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Eigenvalue problem with fractional differential operator: Chebyshev cardinal spectral method

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Abstract. In this paper, we intend to introduce the Sturm-Liouville fractional problem and solve it using the collocation method based on Chebyshev cardinal polynomials. To this end, we first provide an introduction to the Sturm-Liouville fractional equation. Then the Chebyshev cardinal functions are introduced along with some of their properties and the operational matrices of the derivative, fractional integral, and Caputo fractional derivative are obtained for it. Here, for the first time, we solve the equation using the operational matrix of the fractional derivative without converting it to the corresponding integral equation. In addition to efficiency and accuracy, the proposed method is simple and applicable. The convergence of the method is investigated, and an example is presented to show its accuracy and efficiency.

Keywords: collocation method, fractional Sturm-Liouville eigenvalue problem, Chebyshev cardinal functions.
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1 Introduction

One of the most important theories that play a brilliant role in modern mathematical analysis is the general theory of Sturm-Liouville differential equations. This theory was first presented in 1837 in a joint article by Sturm and Liouville and has been used in the analysis of many problems related to mathematics, physics and other branches of science. The results of their research have been widely used over the years and many results in delayed differential equations, differential and functional equations, as well as partial differential equations are obtained by using them [3, 5, 6].

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In Sturm-Liouville theory, the differential equation of the form

$$-\frac{d}{dx}(p(x)\frac{du}{dx})+f(x)u(x)=\lambda w(x)u, \quad x\in(a,b), \tag{1}$$

is studied, in which the coefficients $p(x)$, $f(x)$ and $w(x)$ are called the Sturm-Liouville coefficients. These coefficients should be satisfied the following minimal conditions

- $p,f,w:(a,b)\to\mathbb{R}$,
- $p^{-1},f,w\in L_c^1(a,b)$,
- w is a weight function on (a,b),

where the space $L_c^1(a,b)$ consists of the complex-valued functions on a compact interval, which are Lebesgue integrable on all compact sub-intervals. In this problem, λ is called eigenvalue and finding it is a part of solving the problem.

When the Sturm-Liouville coefficients satisfy the minimal condition, it can be shown that the Sturm-Liouville problem with initial-boundary values has a solution [17]. If the Sturm-Liouville coefficients and p' are continuous functions, and also the functions p and w are positive functions, then the Sturm-Liouville equation with boundary conditions

$$\begin{aligned} c_1u(a)+c_2u'(a)=0, \quad c_1^2+c_2^2>0,\\ d_1u(a)+d_2u'(a)=0, \quad d_1^2+d_2^2>0, \end{aligned}$$

is called the regular Sturm-Liouville problem. For such an equation, the eigenvalues are real, and we have $\lambda_1<\lambda_2<\cdots\to+\infty$. One can find a unique eigenfunction corresponding to the eigenvalue λ_i $(i=1,2,\ldots)$. After normalization through weighted inner product, these eigenfunctions introduce a system of orthonormal basis.

Finding eigenvalues and eigenfunctions for this problem is very valuable considering that Sturm-Liouville equation plays a very important role in both mathematics and mathematical physics. Among the applications of this problem, we can mention the time-independent one-dimensional Schrödinger model, which is one of the subjects raised in quantum mechanics. Also, this equation usually appears in applying the Separation of variables method on partial differential equations such as the wave equation, Laplace's equation and the heat equation [3]. Due to the variable coefficients in this equation, there is no analytical method that can solve a wide range of this type of equation. Therefore, we must introduce appropriate numerical methods to overcome this problem. Among the existing methods, we may mention the Haar wavelet method [7], boundary Value Methods [11], Sinc-Galerkin and differential transform methods [4], differential quadrature method [20], homotopy analysis method [1], Sinc collocation method and differential quadrature method [9].

Due to the importance and application of fractional calculations and fractional equations in various sciences, the study of this branch of mathematics has attracted the attention of mathematicians. The importance of the fractional Sturm-Liouville equation can be found in its relationship with the fractional diffusion in a bounded domain [14]. Here, we refer to some of the numerical methods that are proposed previously, including the Variational method [15], analytical solutions [18], Adomian decomposition method [19], fractional differential transform method [10], and Finite element method [12].

