Journal of Mathematical Modeling
Vol. 11, No. 4, 2023, pp. 605–616. Research Article

JMM

An investigation into the optimal control of the horizontal and vertical incidence of communicable infectious diseases in society

Reza Akbari*

Department of Mathematics, Payame Noor Unvierstiy, Tehran, Iran
Email(s): r.akbari@pnu.ac.ir

Abstract. This article aims at proposing and developing a three-component mathematical model for susceptible, infected and recovered (SIR) population, under the control of vaccination of the susceptible population and drug therapy (antivirus) of the infected population (patient) in case of an infectious disease. The infectious disease under study can be transmitted through direct contact with an infected person (horizontal transmission) and from parent to child (vertical transmission). We investigate the basic reproduction number of the mathematical model, the existence and local asymptotic stability of both the disease free and endemic equilibrium. Using Pontryagin's minimum principle, we investigate the conditions of reducing the susceptible and infected population and increasing the recovered population based on the use of these two controllers in society. A numerical simulation of the optimal control problem shows, using both controllers is much more effective and leads to a rapid increase in the recovered population and prevents the disease from spreading and becoming an epidemic in the society.

Keywords: Optimal control, infectious diseases, basic reproduction number, stability, mathematical model.
AMS Subject Classification 2010: 37N25,92B05, 93D20.

1 Introduction

Generally, human diseases are divided into two categories: Infectious diseases and Non-infectious diseases. Nowadays, the spread of infectious diseases in different parts of the world is considered as one of the main concerns of global health. Infectious diseases ignore geographical and political boundaries and are a global threat that puts every nation and individual at risk and is one of the main causes of death worldwide, especially in developing countries. Infectious diseases are among the health problems that

*Corresponding author
Received: 3 February 2023 / Revised: 26 April 2023 / Accepted: 23 June 2023
DOI: 10.22124/jmm.2023.23774.2122

are transmitted from one infected individual to another and they put a part or the whole body under their negative influence by disrupting its functions. The methods of transmission of infectious diseases are as follows:

(i) Horizontal transmission: Transmission of the disease through direct contact with an infected person, for example, by physical contact and inhalation or ingestion of infectious materials such as Covid-19, flu, etc.

(ii) Vertical transmission: Transmission of infection or other diseases from parents to children (transmission of infection from mother to fetus during the pre-partum, intra-partum, or post-partum periods) such as AIDS, hepatitis B, and hepatitis C.

The following items can be mentioned for preventing the incidence and controlling infectious diseases in society:

(a) Eradicating the reservoir of the disease.

(b) Cutting the transmission lines off.

(c) Protecting the susceptible individuals.

In addition, new vaccines and new treatment technologies, as well as infrastructure improvements can help overcome the combat against infectious diseases.

Expressing disease behaviors through mathematical models is a tool for studying and investigating ways to control and prevent the spread of disease in society. Mathematical models are the most important tools in analyzing the spread of infectious diseases. The main reason for studying the mathematical modeling of disease transmission is to understand the transmission mechanism and the need for more effective control strategies. We can have a better and more comprehensible picture of the transmission and incidence of the disease in society and evaluate and investigate the effects of different control strategies by means of a good mathematical model. Recently, mathematical models of infectious diseases have attracted the attention of many researchers [4, 8, 9, 16–18, 20–24].

The optimal control theory is another branch of mathematics that is widely used to study and control the incidence of infectious diseases and is a powerful tool for decision-making in complex biological situations [5]. One of the methods of controlling infectious diseases that can protect society against these diseases is vaccination of the susceptible population [2,3,5,6,14,15,25]. Historically, vaccines have been very useful in preventing the diseases or death of millions of people. Also, (anti-viruses) drug therapy is another developed control method that can prevent the incidence of disease in society. However, it is not applicable to all viruses including HIV/AIDS, hepatitis B, hepatitis C, and influenza.

In this article, we divided the population into three separate categories:

(a) The population susceptible to infectious diseases.

(b) The infected population to infectious diseases.

(c) The population recovered from infectious disease.

Accordingly, we propose and develop a three-component mathematical model [6], including susceptible, infected and recovered (SIR) population, to investigate and control infectious disease. This infectious disease is transmitted in horizontal and vertical ways in society and can be controlled by vaccination and drug therapy (antivirus). The vaccination of the susceptible population will reduce the horizontal transmission of the disease and drug therapy of the infected population will reduce the vertical transmission in society.

