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Robust computational technique for a class of singularly perturbed nonlinear differential equations with Robin boundary conditions

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Abstract. In this article, a class of singularly perturbed nonlinear differential equations with Robin boundary conditions is considered. A numerical method consists of the classical finite difference operator over a Shishkin mesh with two-mesh algorithm is constructed to solve the problems. The method is proved to be first order convergent uniformly with respect to the perturbation parameter. Experiments are carried out for two different types of Robin boundary conditions and Neumann boundary conditions as a special case of Robin boundary conditions.

Keywords: Singular perturbation problems, Robin boundary conditions, nonlinear differential equations, finite difference scheme, Shishkin mesh, parameter-uniform convergence.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

A singularly perturbed differential equation (DE) is a differential equation whose solution contains regions of rapid variation; the regions which may be apparent in the solution or in its derivatives are called layers. The perturbation technique is a well known tool which is frequently exploited to analyze problems in fluid dynamics. Problems which arise in fluid dynamics are often nonlinear in nature and some techniques are available in the literature to solve such problems. Further, very few works are available in the literature for singularly perturbed nonlinear DEs.

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OMalley [11] considered a partially perturbed nonlinear system of IVPs. In [8], Manikandan et al. considered a singularly perturbed nonlinear system of DEs arising in a two-time scale system. A method is constructed with the standard backward difference operator on a Shishkin mesh to solve the system. In [9], a system of singularly perturbed second order semilinear DEs with Dirichlet boundary conditions is considered and a second order convergent method consists of a classical finite difference(CFD) scheme on a Shishkin mesh is constructed to solve it.

In [10], a partially perturbed nonlinear system of DEs is considered in which one component of the solution exhibits boundary layers due to the presence of perturbation parameter whereas the other component exhibits less-severe layers. A method composed of a CFD scheme applied on a Shishkin mesh is suggested to solve the system. In [2], on a uniform mesh, an exponentially fitted difference scheme is developed for a singularly perturbed nonlinear reaction diffusion BVPs. The method of integral identities with the use of exponential basis functions and interpolating quadrature rules with weight and remainder term in integral form are considered in it. Ishwariya et al. in [6] modelled a biochemical reaction namely Michaelis-Menten kinetics into a system of singularly perturbed first order nonlinear DEs with prescribed initial values. A CFD scheme has been used as the nonlinear solver on an appropriate Shishkin mesh. The method is proved to be parameter uniform almost first order convergent in the maximum norm.

For a system of singularly perturbed semilinear reaction-diffusion DEs, a parameter uniform method is constructed with an appropriate layer-adapted piecewise uniform mesh in [4]. A class of two parameter singularly perturbed nonlinear reaction diffusion equations with initial boundary value conditions is considered in [5]. The asymptotic behaviors of the solution are discussed by constructing the asymptotic expansion of the solution under suitable conditions. In [12], semilinear reaction-diffusion two-point BVPs with multiple solutions which can have boundary or interior layers are considered. In [14], the authors discussed the existence, uniqueness and asymptotic estimates of solutions of the singularly perturbed nonlinear second-order ODEs of convection diffusion type with Robin boundary conditions. Chein-Shan et.al. [7], developed two novel algorithms to find the solution for a second-order nonlinear singularly perturbed BVP of convection diffusion type which satisfies the Robin boundary conditions. The authors introduced a new idea of boundary shape function with two different types of algorithms such that the BVP is transformed into an IVP for the new variable. It should be noted that no work is available in the literature for a singularly perturbed nonlinear DE of reaction-diffusion type with Robin and Neumann boundary conditions.

The behaviour of a singularly perturbed linear DE of convection-diffusion type with Robin boundary conditions is investigated in [1] and a numerical method is also developed. It is worth observing that the layer patterns and the usage of CFD operators for a singularly perturbed linear DE of convection-diffusion type with Robin boundary conditions reported in [1] are preserved correspondingly in the present article for a singularly perturbed nonlinear DE of reaction-diffusion type with Robin and Neumann boundary conditions. Further, the novel aspect of the present article is that no artificial condition on the perturbation parameter ε is imposed.