In this work, the problem we are looking to find its eigenvalues and eigenfunctions is the Sturm-Liouville fractional problem

$$
{}^{C}\mathscr{D}_0^{\alpha}\left(p\mathscr{D}(u)\right)(x)+h(x)\mathscr{D}(u)(x)+q(x)u(x)+\lambda\sigma(x)u(x)=0,\quad x\in(0,1),\alpha\in\mathbb{R}^{+}, \tag{2}
$$

in which $\mathscr{D}$ and ${}^{C}\mathscr{D}_0^{\alpha}$ assign the derivative operator and the Caputo fractional derivative, respectively. In this equation, $p(x)\neq 0$, $q(x)$ is the potential, and $\sigma(x)$ the positive weight function. On the other hand, the functions $p(x)$, $h(x)$, $q(x)$, and $\sigma(x)$ are assumed to be sufficiently regular. This equation includes boundary conditions

$$
\begin{cases}
\sum_{j=0}^{n} d_{i,j}u^{(j)}(a)=0, & i=0,\dots,J-1,\\
\sum_{j=0}^{n} d_{i,j}u^{(j)}(b)=0, & i=J,\dots,n,
\end{cases} \tag{3}
$$

where $J<n$, and the coefficients $\{d_{i,j}\}$ for $i,j=0,\dots,n$ are constant.

The outline of this paper can be summarized as follows. In Section 2, the Chebyshev cardinal functions of the first kind and their properties are briefly introduced. Also, the fractional matrices of derivative, fractional integration and the Caputo fractional derivative are obtained in this section. Section 3 is dedicated to applying the collocation method to solve the desired equation. The convergence analysis also investigates in this section, and its result is mentioned as a Theorem. In Section 4, we illustrate the efficiency and accuracy of the proposed method using a numerical example.

2 Chebyshev cardinal functions of the first kind

Given positive integer number ω, we put $\mathscr{Y}$ as a set of the roots of the Chebyshev polynomial of the first kind $T_{\omega+1}$, i.e.,

$$
\mathscr{Y}:=\{y_j : T_{\omega+1}(y_j)=0, j\in\Omega\},\quad \Omega:=\{1,2,\dots,\omega+1\},
$$

where $\{y_j\}_{j\in\Omega}$ are the roots of the Chebyshev polynomial $T_{\omega+1}$ on $[-1,1]$ and are given by

$$
y_j:=\cos\left(\frac{2j-1}{2\omega+2}\pi\right),\quad \forall j\in\Omega. \tag{4}
$$

To generate the shifted Chebyshev polynomials $T^{*}_{\omega+1}$ on an arbitrary interval $[a,b]$, we take the change of variable as follows

$$
T^{*}_{\omega+1}(x):=T_{\omega+1}\left(\frac{2(x-a)}{b-a}-1\right),\quad x\in[a,b]. \tag{5}
$$

It follows from the change of variable $y=\left(\frac{2(x-a)}{b-a}-1\right)$ that the roots of $T^{*}_{\omega+1}$ can be obtained by $x_j=\frac{(y_j+1)(b-a)}{2}+a$. A significant case of the cardinal functions that use the orthogonal polynomials is the Chebyshev cardinal functions. We define these functions as

$$
\psi_j(x)=\frac{T^{*}_{\omega+1}(x)}{T^{*}_{\omega+1,x}(x_j)(x-x_j)},\quad j\in\Omega, \tag{6}
$$

where the subscript x indicates differentiation with respect to x. The main property of the cardinal functions is that they satisfy

$$\psi_j(x_i) = \delta_{ji}, \quad i \in \Omega, \tag{7}$$

in which δ_{ji} specifies the Kronecker δ-function. Given $n \in \mathbb{N}$, we denote the Sobolev space by $H^n([a,b])$ consisting of all functions $p \in C^n([a,b])$ such that $\mathscr{D}^{n'} p \in L^2([a,b])$ for all $\mathbb{N} \ni n' \leq n$, in which $\mathscr{D}$ assigns the derivative operator. It follows from the definition of the Chebyshev cardinal functions that any function p can be represented as an expansion based on these polynomials, i.e.,

$$p_\omega(x) \approx \sum_{j=1}^{\omega+1} p(x_j)\psi_j(x). \tag{8}$$

Lemma 1 ([8]). *Let $n \in \mathbb{N}$ be given and let $\{x_j\}_{j\in\Omega}$ be the shifted Gauss-Chebyshev points. We say that the error of expansion (8) can be bounded and we have*

$$\|p - p_\omega\|_{L^2([a,b])} \leq C_0 \omega^{-n} |p|_{H^{n,\omega}([a,b])}, \tag{9}$$

where C_0 is a constant and independent of n.

