$f^{1/2}(\mathbf{x}) = \begin{cases} A\cos(c_0 + c_i x_i) + B\sin(c_0 + c_i x_i), & \overline{\beta}_{ij}c_i c_j + \lambda = 0, \lambda \neq 0,\\ A\exp(c_0 + c_i x_i), & \overline{\beta}_{ij}c_i c_j - \lambda = 0, \lambda \neq 0,\\ c_0 + c_i x_i, & \lambda = 0,\end{cases} \tag{19}$$

where the $A, B, c_k, k = 1, 2, \ldots, n$ are constants. Specifically, the quadratic inhomogeneity function $f(\mathbf{x}) = (c_0 + c_i x_i)^2$ in (19) can be written in the form of a sum of a constant and a variable terms as $f(\mathbf{x}) = c_0^2 + \left(2c_0 c_i x_i + c_i^2 x_i^2\right)$ so that the coefficients $\beta_{ij}(\mathbf{x}), \gamma^2(\mathbf{x}), \alpha(\mathbf{x})$ fall within the class of constant-plus-variable coefficients. However, the trigonometric inhomogeneity functions can not be written in a simple form of a sum of a constant and a variable terms.

4 Numerical examples

Numerical solutions to several problems will be sought by employing the combined BEM and Laplace transform. A standard BEM with constant element is derived by discretising the integral equation (17).

Table 1: Values of W_m of the Stehfest formula for $N = 6, 8, 10, 12$.

W_m	$N = 6$	$N = 8$	$N = 10$	$N = 12$
W_1	1	-1/3	1/12	-1/60
W_2	-49	145/3	-385/12	961/60
W_3	366	-906	1279	-1247
W_4	-858	16394/3	-46871/3	82663/3
W_5	810	-43130/3	505465/6	-1579685/6
W_6	-270	18730	-236957.5	1324138.7
W_7		-35840/3	1127735/3	-58375583/15
W_8		8960/3	-1020215/3	21159859/3
W_9			164062.5	-8005336.5
W_{10}			-32812.5	5552830.5
W_{11}				-2155507.2
W_{12}				359251.2

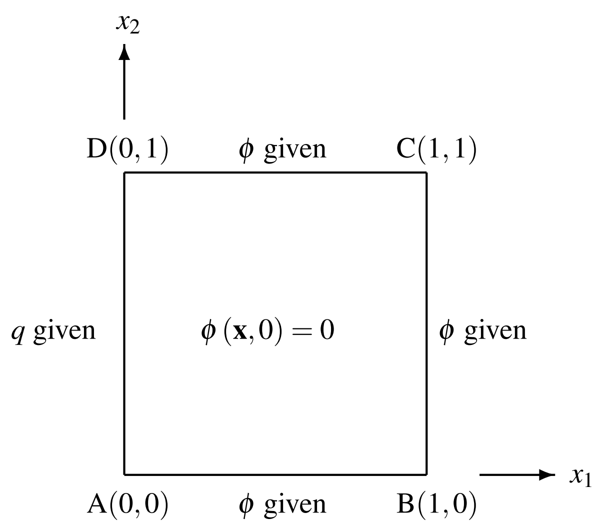

Figure 1: The boundary conditions for the test problems in Section 4.1

For simplicity, the spatial domain Ω is taken to be a unit square. A number of 320 same length boundary elements is utilized. The Bode's quadrature is used to evaluate the line integrals over each element. The time t domain is chosen to be the interval $0 \leq t \leq 5$.

To compute the numerical solutions, a FORTRAN code is developed and a script for calculating the values of the coefficients $W_m, m = 1, 2, \ldots, N$ of the Stehfest formula in (18) for any even number N is embedded in the code. Table (1) shows the values of W_m for $N = 6, 8, 10, 12$ which are obtained from the script.