Consider the following class of nonlinear singularly perturbed BVP with Robin boundary conditions.

$$-\varepsilon y''(t)+f(t,y)=0 \text{ on } \Omega=(0,1), \tag{1}$$

$$y(0)-y'(0)=\phi, \quad y(1)+y'(1)=\psi, \quad 0<\varepsilon\ll 1. \tag{2}$$

where ϕ and ψ are given constants. For all $(t,y)\in\bar{\Omega}\times\mathbb{R}$ and $f(t,y)\in C^4(\bar{\Omega}\times\mathbb{R})$ the following condition is assumed

$$\min_{t\in\bar{\Omega}}\left(\frac{\partial f(t,y)}{\partial y}\right) \geq \alpha > 0, \text{ for some constant } \alpha, \tag{3}$$

where $\bar{\Omega}=[0,1]$. Assumption (3) and the implicit function theorem ensure that $y \in C^4(\bar{\Omega})$. Problem (1)-(2) can be written in the operator form as

$$Ty(t) \;:=\; -\varepsilon y''(t) + f(t,y) = 0 \text{ on } \Omega, \tag{4}$$

$$b_0 y(0) = \phi, \quad b_1 y(1) = \psi, \tag{5}$$

where $b_0 = I - d/dt$ and $b_1 = I + d/dt$.

Throughout the article, C denotes a generic positive constant, which is independent of t, ε and N, the discretization parameter.

2 Analytical results

The reduced problem corresponding to (1)-(2) is defined by

$$f(t,r) = 0 \text{ on } \Omega. \tag{6}$$

The existence of a unique solution for (6) is ensured by condition (3) and the implicit function theorem. Further, the solution r of (6) and its derivatives are bounded independently of ε. Hence,

$$|r^{(k)}(t)| \leq C \text{ for } k = 0,1,2,3, \quad t \in \bar{\Omega}. \tag{7}$$

A decomposition of the solution $y(t)$ of (1) into a smooth component $p(t)$ and a singular component $q(t)$ is considered as $y(t) = p(t) + q(t)$, where

$$Tp(t) := -\varepsilon p''(t) + f(t,p) = 0 \text{ on } \Omega, \tag{8}$$

$$b_0 p(0) = b_0 r(0), \qquad b_1 p(1) = b_1 r(1), \tag{9}$$

$$Tq(t) := -\varepsilon q''(t) + f(t,p+q) - f(t,p) = 0 \text{ on } \Omega, \tag{10}$$

$$b_0 q(0) = b_0 (y-p)(0), \qquad b_1 q(1) = b_1 (y-p)(1). \tag{11}$$

Theorem 1. *For all* $t \in \bar{\Omega}$,

$$|p^{(k)}(t)| \leq C, \textit{ for } k = 0,1,2,3, \quad |p^{(4)}(t)| \leq C\varepsilon^{-1/2}.$$

Proof. For convenience, $p(t)$ is further decomposed as $p(t) = \hat{v}(t) + \tilde{v}(t)$, where $\hat{v}(t)$ is the solution of

$$-\varepsilon \hat{v}''(t) \;+\; f(t,\hat{v}) = 0, \;\; t \in \Omega, \tag{12}$$

$$b_0 \hat{v}(0) \;=\; b_0 p(0), \quad b_1 \hat{v}(1) = b_1 p(1), \tag{13}$$

and $\tilde{v}(t)$ is the solution of

$$-\varepsilon \tilde{v}''(t) \;+\; f(t,\hat{v}+\tilde{v}) - f(t,\hat{v}) = 0, \;\; t \in \Omega, \tag{14}$$

$$b_0 \tilde{v}(0) \;=\; 0, \quad b_1 \tilde{v}(1) = 0. \tag{15}$$

Let $t \in \Omega$. Using (6) and (12), we get

$$-\varepsilon(\hat{v}-r)''(t)+a_1(t)(\hat{v}-r)(t)=\varepsilon r''(t), \tag{16}$$

where $a_1(t)=\dfrac{\partial f}{\partial y}(t,\chi(t))$ is the intermediate value. Consider the linear operator

$$T_1 z(t)=-\varepsilon z''(t)+a_1(t)z(t)=\varepsilon r''(t), \tag{17}$$

where $z=\hat{v}-r$. From (9) and (13) we drive

$$b_0 z(0)=0, \quad b_1 z(1)=0. \tag{18}$$

The operator T_1 together with (18) satisfies the maximum principle in [13]. Thus,

$$|z(t)| \leq C\varepsilon. \tag{19}$$

On differentiating (17) once, we get

$$T_1 z'(t)=-\varepsilon z'''(t)+a_1(t)z'(t)=\varepsilon r'''(t)-a_1'(t)z(t). \tag{20}$$