2.1 Operational matrix of derivative

The purpose of introducing the operational matrix is to simplify calculations. In fact, by finding the general form to represent the derivative operator based on bases, we will no longer need to find the derivative when applying the method. To this end, we introduce the $(\omega+1)$-dimensional vector function $\Psi(x)$ whose j-th element is equal to $\psi_j(x)$. The operational matrix of the derivative satisfies the relation

$$\mathscr{D}(\Psi)(x) = D\Psi(x), \tag{10}$$

where $\mathscr{D}$ denotes the derivative operator. Motivated by (8) and (10), it is easy to demonstrate that the entries of the matrix D are computed by

$$D_{j,i} = \mathscr{D}(\psi_j)(x_i). \tag{11}$$

To simplify the computations, there is another expression for Chebyshev cardinal polynomials [2], viz

$$\psi_j(x) = \rho \prod_{k=1,k\neq j}^{\omega+1} (x - x_k), \tag{12}$$

where $\rho = 2^{2\omega+1}/((b-a)^{\omega+1} T^*_{\omega+1,x}(x_j))$. To derive the elements of D, by taking the derivative from both sides of equation (12) with respect to x, we obtain

$$\begin{aligned}
\mathscr{D}(\psi_j)(x) &= \rho \mathscr{D} \prod_{\substack{k=1\\k\neq j}}^{\omega+1} (x - x_k) = \rho \sum_{\substack{l=1\\l\neq j}}^{\omega+1} \prod_{\substack{k=1\\k\neq j,l}}^{\omega+1} (x - x_k) \\
&= \sum_{\substack{l=1\\l\neq j}}^{\omega+1} \frac{T^*_{\omega+1}(x)}{(x-x_j)(x-x_l)T^*_{\omega+1,x}(x_j)} \\
&= \sum_{\substack{l=1\\l\neq j}}^{\omega+1} \frac{1}{(x-x_l)} \psi_j(x).
\end{aligned} \tag{13}$$

Now, it is easy to verify that if $i = j$, we have

$$\mathscr{D}(\psi_j)(x_i) = \sum_{\substack{l=1\\ l\neq i}}^{\omega+1} \frac{1}{(x_i - x_l)}, \tag{14}$$

and if $i \neq j$, we got

$$\mathscr{D}(\psi_j)(x_i) = \rho \prod_{\substack{l=1\\ l\neq i,j}}^{\omega+1} (x_i - x_l). \tag{15}$$

2.2 Operational matrix of fractional integration

Before introducing the operational matrix of fractional integration, let us present the definition of fractional integration.

Definition 1. *The Riemann-Liouville fractional integral operator $\mathscr{I}_0^\alpha$ of order $\alpha > 0$ $(\Re(\alpha) > 0)$ is defined as*

$$(\mathscr{I}_a^\alpha u)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \frac{u(t)}{(x-t)^{1-\alpha}} dt, \quad (x \in [0,1]; \quad \Re(\alpha) > 0), \tag{16}$$

One can verify that the fractional integral of the power function is also a power function, by simplicity, i.e.,

$$\left(\mathscr{I}_0^\alpha (t)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (x)^{\beta+\alpha-1}. \tag{17}$$

Lemma 2 ([13])**.** *Assuming $\Re(\alpha) > 0$, the fractional integration operators $\mathscr{I}_0^\alpha$ can be bounded in $L_p(0,1)$ $(1 \leq p \leq \infty)$*

$$\begin{aligned} \|\mathscr{I}_0^\alpha u\|_p &\leq K \|u\|_p, \\ K &= \frac{(1-0)^{\Re(\alpha)}}{\Re(\alpha)\,|\Gamma(\alpha)|}. \end{aligned} \tag{18}$$