4.1 Examples with analytical solutions

4.1.1 Problem 1

In order to see the accuracy of the BEM and the Stehfest formula we will consider some problems with analytical solutions. We take a mutual constant coefficients $\overline{\beta}_{ij}$ and $\overline{\gamma}^2$ for all problems

$$\overline{\beta}_{ij} = \begin{bmatrix} 0.65 & 0.25 \\ 0.25 & 1 \end{bmatrix}, \quad \overline{\gamma}^2 = 0.5,$$

and a mutual set of boundary conditions (see Figure 1)

ϕ is given on side AB, BC, CD,
q is given on side AD.

For each problem, numerical solutions ϕ and the derivatives $\partial\phi/\partial x_1$ and $\partial\phi/\partial x_2$ are calculated at 19×19 interior points $(x_1, x_2) = \{.05, .1, .15, \ldots, .9, .95\} \times \{.05, .1, .15, \ldots, .9, .95\}$ and 11 time-steps $t = 0.0005, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5$. The value $t = 0.0005$ is supposed to be the approximating value of $t = 0$ as the singular point of the Stehfest formula. The aggregate relative error E of the numerical solutions for each time t is computed using the formula

$$E = \left[\frac{\sum_{i=1}^{19\times 19} (\phi_{n,i} - \phi_{a,i})^2}{\sum_{i=1}^{19\times 19} \phi_{a,i}^2} \right]^{\frac{1}{2}},$$

where ϕ_n and ϕ_a are respectively the numerical and analytical solutions ϕ or the derivatives $\partial\phi/\partial x_1$ and $\partial\phi/\partial x_2$.

Case 1: trigonometrically graded material We assume the inhomogeneity function $f(\mathbf{x})$ is a trigonometric function

$$f(\mathbf{x}) = \left[\cos(0.55 - 0.2x_1 - 0.25x_2) + \sin(0.55 - 0.2x_1 - 0.25x_2)\right]^2,$$

so that the medium under consideration is a trigonometrically graded material. For $f(\mathbf{x})$ to satisfy (19) $\lambda = -0.1135$. We take $\overline{\alpha}(t) = 1.227t(t-5.5)/(4t-11)$. The analytical solution is

$$\phi(\mathbf{x}, t) = \frac{t(5.5-t)(1 - 0.15x_1 - 0.75x_2)}{\cos(0.55 - 0.2x_1 - 0.25x_2) + \sin(0.55 - 0.2x_1 - 0.25x_2)}.$$

Figure 2 shows the aggregate relative errors E of the numerical solutions ϕ with $N = 6, 8, 10, 12$ for the Stehfest formula (18). It indicates convergence of the Stehfest formula when the value of N changes from $N = 6$ to $N = 10$. For Case 1 we may assume that the value of N is optimized at $N = 10$. Increasing N to $N = 12$ does not give more accurate solutions. See for example Hassanzadeh and Pooladi-Darvish [2] for examples of convergence of different numerical Laplace inversion methods. According to Hassanzadeh and Pooladi-Darvish [2] the parameter N should be optimized by trial and error. Increasing N will increase the accuracy up to a point, and then the accuracy will decline due to round-off errors.

For the derivative solution $\partial\phi/\partial x_1$, Figure 3 shows that $N = 8$ is the optimized value of N for the aggregate relative errors E. Whereas for the derivative solution $\partial\phi/\partial x_2$, Figure 4 shows that $N = 10$ is the optimized value of N for the aggregate relative errors E.