The article includes the following sections. In Section 2 , we will introduce the (SIR) mathematical model for the transmission of the infectious disease. In Section 3, we will examine the equilibrium points and base reproduction rate as well as local asymptotic stability. In Section 4, the optimal control problem will be defined and studied. In Section 5, we will show the numerical results with a numerical example. We conclude the paper in Section 6.

2 Mathematical model of infectious disease transmission

In this section, we propose and develop an (SIR) model with vaccine and antiviral drug. In this model, the population N(t) is divided into three sub-populations: the susceptible $(S(t))$, infected $(I(t))$, recovered $(R(t))$. Thus, $N(t) = S(t) + I(t) + R(t)$. In this model, the infectious disease under study can be transmitted through direct contact with an infected person (horizontal transmission) and from parents to child, transmission of infection from mother to fetus during the prepartum, intrapartum or postpartum periods (vertical transmission). Thus, the (SIR) model describing the infectious disease transmission dynamics is given by the following system differential equations:

$$\begin{cases} \dot{S}(t) = \Lambda - p\Lambda I - q\Lambda R - \varphi S - \rho SI - mu_1 S + \theta R, \\ \dot{I}(t) = \rho SI + p\Lambda I - (\varphi + \varphi_1)I - \xi I - nu_2 I, \\ \dot{R}(t) = q\Lambda R + mu_1 S + nu_2 I - \varphi R - \theta R + \xi I, \\ S(0) \geqslant 0, \;\; I(0) \geqslant 0, \;\; R(0) \geqslant 0. \end{cases} \tag{1}$$

In these equations, all parameters are nonnegative, and their definitions are given in Table 1.

Lemma 1. *The solutions of system* (1) *are bounded.*

Proof. Since $N(t) = S(t) + I(t) + R(t)$, we have

$$\dot{N}(t) = \dot{S}(t) + \dot{I}(t) + \dot{R}(t) = \Lambda - \varphi N - \varphi_1 I \leqslant \Lambda - \varphi N(t).$$

Now integrating both sides of the above inequality and using the theory of differential inequality due to [7], we get

$$0 \leqslant N(t) \leqslant \frac{\varphi}{\Lambda} + N_0 e^{-\Lambda t},$$

where N_0 is the initial value of the total population system (1). Now by taking $t \longrightarrow +\infty$, we have

$$0 \leqslant N(t) \leqslant \frac{\varphi}{\Lambda}.$$

Therefore, every solution of model (1) initiating in R^3_{+0} are confined in the region

$$\mathscr{Y} = \{(S,I,R) \in \mathbb{R}^3_{+0} |\; 0 \leqslant S(t) + I(t) + R(t) \leqslant \frac{\varphi}{\Lambda}\}$$

for all $t > 0$. □

The region $\mathscr{Y}$ is a positive invariant set, thus it is sufficient to focus our attention only in this region as the model is epidemiologically and mathematically well posed in the sense of [12].

3 Analysis of the system

Throughout this section, we assume that the control parameters are constant.

3.1 Basic reproduction number and equilibria

For epidemic models, the basic reproduction number $\mathscr{R}_0$, is the expected number of secondary cases produced, in a completely susceptible population, by a typical infective individual. According to the notation in [10], we have

$$\mathscr{R}_0 = \frac{\rho\Lambda(\theta+\varphi-q\Lambda)}{\varphi(\theta+\varphi+mu_1-q\Lambda)(\varphi+\varphi_1+\xi+nu_2-p\Lambda)}.$$

By simple calculation, it's easy to see that system (1) has two equilibria

(i) The disease-free equilibria (DFE), is given as follows:

$$\mathscr{P}_f = (S_f, I_f, E_f) = (\frac{\Lambda(-q\Lambda+\varphi+\theta)}{\varphi(-q\Lambda+\varphi+\theta+mu_1)}, 0, \frac{mu_1 S_f}{-q\Lambda+\varphi+\theta}),$$

and it exists if $\theta \geqslant q\Lambda$.