Considering the function $u \in L_1[0,1]$ and $\Re(\alpha) > 0$, then it is easy to confirm that the function $\mathscr{I}_0^\alpha u(x)$ is an element of $L_1[0,1]$ [13]. It follows from the definition of Chebyshev cardinal functions that the fractional integral from the Chebyshev cardinal functions is an $L_1[0,1]$ function. So, similar to the operational matrix of derivative, there exists a square matrix of dimension $(\omega+1)\times(\omega+1)$ such that the fractional integral of Chebyshev cardinal functions can be represented by it, viz

$$\mathscr{I}_0^\alpha \Psi(x) = I^\alpha \Psi(x). \tag{19}$$

Thus, our aim is to find the elements of matrix I^α. Motivated by (8), one can show that the elements of matrix I^α can be obtained by

$$I_{j,i}^\alpha = \mathscr{I}_0^\alpha \psi_j(x_i). \tag{20}$$

Before calculating these integrals, we can demonstrate

$$\prod_{\substack{k=1\\ k\neq i}}^{\omega+1} (x - x_k) = \sum_{k=0}^{\omega} r_{i,k} x^{\omega-k}, \tag{21}$$

in which

$$r_{i,0} = 1, r_{i,k} = \frac{1}{k}\sum_{l=0}^{k} s_{i,l} r_{i,k-l},\ i = 1,\ldots,\omega+1, k = 1,\ldots,\omega,$$

and

$$s_{i,k} = \sum_{\substack{j=1\\ j\neq i}}^{\omega+1} x_j^k,\ i = 1,\ldots,\omega+1, k = 1,\ldots,\omega.$$

Therefore, the Chebyshev cardinal functions can be re-determined as follows

$$\psi_j(x) = \rho\sum_{k=0}^{\omega} r_{j,k} x^{\omega-k}. \tag{22}$$

Substituting (22) in (20), one can write

$$\mathscr{I}_0^\alpha \psi_j(x) = \rho \mathscr{I}_0^\alpha (\sum_{k=0}^{\omega} r_{j,k} x^{\omega-k}) = \rho \sum_{k=0}^{\omega} r_{j,k} \mathscr{I}_0^\alpha (x^{\omega-k}) = \rho \sum_{k=0}^{\omega} r_{j,k} \frac{\Gamma(\omega-k+1)}{\Gamma(\omega-k+\alpha+1)} x^{\omega-k+\alpha}.$$

This gives rise to finding the elements of I^α, viz

$$I_{j,i}^\alpha = \rho \sum_{k=0}^{\omega} r_{j,k} \frac{\Gamma(\omega-k+1)}{\Gamma(\omega-k+\alpha+1)} x_i^{\omega-k+\alpha}. \tag{23}$$

2.3 Operational matrix of the Caputo fractional derivative

Given a finite interval $[a,b]$ $(-\infty < a < b < \infty)$, we specify the space of absolutely continuous functions on $[a,b]$ by $AC[a,b]$.

Definition 2. *Let $n \in \mathbb{N}$. We say that $u \in AC^n[a,b]$, if the function u has continuous derivatives up to order $n-1$ such that $u^{(n-1)} \in AC[a,b]$;*

$$AC^n[a,b] = \{u : [a,b] \to \mathbb{C}, \quad \& \quad \mathscr{D}^{(n-1)}(u) \in AC[a,b]\}.$$

Definition 3. *Let $\Re(\alpha) > 0$ and the number n is determined by*

$$n = \begin{cases} [\Re(\alpha)] + 1, & \alpha \notin \mathbb{N}_0, \\ \alpha, & \alpha \in \mathbb{N}_0. \end{cases} \tag{24}$$

If $u(x) \in AC^n[0,1]$, the Caputo fractional derivative $({}^c\mathscr{D}_0^\alpha u)(x)$ exists for almost every $x \in [0,1]$ [13], and we have

$$({}^c\mathscr{D}_0^\alpha u)(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} =: (I_{0+}^{n-\alpha} \mathscr{D}^n u)(x), \tag{25}$$

Lemma 3. *[13] Let $\Re(\alpha) > 0$ and $n = -[\alpha]$. The Caputo fractional derivative of power function is also a power function, i.e.,*

$$({}^c\mathscr{D}_0^\alpha (x)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (x)^{\beta-\alpha}, \quad (\Re(\beta) > n). \tag{26}$$

In the sequel, we aim to find a square matrix D^{α} that satisfies the following relation

$$ {}^{c}\mathscr{D}_{0}^{\alpha}(\Psi(x)) \approx D^{\alpha}\Psi(x). \tag{27} $$

To find the entries of the matrix D^{α}, the equation (25) can be used as follows

$$ {}^{c}\mathscr{D}_{0}^{\alpha}(\Psi(x)) = \mathscr{I}_{0}^{n-\alpha}\mathscr{D}^{n}(\Psi(x)) \approx D^{n}(I^{n-\alpha})\Psi(x). $$