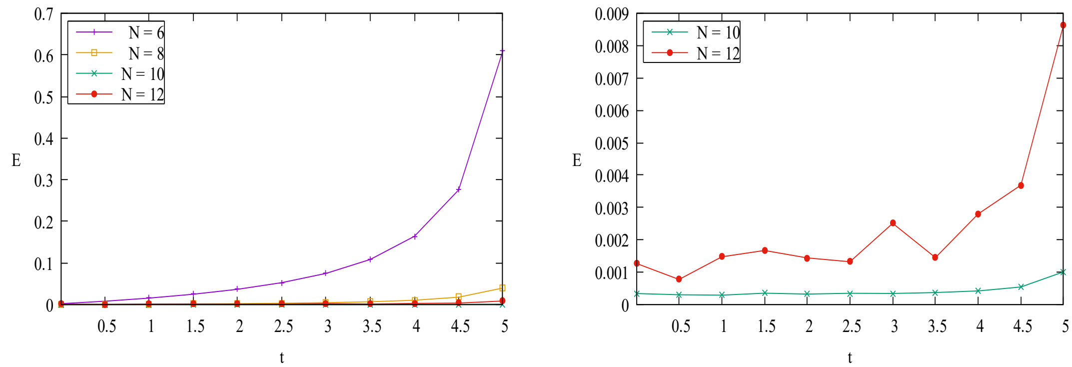

Figure 2: Top: The aggregate relative error E of the numerical solutions ϕ with $N = 6, 8, 10, 12$ for Case 1 (left) and zoom-in view for $N = 10, 12$ (right).

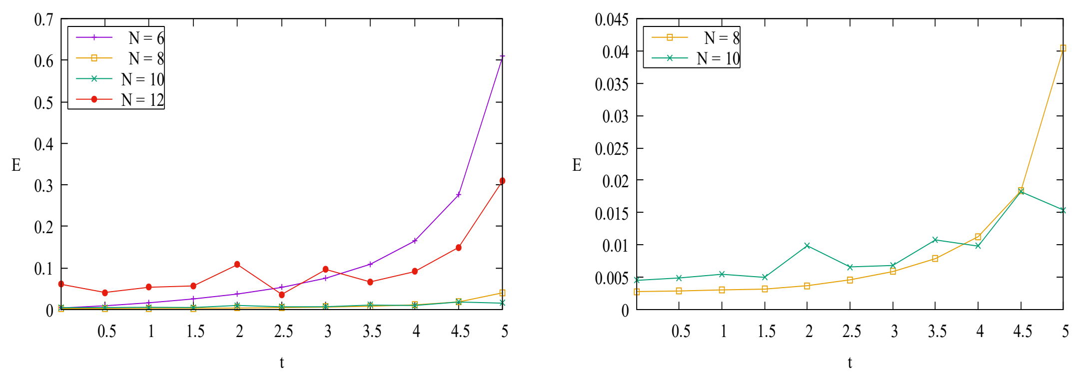

Figure 3: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_1$ with $N = 6, 8, 10, 12$ for Case 1 (left) and zoom-in view for $N = 8, 10$ (right).

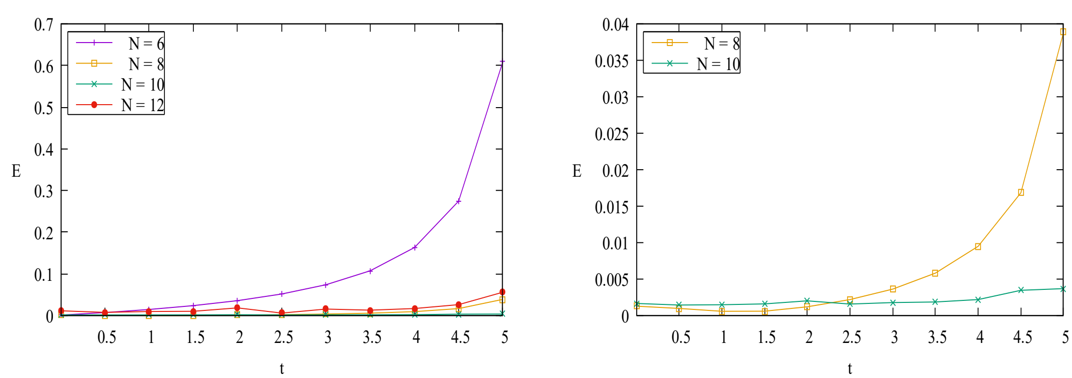

Figure 4: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_2$ with $N = 6, 8, 10, 12$ for Case 1 (left) and zoom-in view for $N = 8, 10$ (right).