(ii) The endemic equilibrium (EE) is $\mathscr{P}_e = (S_e, I_e, R_e)$, where

$$\begin{aligned} S_e &= \frac{\varphi+\varphi_1+\xi+nu_2-p\Lambda}{\rho}, \\ I_e &= \frac{\Lambda\beta(\theta+\varphi-q\Lambda)-\varphi(\theta+\varphi+mu_1-q\Lambda)(\varphi+\varphi_1+\xi+nu_2-p\Lambda)}{\rho[(\varphi+\varphi_1)(\theta+\varphi-q\Lambda)+\varphi(nu_2+\xi)]}, \\ R_e &= \frac{mu_1 S_e}{\theta+\varphi-q\Lambda} + \frac{(nu_2+\xi)I_e}{\theta+\varphi-q\Lambda}. \end{aligned}$$

So, $\mathscr{P}_e = (S_e, I_e, R_e)$ exists if $\mathscr{R}_0 > 1$.

According to Figure 1, the following can be stated.

(a) If there is no control in society, $(u1 = 0\ ,\ u2 = 0)$, then $\mathscr{R}_0 > 1$. That is, the disease has turned into an epidemic and some control measures should be taken to prevent the spread and epidemic of it.

(b) If we only use drug to control the spread of the disease (controlling only the infected population), $(u1 = 0\ ,\ u2 \neq 0)$, then $\mathscr{R}_0$ will significantly decrease but still $\mathscr{R}_0 > 1$. That is, the disease has become endemic in society.

(c) If we only use vaccination to control the spread of the disease (vaccination of the susceptible population of society), $(u1 \neq 0\ ,\ u2 = 0)$, then the basic reproduction rate will become downward and the control of susceptible population will lead to $\mathscr{R}_0 < 1$. That is, the disease will disappear and will not spread in society. Therefore, the presence of control in society will reduce to $\mathscr{R}_0$.

(d) If we use both vaccination control and drug therapy to control the spread of the disease, $(u_1 \neq 0\ ,\ u_2 \neq 0)$, then the basic reproduction rate will become downward and the control of susceptible and infected populations will lead to $\mathscr{R}_0 < 1$. That is, the disease will disappear and will not spread in society.

Therefore, it is concluded that the simultaneous application of two controllers, vaccination and drug therapy, in society will cause a rapid and large reduction of $\mathscr{R}_0$.

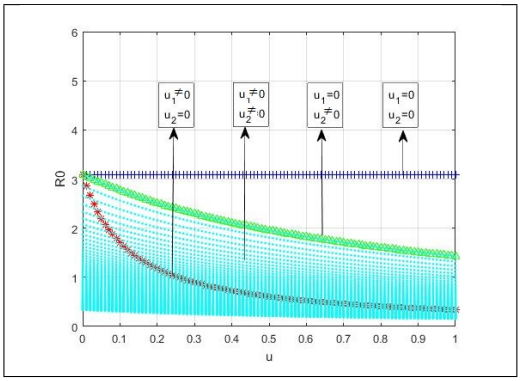

Figure 1: The behavior of $\mathscr{R}_0$ both with control and without control

3.2 Stability of equilibria

In this section, we discuss the local stability of equilibria whose existence have been stated in the previous analysis. The following lemma is used to demonstrate the local stability of equilibria.

Lemma 2. *(Lemma 3, [17]). Let M be a 3×3 real matrix. If $tr(M)$, $\det(M)$ and $\det(M^{[2]})$ are all negative, then all eigenvalues of M have negative real part.*

Theorem 1. *(i) If $\mathscr{R}_0 < 1$, then DFE of (1) is locally asymptotically stable.*

(ii) If $\mathscr{R}_0 > 1$, then EE of (1) is locally asymptotically stable.

Proof. (i) If $\mathscr{R}_0 < 1$, then

$$\rho\Lambda(\theta+\varphi-q\Lambda)-\varphi(\theta+\varphi+mu_1-q\Lambda)(\varphi+\varphi_1+\xi+nu_2-p\Lambda)<0.$$

The Jacobian matrix evaluated at $\mathscr{P}_f$ is

$$J_{\mathscr{P}_f}=\begin{bmatrix} -\varphi-mu_1 & -p\Lambda-\rho S_f & -q\Lambda+\theta \\ 0 & p\Lambda-\varphi-\varphi_1-\xi-nu_2+\rho S_f & 0 \\ mu_1 & nu_2+\xi & q\Lambda-\theta-\varphi \end{bmatrix}.$$