Thus, the operational matrix of Caputo fractional derivative D^{α} is obtained via

$$ D^{\alpha} := D^{n}(I^{n-\alpha}). \tag{28} $$

3 Collocation method

Assume that $\mathscr{P}$ is the projection operator that maps any continuous function onto the space $\prod_{\omega}$ where $\prod_{\omega}$ denotes the space of all polynomials of degree ω. In the first step, we suppose that the solution of the Sturm-Liouville equation (2) can be approximated using the Chebyshev cardinal functions, i.e.,

$$ u(x) \approx \mathscr{P}(u)(x) = U^{T}\Psi(x) := u_{\omega}(x), \tag{29} $$

where U is an $(\omega+1)$-dimensional vector whose elements should be specified. Using the operational matrix of derivative D, one can also approximate the derivative of the unknown function u, viz

$$ \mathscr{D}(u)(x) \approx \mathscr{P}(\mathscr{D}(u))(x) = U^{T}D\Psi(x) := u'_{\omega}(x). \tag{30} $$

Putting (29) and (30) back into (2), we have

$$ {}^{C}\mathscr{D}_{0}^{\alpha}\left(pu'_{\omega}\right)(x) + h(x)u'_{\omega}(x) + q(x)u_{\omega}(x) + \lambda\sigma(x)u_{\omega}(x) = 0, \quad x \in (0,1). \tag{31} $$

To give rise to the collocation method, we have to map all terms in (31) onto the space $\prod_{\omega}$ using the projection operator $\mathscr{P}$, as follows.

- Let $f_1(x) := {}^{C}\mathscr{D}_{0}^{\alpha}(w_1)(x)$ where $w_1 := pu'_{\omega}$. Firstly, we map the function $w_1(x)$ onto the space $\prod_{\omega}$ using $\mathscr{P}$, viz

  $$ w_1(x) \approx \mathscr{P}(w_1)(x) = W_1^{T}\Psi(x) = U^{T}G_1\Psi(x), \tag{32} $$

  where G_1 is a square matrix of order $(\omega+1)$, and W_1 is an $(\omega+1)$-dimensional vector whose ith element is obtained by $p(x_i)u'(x_i)$. We can obtain such an equation due to the linearity of w_1. The function $f_1(x)$ can also be approximated using the fractional derivative matrix D^{α}, i.e.,

  $$ f_1(x) \approx \mathscr{P}(f_1)(x) = F_1^{T}\Psi(x) = {}^{C}\mathscr{D}_{0}^{\alpha}(U^{T}G_1\Psi)(x) = U^{T}G_1D^{\alpha}\Psi(x), \tag{33} $$

  in which F_1 is an $(\omega+1)$-dimensional vector.

- Putting $f_2(x) := h(t)u'_{\omega}(x)$, we have

  $$ f_2(x) \approx \mathscr{P}(f_2)(x) = F_2^{T}\Psi(x) = U^{T}G_2\Psi(x), \tag{34} $$

  where G_2 is a square matrix of order $(\omega+1)$, and F_2 is an $(\omega+1)$-dimensional vector whose ith element is obtained by $h(x_i)u'_{\omega}(x_i)$.

- The third and fourth terms can also be mapped onto space $\prod_\omega$ using the projection operator $\mathscr{P}$, viz

$$q(x)u_\omega(x) =: f_3(x) \approx \mathscr{P}(f_3)(x) = F_3^T \Psi(x) = U^T G_3 \Psi(x),$$
$$\sigma(x)u_\omega(x) =: f_4(x) \approx \mathscr{P}(f_4)(x) = F_4^T \Psi(x) = U^T G_4 \Psi(x), \quad (35)$$

in which matrices G_3 and G_4 are square matrix of order $(\omega+1)$, and the vectors F_3 and F_4 are $(\omega+1)$-dimensional vectors whose ith element is obtained by

$$(F_3)_i = q(x_i)u_\omega(x_i),$$
$$(F_4)_i = \sigma(x_i)u_\omega(x_i),$$

respectively.