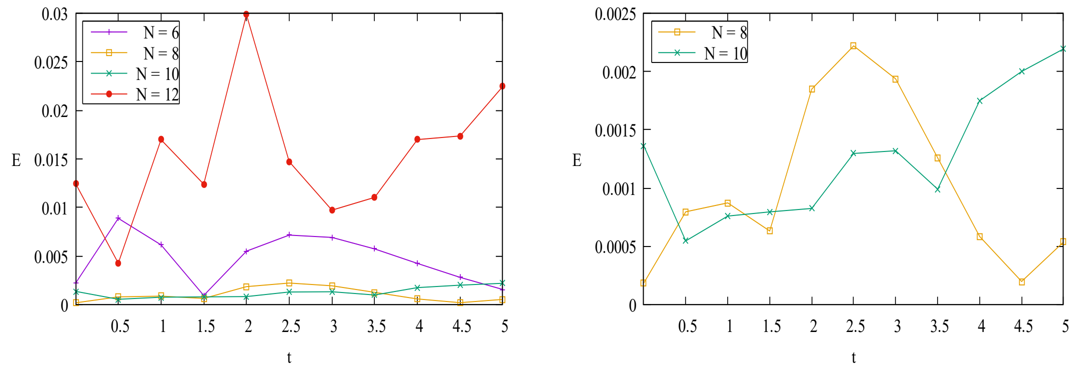

Figure 5: Top: The aggregate relative error E of the numerical solutions ϕ with $N = 6, 8, 10, 12$ for Case 2 (left) and zoom-in view for $N = 8, 10$ (right).

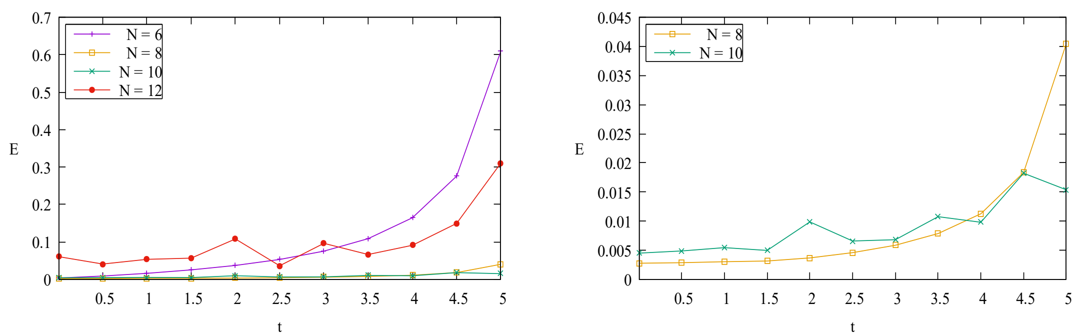

Figure 6: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_1$ with $N = 6, 8, 10, 12$ for Case 2 (left) and zoom-in view for $N = 8, 10$ (right).

Case 2: exponentially graded material Now we take the analytical solution

$$\phi(\mathbf{x}, t) = \frac{[1 - \exp(-1.15t)]\cos(1 - 0.15x_1 - 0.75x_2)}{\exp(0.55 - 0.2x_1 - 0.25x_2)},$$

for an exponentially graded material with gradation function $f(\mathbf{x}) = [\exp(0.55 - 0.2x_1 - 0.25x_2)]^2$, so that from (19) $\lambda = 0.1135$. The rate of change is $\overline{\alpha}(t) = 0.2146739131[1 - \exp(1.15t)]$.

Figure 5 indicates that $N = 10$ may be taken as the optimized value of N for the aggregate relative errors E of the numerical solutions ϕ. Increasing N to $N = 12$ gives worse solutions. Whereas for the solutions $\partial\phi/\partial x_1$ and $\partial\phi/\partial x_2$, $N = 8$ and $N = 10$ are the optimized value of N respectively (see Figures 6 and 7).