Also the second compound of the Jacobian matrix is

$$J^{[2]}_{\mathscr{P}_f}=\begin{bmatrix} J^{[2]}_{\mathscr{P}_{f_{1,1}}} & 0 & q\Lambda-\theta \\ nu_2+\xi & J^{[2]}_{\mathscr{P}_{f_{2,2}}} & -p\Lambda-S_f \\ -mu_1 & 0 & J^{[2]}_{\mathscr{P}_{f_{3,3}}} \end{bmatrix},$$

where

$$\begin{aligned} J^{[2]}_{\mathscr{P}_{f_{1,1}}} &= -\varphi-mu_1+p\Lambda-\varphi-\varphi_1-\xi-nu_2+\rho S_f, \\ J^{[2]}_{\mathscr{P}_{f_{2,2}}} &= -2\varphi-mu_1+q\Lambda-\theta, \\ J^{[2]}_{\mathscr{P}_{f_{3,3}}} &= p\Lambda-2\varphi-\varphi_1-\xi-nu_2+\rho S_f+q\Lambda-\theta. \end{aligned}$$

Therefore

$$\begin{aligned} tr(J_{\mathscr{P}_f}) &= \frac{-\varphi(\theta+\varphi+mu_1-q\Lambda)^2-\varphi^2(\theta+\varphi+mu_1-q\Lambda)-\theta(J_{\mathscr{P}_f})}{\varphi(\theta+\varphi+mu_1-q\Lambda)} < 0, \\ \det(J_{\mathscr{P}_f}) &= \rho\Lambda(\theta+\varphi-q\Lambda)-\varphi(\theta+\varphi+mu_1-q\Lambda)(\varphi+\varphi_1+\xi+nu_2-p\Lambda) < 0, \\ \det(J^{[2]}_{\mathscr{P}_f}) &= -(\varphi+mu_1+\theta+\varphi-q\Lambda)(J^{[2]}_{\mathscr{P}_{f_{1,1}}}J^{[2]}_{\mathscr{P}_{f_{3,3}}}+(q\Lambda-\theta)(mu_1)) < 0. \end{aligned}$$

(ii) If $\mathscr{R}_0 > 1$, then

$$\Lambda\rho(\theta+\varphi-q\Lambda)-\varphi(\theta+\varphi+mu_1-q\Lambda)(\varphi+\varphi_1+\xi+nu_2-p\Lambda) > 0.$$

The Jacobian matrix evaluated at $\mathscr{P}_e$ is

$$J_{\mathscr{P}_e} = \begin{bmatrix} -\varphi-mu_1-\rho I_e & -p\Lambda-\rho S_e & -q\Lambda+\theta \\ \rho I_e & 0 & 0 \\ mu_1 & nu_2+\xi & q\Lambda-\theta-\varphi \end{bmatrix},$$

and

$$J^{[2]}_{\mathscr{P}_e} = \begin{bmatrix} -\varphi-mu_1-\rho I_e & 0 & q\Lambda-\theta \\ nu_2+\xi & J^{[2]}_{\mathscr{P}_{e_{2,2}}} & -p\Lambda-S_e \\ -mu_1 & \rho I_e & q\Lambda-\theta-\varphi \end{bmatrix},$$

where

$$J^{[2]}_{\mathscr{P}_{e_{2,2}}} = -d-mu_1-\rho I_e+q\Lambda-\delta-d.$$

Therefore

$$\begin{aligned} tr(J_{\mathscr{P}_e}) &= -2\varphi-mu_1-\rho I_e+q\Lambda-\theta < 0, \\ \det(J_{\mathscr{P}_e}) &= -\rho I_e\Big[(\varphi+\varphi_1)(-q\Lambda+\theta+\varphi)+(\xi+nu_2)(-q\Lambda+\theta+\varphi-q\Lambda+\theta)+ \\ &\quad (\varphi+\varphi_1)(-q\Lambda+\theta+\varphi)+\varphi(\xi+m_2u_2)\Big] < 0, \\ \det(J^{[2]}_{\mathscr{P}_e}) &= -(\varphi+mu_1+\rho I_e-q\Lambda+\theta+\varphi)\Big[(\varphi+\rho I_e)(-q\Lambda+\theta+\varphi)+mu_1\varphi\Big] \\ &\quad -\rho I_e\Big[(\varphi+mu_1+\rho I_e)(p\Lambda+\rho S_e)+(\theta-q\Lambda)(nu_2+\xi)\Big] < 0. \end{aligned}$$

Therefore, by Lemma 2 the eigenvalues of $J_{\mathscr{P}_f}$ and $J_{\mathscr{P}_e}$ all have negative real parts and hence

(i) If $\mathscr{R}_0 < 1$, then DFE of (1) is locally asymptotically stable.