Substituting equations (33)-(35) into (31), one can obtain the residual function $r(x)$ as

$$r(x) := \left(U^T G_1 D^\alpha + U^T G_2 + U^T G_3 + \lambda U^T G_4\right)\Psi(x) = 0. \quad (36)$$

In the collocation method, we select a collection of points in the domain called the collocation points to minimize the residual using these points. Setting the points $\{x_i\}_{i=1}^{\omega+1}$ as the collocation points and using (7), we have

$$U^T G_1 D^\alpha + U^T G_2 + U^T G_3 + \lambda U^T G_4 = 0, \quad (37)$$

Putting $\Upsilon(\lambda) := (G_1 D^\alpha + G_2 + G_3 + \lambda G_4)^T$, one can write

$$\Upsilon(\lambda)U = 0, \quad (38)$$

where $\Upsilon(\lambda)$ is a square matrix that depends on λ.

Notice that Sturm-Liouville equation (2) contains non-zero eigenvectors [2, 4, 16], so matrix $\Upsilon(\lambda)$ must be an invertible matrix when λ is an eigenvalue of Sturm-Liouville equation (2). This is equivalent to having

$$det(\Upsilon(\lambda)) = 0. \quad (39)$$

As we know, $det(\Upsilon(\lambda))$ is the characteristic polynomial of the matrix $\Upsilon(\lambda)$, and so λ is the root of this characteristic polynomial. Given $Q := \mathbb{R}$ or $Q := \mathbb{C}$, it is convenient to demonstrate that

$$U \in ker\{\Upsilon(\lambda)\} = \{U \in Q^{\omega+1+L} : \Upsilon(\lambda)U = 0\},$$

The vector U is forced to be a nonzero vector, so $\Upsilon(\lambda)$ has a nonzero kernel. Since the roots of the characteristic polynomial are calculated approximately, then $det(\Upsilon(\lambda))$ is not equal to zero for any selected λ exactly. Therefore, the eigenfunction is selected corresponding to the smallest eigenvalue of $\Upsilon(\lambda)$ as

$$u_\omega(x) = \frac{\sum_{i=1}^{\omega+1} U_i \psi_i(x)}{\|\sum_{i=1}^{\omega+1} U_i \psi_i(x)\|_2},$$

Here, the eigenfunction is divided by its norm to obtain its normalized state.

3.1 Convergence analysis

It follows from [8] that there is an optimal error approximation between $\mathscr{P}(\mathscr{D}u)$ and $\mathscr{D}u$, viz,

$$\|\mathscr{D}u - \mathscr{P}(\mathscr{D}u)\|_2 \le C_D \omega^{1-n} |u|_{H^{n,\omega}([a,b])}, \tag{40}$$

in which C_D is a constant.

Suppose that u_ω is an approximate solution obtained using the method presented in the previous section for the fractional Sturm-Liouville equation (2). Subtracting (2) from (31) and introducing $z_\omega := u - u_\omega$ as a difference between the exact and approximate solutions, we have

$$R_\omega(x) := {}^C\mathscr{D}_a^\alpha \left(p z'_\omega\right)(x) + h(x) z'_\omega(x) + q(x) z_\omega(x) + \lambda \sigma(x) z_\omega(x). \tag{41}$$

Now we present a theorem to prove the convergence of the proposed method as follows.

Theorem 1. *Consider the exact solution $u(x) \in H^n([0,1])$ of Eq. (2) is a sufficiently smooth function. Also, assume that the function $u_\omega(x)$ be an approximate solution obtained using the proposed method. Then, the residual decreases as ω tend to infinity, and we have*

$$\lim_{\omega \to \infty} \|R_\omega(x)\|_2 = 0. \tag{42}$$

Proof. Taking the L_2-norm from both sides of (41), we get

$$\begin{aligned} \|R_\omega(x)\|_2 &= \|{}^C\mathscr{D}_a^\alpha \left(p z'_\omega\right)(x) + h(x) z'_\omega(x) + q(x) z_\omega(x) + \lambda \sigma(x) z_\omega(x)\|_2 \\ &\le \|{}^C\mathscr{D}_a^\alpha \left(p z'_\omega\right)(x)\|_2 + \|h(x) z'_\omega(x)\|_2 + \|q(x) z_\omega(x)\|_2 + |\lambda| \|\sigma(x) z_\omega(x)\|_2, \end{aligned} \tag{43}$$

in which the triangle inequality is applied. Given that the functions $p(x)$, $q(x)$, $\sigma(x)$, and $h(x)$ are assumed to be continuous, therefore these functions have a maximum value in $[0,1]$. Assume that M is a fixed number such that

$$\max\{p(x), h(x), q(x), \sigma(x)\}_{x\in[0,1]} \le M. \tag{44}$$

Considering Lemma 1, 3 and using (40), one can write

$$\begin{aligned} \|R_\omega(x)\|_2 &\le M\left(\|{}^C\mathscr{D}_a^\alpha z'_\omega(x)\|_2 + \|h(x) z'_\omega(x)\|_2 + (1+|\lambda|)\|z_\omega(x)\|_2\right) \\ &\le KMC_D \omega^{1-n} |u(x)|_{H^{n,\omega}([a,b])} + (1+|\lambda|) MC\omega^{-n} |u(x)|_{H^{n,\omega}([a,b])}. \end{aligned}$$