Case 3: quadratically graded material Next, we assume that the material is quadratically graded, with function of gradation $f(\mathbf{x}) = (0.55 - 0.2x_1 - 0.25x_2)^2$, so that from (19) $\lambda = 0$. The rate of change is $\overline{\alpha}(t) = 1.133375t$. The analytical solution is

$$\phi(\mathbf{x}, t) = \frac{(t/5)\exp(1 - 0.15x_1 - 0.75x_2)}{0.55 - 0.2x_1 - 0.25x_2}.$$

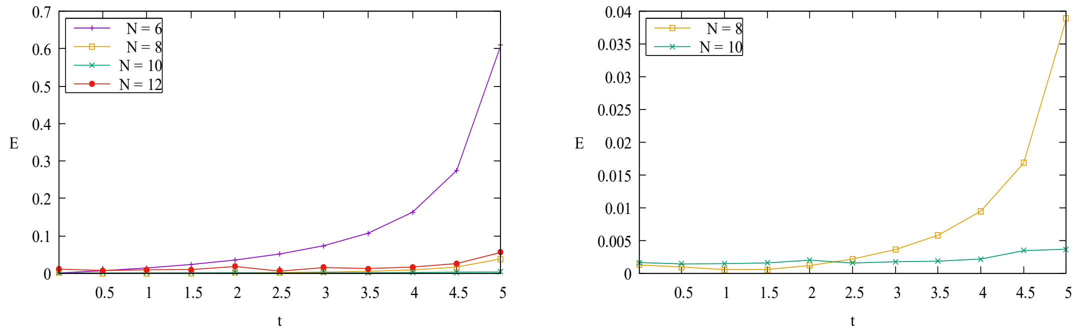

Figure 7: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_2$ with $N = 6, 8, 10, 12$ for Case 2 (left) and zoom-in view for $N = 8, 10$ (right).

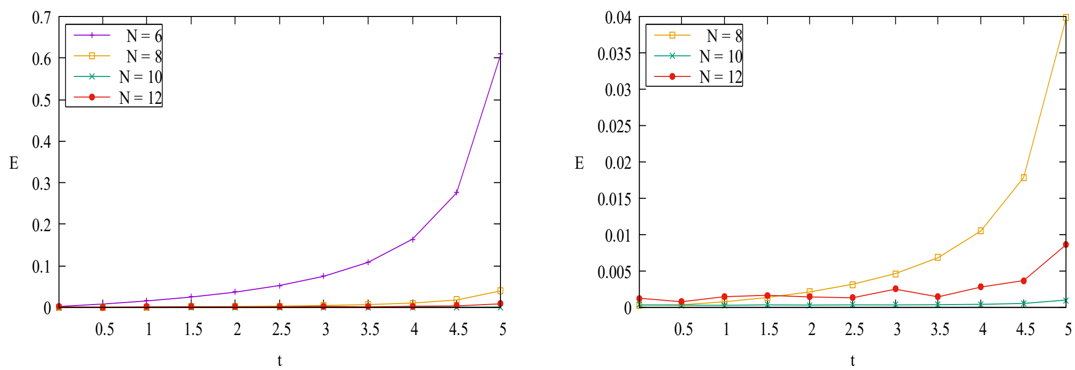

Figure 8: Top: The aggregate relative error E of the numerical solutions ϕ with $N = 6, 8, 10, 12$ for Case 3 (left) and zoom-in view for $N = 8, 10, 12$ (right).

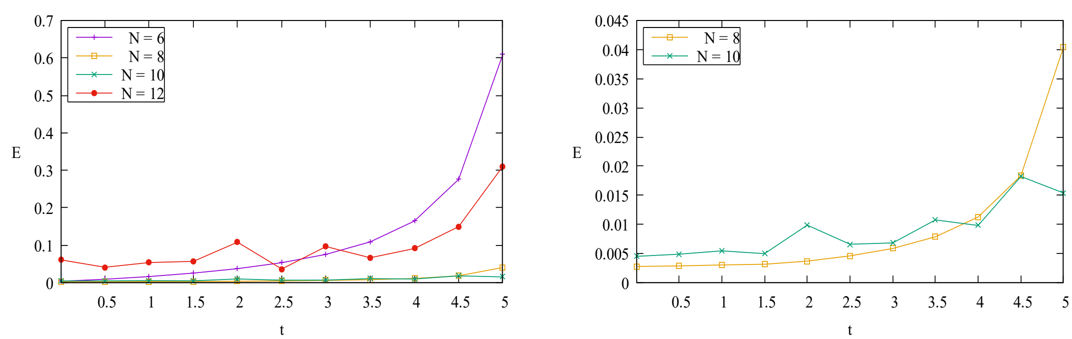

Figure 9: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_1$ with $N = 6, 8, 10, 12$ for Case 3 (left) and zoom-in view for $N = 8, 10$ (right).