(ii) If $\mathscr{R}_0 > 1$, then EE of (1) is locally asymptotically stable.

□

4 Optimal control problem

In the previous section, we studied the asymptotic stability of DFE and EE equilibrium points under the conditions for system (1) parameters. Applying the controllers of vaccination and drug therapy to control the disease in society would impose some costs on society. These costs can be either material or moral. The vaccination costs include the cost of purchasing and maintaining it as well as the negative effects of the vaccine, and drug therapy costs are the cost of medicine, hospital, testing and negative effects of treatment. Our duty is to optimize these costs. The best and most useful tool to achieve this goal is an optimal control problem, which will create limitations for the parameters of the problem. Hence, we define a control set as follows:

$$\mathscr{U} = \Big\{(u_1(t),u_2(t))|\ \ 0 \leqslant u_i \leqslant u_{i\max} \leqslant 1\ ,\ i=1,2\ ,\ t\in[0,t_f]\Big\},$$

which is Lebesgue measurable.

Based on these considerations, we define the following optimal control problem:

$$\min_{u_1,u_2\in\mathscr{U}} \quad J(u_1,u_2) = \int_0^{t_f} [A_1S(t)+A_2I(t)+\frac{1}{2}(C_1u_1^2+C_2u_2^2)]dt \tag{2}$$

$$s.t:$$

$$\begin{cases} \dot{S}(t) = \Lambda - p\Lambda I - q\Lambda R - \varphi S - \rho SI - mu_1S + \theta R, \\ \dot{I}(t) = \rho SI + p\Lambda I - (\varphi+\varphi_1)I - \xi I - nu_2I, \\ \dot{R}(t) = q\Lambda R + mu_1S + nu_2I - \varphi R - \theta R + \xi I, \\ S(0)\geqslant 0,\ \ I(0)\geqslant 0,\ \ R(0)\geqslant 0. \end{cases}$$

Here, t_f is the final time and A_1, A_2 are positive weights to keep a balance in the size of susceptible and infected population, C_1 and C_2 are the costs associated with vaccination and treatment respectively. The square of the disease control parameter is taken to remove some unwanted side effects of the disease as well as to consider the overdoses of the control [13]. Our goal is to minimize the objective function (2), that is, we need to seek the optimal control function $(u_1^*(t),u_2^*(t))\in\mathscr{U}$ satisfying

$$J(u_1^*,u_2^*) = \min_{u_1,u_2\in\mathscr{U}} J(u_1,u_2).$$

The existence of an optimal control pair is guaranteed by the compactness of the control and the states spaces, and the convexity in the problem based on Theorem 4.1 in [11].

For optimality conditions, first we find the Lagrangian and Hamiltonian for problem (2) [13, 15, 19]:

$$L(S(t),I(t),u) = A_1S(t)+A_2I(t)+\frac{1}{2}C_1u_1^2+\frac{1}{2}C_2u_2^2,$$

and

$$\begin{aligned} H(S,I,R,u_1,u_2,\lambda_1,\lambda_2,\lambda_3,t) &= L(S(t),I(t),u_1,u_2)+\lambda_1\dot{S}(t)+\lambda_2\dot{I}(t)+\lambda_3\dot{R}(t) \\ &= A_1S(t)+A_2I(t)+\frac{1}{2}C_1u_1^2+\frac{1}{2}C_2u_2^2 \\ &\quad +\lambda_1(\Lambda - p\Lambda I - q\Lambda R - \varphi S - \rho SI - mu_1S + \theta R) \\ &\quad +\lambda_2(\rho SI + p\Lambda I - (\varphi+\varphi_1)I - \xi I - nu_2I) \\ &\quad +\lambda_3(q\Lambda R + mu_1S + nu_2I - dR - \theta R + \xi I), \end{aligned}$$

where λ_i, $i = 1,2,3$ are the adjoint variables, which are determined by solving the following equations:

$$\frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial S} = -A_1 + \lambda_1(\varphi + \rho I + mu_1) - \lambda_2 \rho I - \lambda_3 mu_1,$$
$$\frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial I} = -A_2 + \lambda_1(\rho S + p\Lambda) - \lambda_2(\rho S + p\Lambda - \xi - \varphi - \varphi_1 - nu_2) - \lambda_3(\xi + nu_2),$$
$$\frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial R} = -\lambda_1 \theta - \lambda_3(q\Lambda - \theta - \varphi),$$

and the transversal conditions, $\lambda_i(t_f) = 0$ $i = 1,2,3$. By using Pontryagin minimum principle, we can obtain the optimal conditions as follows:

$$\frac{\partial H}{\partial u_i} = 0, \quad i = 0,1.$$

Then

$$u_1^* = \min\{\max\{0, \frac{(\lambda_1 - \lambda_3)mS^*}{C_1}\}, u_{1\max}\},$$
$$u_2^* = \min\{\max\{0, \frac{(\lambda_2 - \lambda_3)nI^*}{C_2}\}, u_{2\max}\}.$$

The above analysis can be expressed as the following theorem

Theorem 2. *Let (S^*, I^*, R^*) be the optimal state solution related to optimal controls (u_1^*, u_2^*) for the optimal control problem (2). Then there exist adjoint variables λ_i $(i = 1,2,3)$ such that*

$$\frac{d\lambda_1(t)}{dt} = -A_1 + \lambda_1(\varphi + \rho I + mu_1) - \lambda_2 \rho I - \lambda_3 mu_1,$$
$$\frac{d\lambda_2(t)}{dt} = -A_2 + \lambda_1(\rho S + p\Lambda) - \lambda_2(\rho S + p\Lambda - \xi - \varphi - \varphi_1 - nu_2) - \lambda_3(\xi + nu_2),$$
$$\frac{d\lambda_3(t)}{dt} = -\lambda_1 \theta - \lambda_3(q\Lambda - \theta - \varphi),$$

with transversally conditions, $\lambda_i(t_f) = 0$, $i = 1,2,3$. Moreover, the optimal controls (u_1^*, u_2^*) which minimizes problem (2) over the region $\mathscr{U}$ can be shown as following:

$$u_1^* = \min\{\max\{0, \frac{(\lambda_1 - \lambda_3)mS^*}{C_1}\}, u_{1\max}\},$$
$$u_2^* = \min\{\max\{0, \frac{(\lambda_2 - \lambda_3)nI^*}{C_2}\}, u_{2\max}\}.$$

5 Numerical simulations

In this section, we discuss control problem (2) numerically where parameters, values are as in Table 1, and we consider the initial values as in [2]:

$$S_0 = 0.493 \qquad I_0 = 0.035 \qquad R_0 = 0.0035.$$

To investigate the problem, we will consider two different situations including with and without control. The RK4 method will be used to solve the problem. The results are as follows:

Table 1: Definitions of parameters used in model (1)

Symbol	Description	Values range	Reference
p	Fraction of newborns are infected	0.11	[2]
q	Fraction of newborns are immune	0.1	[2]
φ	Rate of death population	0.002	[1]
φ_1	Rate of death population by infectious disease	0.0008	[1]
Λ	Rate of birth population	0.0121	[2]
θ	Loss of immunity rate	0.1	[1]
ρ	Transmission coefficient	0.125	[1]
ξ	Rate moving from I to R	0.025	[1]
u_1	Rate of vaccination per year	$0 \leq u_1 \leq 1$	-
u_2	Rate of antiviral drug per year	$0 \leq u_2 \leq 1$	-
m	Efficacy of vaccine	0.81	[6]
n	Efficacy of antiviral drug	0.71	[6]

(a) According to Figure 2, the susceptible population will increase if it is not vaccinated in order to control the incidence of the disease in society, which will increase the horizontal transmission and spread and epidemic of the disease in society. In this case, the disease will turn into an epidemic. But the susceptible population will decrease if the control and vaccination increase and as a result, the infected population will also decrease, and with the passage of time and the continuation of vaccination, the horizontal transmission will decrease and the disease will disappear.

(b) According to Figure 3, the infected population will increase if it is not treated in order to control the incidence of the disease in society, and disease will turn into an epidemic, which will increase the vertical transmission in society. But the infected population will decrease if the control and drug therapy increase, and with the passage of time and the continuation of drug therapy, vertical transmission will decrease and the disease will disappear.