Assuming $C_{\max} := \max\{C, C_D\}$, we can write this error bound more simply, i.e.,

$$\|R_\omega(x)\|_2 \le KM\delta C_{\max}(1+|\lambda|)\omega^{1-n} |u(x)|_{H^{n,\omega}([a,b])}.$$

According to the presented error limit, if $u(X)$ is a sufficiently smooth function and $n \ge 1$, then the error $\|R_\omega(x)\|_2$ will decrease exponentially and we have $\lim\limits_{\omega\to\infty} \|R_\omega(x)\|_2 = 0$. □

Table 1: An approximation of the first three eigenvalues for Example 1.

	Approximate λ					Exact λ
x	$\alpha=0.5$	$\alpha=0.6$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	$\alpha=1$
λ_1	2.121490	2.120820	2.231887	2.334695	2.467401	2.467401
λ_2	13.859078	14.084862	16.895113	19.211283	22.206609	22.206611
λ_3	24.238036	29.790027	41.982423	50.566853	61.685027	61.685029

4 Numerical experiments

In this section, we want to show the efficiency and accuracy of the proposed method by providing a numerical example.

Example 1. Consider the following fractional Sturm-Liouville equation

$$^{C}\mathscr{D}_0^{\alpha} u'(x) + \lambda u(x) = 0, \quad x \in [0,1],$$

with boundary conditions $u'(0)=0$ and $u(1)=0$. We can find the exact eigenvalues and eigenfunctions when $\alpha=1$ they are equal to

$$u_i = \cos(\lambda_i x), \quad \lambda_i = (i\pi + \pi/2)^2,$$

To demonstrate the accuracy and efficiency of the method, we have provided tables and figures for this example using the presented method. Table 1 is reported to show the approximate solution of the 3 first eigenvalue of this example for different choices of α . According to the approximations presented in this table for $\alpha=1$ and comparing it with the exact value, the accuracy of the presented method is obvious. We know that when the order of the fractional derivative tends to the integer value, the fractional derivative will also tend to corresponding integer derivative, i.e.,

$$\lim_{\alpha\to n} {}^{c}D^{\alpha} f(x) = f^{(n)}(x),$$
$$\lim_{\alpha\to n-1} {}^{c}D^{\alpha} f(x) = f^{(n-1)}(x) - f^{(n-1)}(0).$$

This can also be easily seen based on the results presented in Figure 1. In this Figure, the eigenfunctions for the first and second eigenvalues and for different choices of α are presented. This figure also confirms the convergence of the proposed method. We have also presented the Figure 2 to illustrate the eigenfunctions for different values of α and λ.

Example 2. Consider the following fractional Sturm-Liouville equation

$$^{C}\mathscr{D}_0^{\alpha} u'(x) + x^2 u'(x) + \sin(x)u(x) + \lambda u(x) = 0, \quad x \in [0,1],$$

with boundary conditions $u(0)=0$ and $u'(1)=0$.

Table 2 is reported to show the approximate solution of the 3 first eigenvalue of this example for different choices of α. In Figure 3, the eigenfunctions for the first and second eigenvalues and for different choices of α are presented. This figure also confirms the convergence of the proposed method.