Figures 8, 9 and 10 indicate that we may choose $N = 10$ as the optimized value of N.

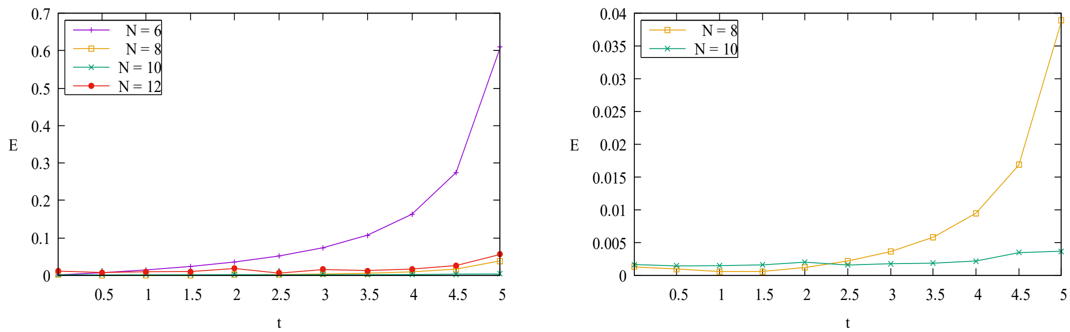

Figure 10: Top: The aggregate relative error E of the numerical solutions $\partial\phi/\partial x_2$ with $N = 6, 8, 10, 12$ for Case 3 (left) and zoom-in view for $N = 8, 10$ (right).

4.2 Examples without analytical solutions

4.2.1 Problem 2

Further, we will show that the anisotropy and inhomogeneity of materials give impacts on the solutions. We will use $\overline{\beta}_{ij}, \overline{\gamma}^2, f(\mathbf{x})$ in Case 1 of Problem 4.1 for this problem, which are

$$\begin{aligned}
\overline{\beta}_{ij} &= \begin{bmatrix} 0.65 & 0.25 \\ 0.25 & 1 \end{bmatrix}, \\
\overline{\gamma}^2 &= 0.5, \\
f(\mathbf{x}) &= \left[\cos(0.55 - 0.2x_1 - 0.25x_2) + \sin(0.55 - 0.2x_1 - 0.25x_2)\right]^2.
\end{aligned}$$

We choose $\hat{\alpha}(t) = 1$. As we aim to show the impacts of the anisotropy and inhomogeneity of the material, we need to consider the case of homogeneous material and the case of isotropic material. We assume that when the material is homogeneous then $f(\mathbf{x}) = 1$ and if an isotropic material is under consideration then

$$\overline{\beta}_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore from (18) we have $\lambda = 0$ when the considered material is homogeneous, $\lambda = -0.1025$ if the material is isotropic inhomogeneous, and $\lambda = -0.1135$ if it is anisotropic inhomogeneous. The boundary conditions are (see Figure 11)

$$\begin{aligned}
&q = q(t) \text{ on side AB}, \\
&q = 0 \text{ on side BC}, \\
&\phi = 0 \text{ on side CD}, \\
&q = 0 \text{ on side AD}.
\end{aligned}$$

where $q(t)$ is associated with four cases, namely

$$\begin{aligned}
&\text{Case 1:} \quad q(t) = 1, \\
&\text{Case 2:} \quad q(t) = \exp(-t), \\
&\text{Case 3:} \quad q(t) = t, \\
&\text{Case 4:} \quad q(t) = t/(t + 0.01).
\end{aligned}$$

For all cases we take $N = 10$ in the Stehfest formula (18).