(c) According to Figure 4:

(i) If we only use drug therapy to control the infected population and vaccination is at zero, the number of recovered population will increase in a certain period of time, but after a while, the number of recovered population will decrease and the more control we increase, the population will increase. The improvement will be more downward, and the control and drug therapy will give the opposite result from time to time.

(ii) If we only use the vaccination of the susceptible population to control the disease and drug therapy is at zero, with the increase of vaccination and the continuation of it at the social level, the number of recovered population will increase.

(iii) If we simultaneously use both vaccination and drug therapy to control and prevent the incidence of the disease at the social level, the speed in the increase of recovered population

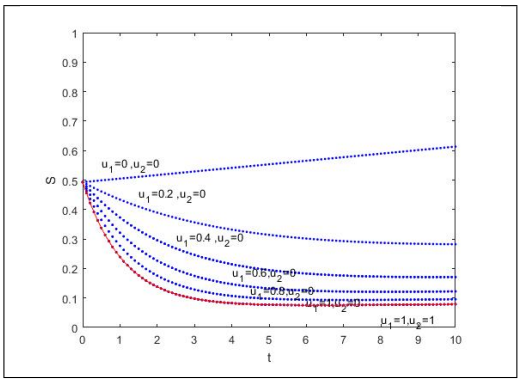

Figure 2: Dynamical behavior of the susceptible populations for different values of u_1 and u_2.

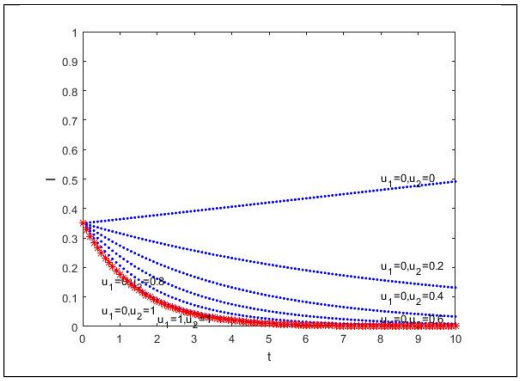

Figure 3: Dynamical behavior of the infected populations for different values of u_1 and u_2.

increase will be very high compared to the previous two cases and this method of controlling infectious diseases at the social level will have good results.

6 Conclusions

This article presented and explained a model for the horizontal and vertical transmission as well as control of an infectious disease in society. Initially, we investigated the dynamic behavior of system and calculated the equilibrium points and reproduction rate of this system which includes two equilibrium points, disease-free and endemic. We proved that based on $(\mathscr{R}_0 < 1)$ and $(\mathscr{R}_0 > 1)$, the equilibrium points are locally asymptotically stable. With a numerical example, we showed that base multiplication rate will decrease if the vaccination and drug therapy increase in society which will be a reason for the disappearance of the disease in society. Then, we formulated the optimal control problem and solved the problem using Pontryagin's minimum principle. With a numerical example, we showed that increasing vaccination in society will lead to the decrease in horizontal transmission and increasing drug treatment causes a decrease in the infected population and vertical transmission. The simultaneous use of these two

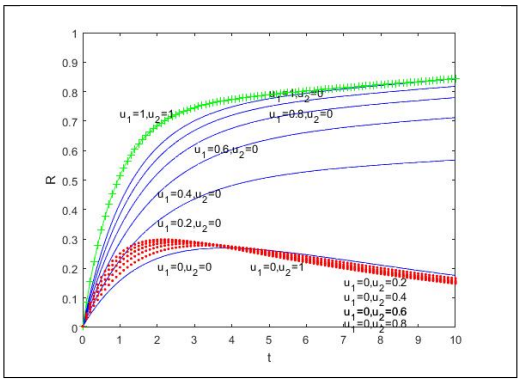

Figure 4: Dynamical behavior of the recovered populations for different values of u_1 and u_2.

controllers will be the best way to control the disease in society, which will cause a sharp reduction in the horizontal and vertical transmission of the disease and a rapid increase in the recovered population.

References

[1] A.S. Ahmad, S. Owyed, A.H. Abdel-Aty, E.E. Mahmoud, K. Shah, H. Alrabaiahg, *Mathematical analysis of COVID-19 via new mathematical model*, Chaos Solit. **143** (2021) 110585.