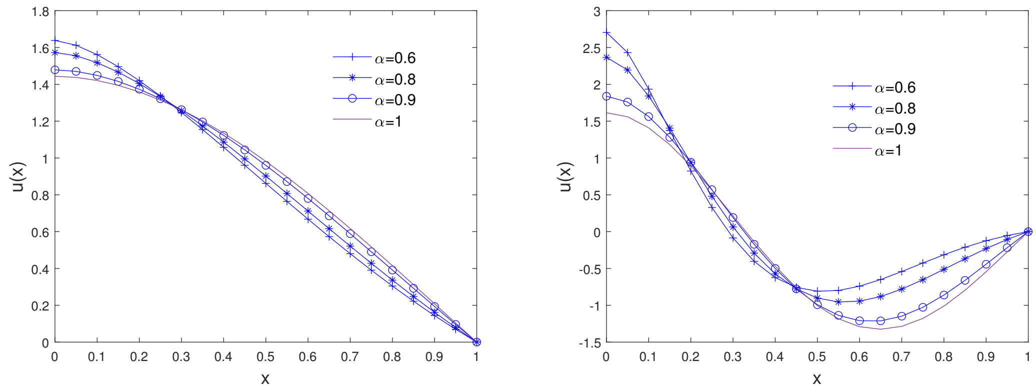


Figure 1: Figures related to eigenfunctions corresponding to the first (left) and second (right) eigenvalue for different α for Example 1.

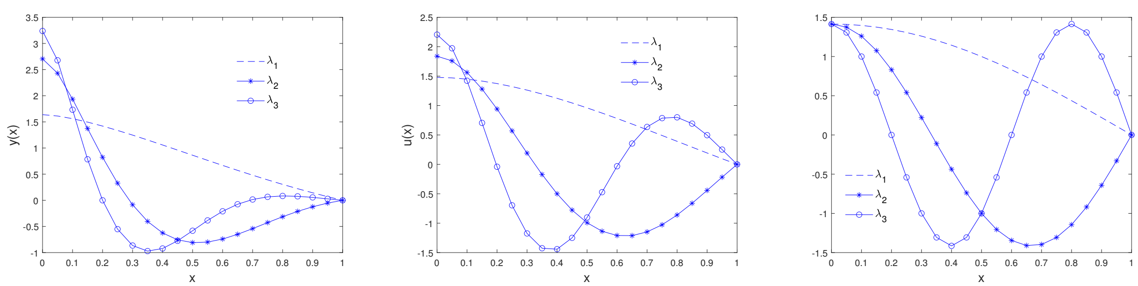
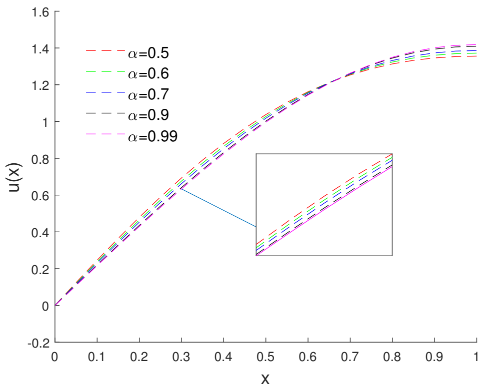


Figure 2: The first three eigenfunctions corresponding to eigenvalues $\alpha = 0.5$ (left), $\alpha = 0.8$ (middle) and $\alpha = 1$ (right) for Example 1.

Table 2: An approximation of the first three eigenvalues for Example 2, taking $\omega = 15$.

	Approximate λ				
x	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 99$
λ_1	1.175793	1.174806	1.212850	1.391709	1.515095
λ_2	15.182269	14.462148	15.071249	18.502191	20.983234
λ_3	24.762867	30.376813	35.517872	49.971304	59.556459



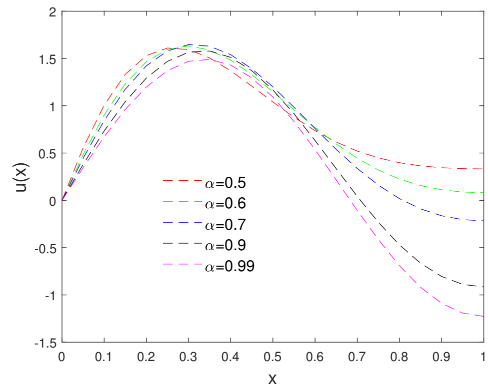


Figure 3: Figures related to eigenfunctions corresponding to the first (left) and second (right) eigenvalue for different α for Example 2.

5 Conclusion

This paper is dedicated to the numerical solution of the fractional Sturm-Liouville equation based on the Chebyshev cardinal polynomials. For the first time, the operational matrix of the Caputo fractional derivative for the Chebyshev cardinal polynomials is introduced in this work and applied to solve the fractional Sturm-Liouville equation. The bound of error for the method is obtained, and the convergence analysis is investigated. The convergence analysis is investigated, and an example confirms the accuracy and efficiency of the proposed method for solving the desired equation.

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