Figure 12 shows for all cases when the material is isotropic and homogeneous the solutions $\phi(0.1, 0.5, t)$ and $\phi(0.9, 0.5, t)$ coincide. This is to be expected as the problem is geometrically symmetric at $x_1 = 0.5$ when the material is isotropic and homogeneous. The results in Figure 12 also indicate that the anisotropy and inhomogeneity of the material give effects on the solutions. Moreover, as also expected, the change of the solution with respect to t mimics the variation of the function $q(t)$ as the boundary condition on side AD.

Whereas, the results in Figure 13 show that the Case 1 of $q(t) = 1$ and Case 4 of $q(t) = t/(t+0.01)$ have the same steady state solution. This is to be expected as $q(t) = t/(t+0.01)$ will converge to 1 when t approaches infinity.

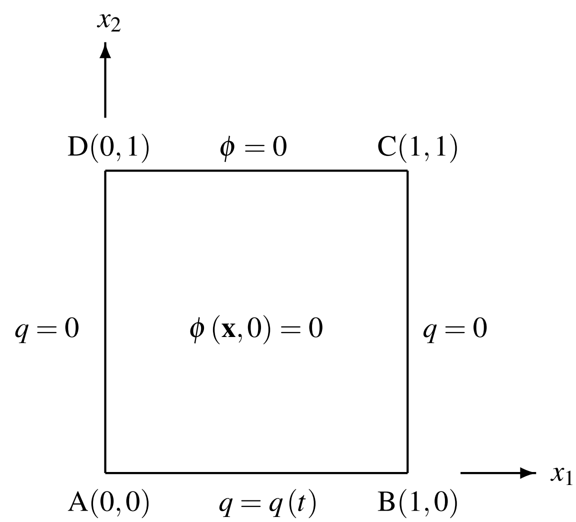

Figure 11: The boundary conditions for Problem 4.2

5 Conclusion

Several problems for a class of anisotropic FGMs (quadratically, exponentially and trigonometrically graded materials) have been solved using a combined BEM and Laplace transform. From the results of both Problem 1 and Problem 2, we may conclude that the analysis of reduction to constant coefficients equation (in Section 3) for deriving the boundary-only integral equation (17) is valid, and the BEM and Stehfest formula is appropriate for solving such problems as defined in Section 2. Moreover, the results of Problem 1 show the accuracy of the method, whereas the results of Problem 2 exhibit the consistency of the numerical solutions. The effect of the inhomogeneity and anisotropy of materials as well as the obtained steady-state solutions are as expected.