[2] R. Akbari, A.V. Kamyad, A.A. Heydari, A. Heydari, *The analysis of a disease-free equilibrium of hepatitis B model*, Sahand Commun. Math. Anal. **3** (2016) 1–11.

[3] R. Akbari, A.V. Kamyad, A.A. Heydari, A. Heydari, *Stability analysis of the transmission dynamics of an HBV model*, Int. J. Ind. Math. **8** (2016) 119–129.

[4] R.M. Anderson, R.M. May, *Infectious Diseases of Humans Dynamics and Control*, Oxford University Press, 1991.

[5] E.A. Bakar, A. Nwagwo, E. Danso-Addo, *Optimal control analysis of an SIR epidemic model with constant recruitment*, Int. J. Appl. Math. Res. **3** (2014) 273–285.

[6] S. Bhattacharyyaa, S. Ghosh, *Optimal control of vertically transmitted disease*, Comput. Math. Methods Med. **11** (2010) 369–387.

[7] G. Birkhoff, G.C.C. Rota, G. Birkoff, G. C. Rota, *Ordinary Differential Equations*, Ginn, Boston, 1982.

[8] S. Boccaletti, G. Bianconi, R. Criado, C.I. del Genio, J. Gmez-Gardees, M. Romance, I. Sendia-Nadal, Z. Wang, M. Zanin, *The structure and dynamics of multilayer networks*, Phys. Rep. **544** (2014) 1–122.

[9] O. Diekmann, J.A.P. Heesterbeek, *Mathematical Epidemiology of Infective Diseases: Model Building, Analysis and Interpretation*, Wiley, New York, 2000.

[10] P.V.D. Driessche, J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental systems of disease transmission*, Math. Biosci. **180** (2002) 29–48.

[11] W. Fleming, R. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

[12] H.W. Hethcote, *The mathematics of infectious diseases*, SIAM Rev. **42** (2000) 599–653.

[13] S. Janaa, P. Haldarb, T.K. Kar, *Optimal control and stability analysis of an epidemic model with population dispersal*, Chaos, Solitons & Fractals **83** (2016) 67–81.

[14] A.V. Kamyad, R. Akbari, A.A. Heydari, A. Heydari, *Mathematical modeling of transmission dynamics and optimal control of vaccination and treatment for hepatitis B Virus*, Comput. Math. Methods Med. **2014**, Article ID 475451.

[15] T.K. Kar, A. Batabyal, *Stability analysis and optimal control of an SIR epidemic model with vaccination*, Biosyst. **104** (2011) 127–135.

[16] M.J. Keeling, P. Rohani, *Modeling Infectious Diseases in Humans and Animals*, Princeton University Press, Princeton, NJ, 2007.

[17] C.C. McCluskey, P.V.D. Driessche, *Global analysis of two tuberculosis models*, J. Dyn. Differ. Equ. **16** (2004) 139–166.

[18] J.S. Muldowney, *Compound matrices and ordinary differential equations*, Rocky Mt. J. Math. **20** (1990) 857-872.

[19] R. Shi, Y. Li, C. Wang, *Stability analysis and optimal control of a fractional-order model for African swine fever*, Virus Res. **288** (2020) 198111.

[20] L. Wang, X. Li, *Spatial epidemiology of net worked metapopulation: an overview*, Chin. Sci. Bull. **59(28)** (2014) 3511–3522.

[21] L. Wang, Z. Wang, Y. Zhang, X. Li,*How human location-specific contact patterns impact spatial transmission between populations*, Sci. Rep. **3** (2013) 1468.

[22] C. Xia, Z. Wang, J. Sanz, *Effects of delayed recovery and nonuniform transmission on the spreading of diseases in complex networks*, Physica A. **392** (2013) 1577-1585.

[23] J. Zhang, *The Geometric Theory and Bifurcation Problem of Ordinary Differential Equations*, Peking University Press, Beijing, 1987.

[24] G. Zhang, Q. Sun, L. Wang, *Noise-induced enhancement of network reciprocity in social dilemmas*, Chaos, Solitons & Fractals **51** (2013) 31–35.

[25] S. Zhang, Y. Zhou, *The analysis and application of an HBV model*, Appl. Math. Model. **36** (2012) 1302–1312