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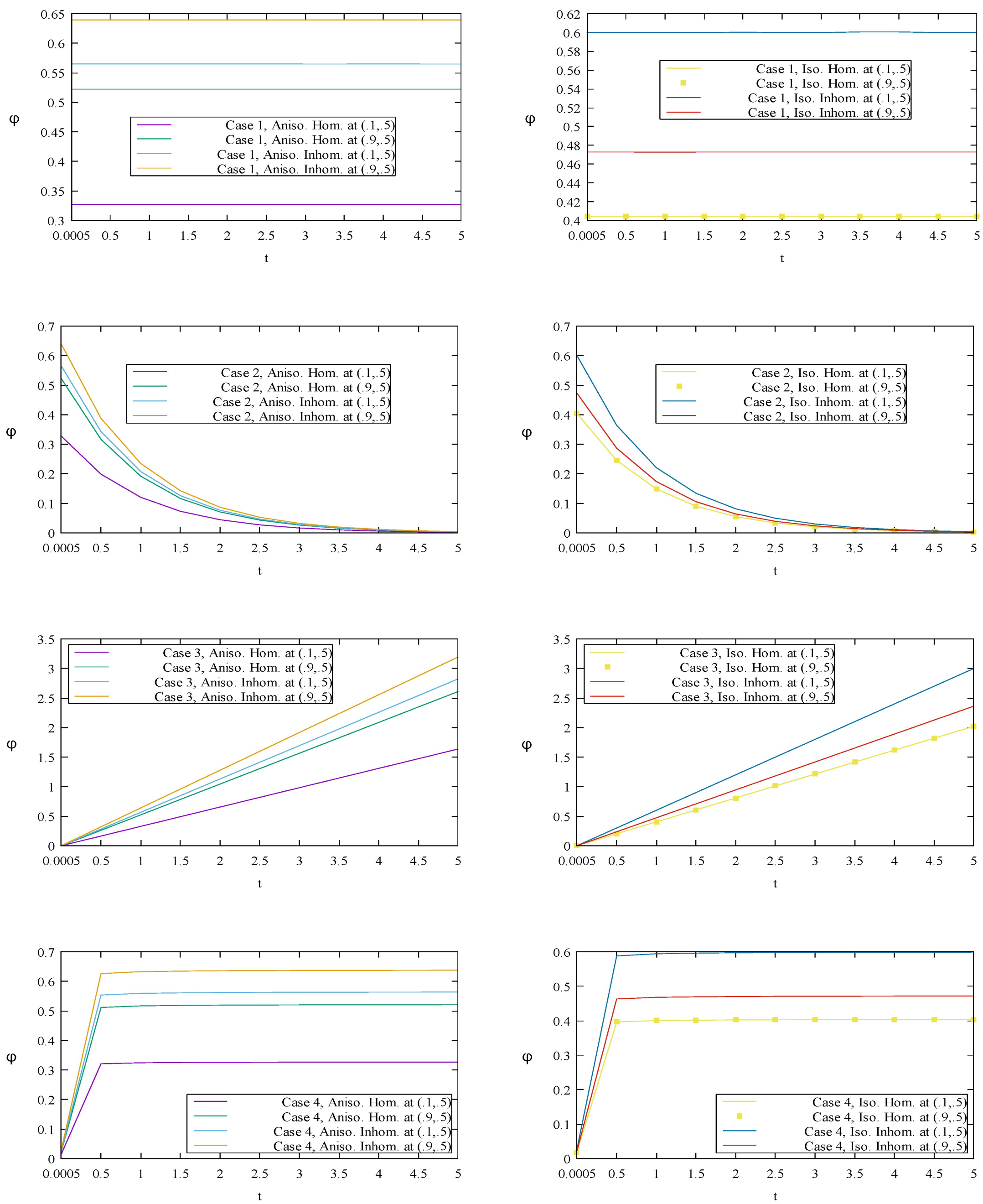

Figure 12: Solutions $\phi\,(0.1, 0.5, t)$ and $\phi\,(0.9, 0.5, t)$ for all cases of Problem 4.2

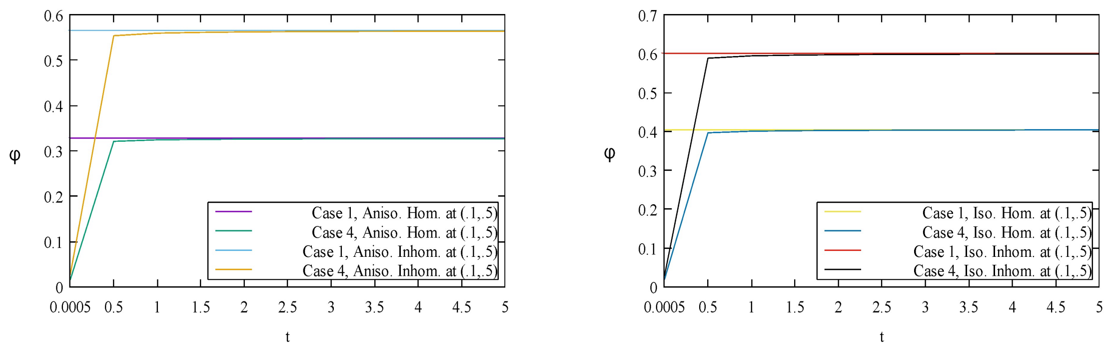

Figure 13: Solutions $\phi(0.1, 0.5, t)$ for Case 1 and 4 of Problem 4.2

suggestions that have improved the paper.

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