Rearranging (18), we get

$$z'(0)=z(0), \quad z'(1)=-z(1). \tag{21}$$

Denoting z' by h in (20) and (21), we get

$$T_1 h(t)=-\varepsilon h''(t)+a_1(t)h(t)=\varepsilon r'''(t)-a_1'(t)z(t), \tag{22}$$

$$h(0)=z(0), \quad h(1)=-z(1). \tag{23}$$

Problem (22)-(23) satisfies the maximum principle in [13]. Thus,

$$|h(t)| \leq C\varepsilon. \tag{24}$$

Rearranging (22) and using (24), we get

$$|h''(t)| \leq C. \tag{25}$$

Using mean-value theorem,

$$|h'(t)| \leq C\varepsilon^{1/2}. \tag{26}$$

Differentiating (22) once and rearranging, we get

$$|h'''(t)| \leq C(1+\varepsilon^{-1/2}). \tag{27}$$

From (19), (24)-(27), we get

$$|\hat{v}^{(k)}(t)| \leq C, \quad k=0,1,2,3, \quad |\hat{v}^{(4)}(t)| \leq C(1+\varepsilon^{-1/2}).$$

From (14), we get

$$-\varepsilon \tilde{v}''(t)+a_2(t)\tilde{v}(t)=0, \tag{28}$$

where $a_2(t)=\dfrac{\partial f}{\partial y}(t,\eta(t))$ is the intermediate value. Problem (28) together with (15), is similar to the problem in [13] and hence

$$|\tilde{v}^{(k)}(t)| \leq C, \quad k=0,1,2,3, \quad |\hat{v}^{(4)}(t)| \leq C(1+\varepsilon^{-1/2}).$$

The bounds for p and its derivatives follow from the bounds of $\hat{v}$ and $\tilde{v}$. □

The layer function $B(t)$ related with the solution $y(t)$ of (1)-(2) is defined by

$$B(t) = B_1(t) + B_2(t), \quad B_1(t) = e^{-t\sqrt{\alpha/\varepsilon}}, \quad B_2(t) = e^{-(1-t)\sqrt{\alpha/\varepsilon}}, \quad t \in \bar{\Omega}.$$

Theorem 2. *For any* $t \in \bar{\Omega}$,

$$|q^{(k)}(t)| \le CB(t), \;\; k = 0, 1,$$

$$|q^{(k)}(t)| \le C\varepsilon^{-\frac{(k-1)}{2}} B(t), \;\; k = 2, 3, 4.$$

Proof. From (10), we get

$$-\varepsilon q''(t) + s(t)q(t) = 0, \tag{29}$$

where $s(t) = \dfrac{\partial f}{\partial y}(t, \lambda(t))$ is the intermediate value. Problem (29) together with (11) is similar to the problem in [13]. Thus the bounds for w and its derivatives hold. □

3 The Shishkin mesh

A Shishkin mesh with N mesh-intervals is constructed on $\bar{\Omega}$ as follows. Let $\Omega^N = \{t_j\}_{j=1}^{N}$ and $\bar{\Omega}^N = \{t_j\}_{j=0}^{N}$. The interval $\bar{\Omega}$ is subdivided into 3 sub-intervals $[0, \tau]$, $(\tau, 1-\tau]$ and $(1-\tau, 1]$ such that $\bar{\Omega} = [0, \tau] \cup (\tau, 1-\tau] \cup (1-\tau, 1]$. The parameter τ is defined by

$$\tau = \min\left\{\frac{1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}} \ln(N)\right\}.$$

On the outer domain $(\tau, 1-\tau]$ a uniform mesh with $\frac{N}{2}$ mesh points is placed and on each of the inner domains $[0, \tau]$ and $(1-\tau, 1]$ a uniform mesh of $\frac{N}{4}$ mesh points is placed.

4 Discrete problem and error analysis

The discrete BVP associated with (1)-(2) is defined to be

$$T^N Y(t_j) = -\varepsilon \delta^2 Y(t_j) + f(t_j, Y(t_j)) = 0, \quad \text{for} \quad t_j \in \Omega^N, \tag{30}$$

$$b_0^N Y(0) = b_0 y(0), \qquad b_1^N Y(1) = b_1 y(1) \tag{31}$$

where $b_0^N = I - D^+$, $b_1^N = I + D^-$,

$$D^+\Theta(t_j) = \frac{\Theta(t_{j+1}) - \Theta(t_j)}{h_j}, \quad D^-\Theta(t_j) = \frac{\Theta(t_j) - \Theta(t_{j-1})}{h_j},$$

$$\delta^2\Theta(t_j) = \frac{\Theta(t_{j+1}) - \Theta(t_{j-1})}{2h_j}, \quad h_j = t_j - t_{j-1}.$$

Theorem 3. *For any mesh functions* U *and* Z *with* $b_0^N U(0) = b_0^N Z(0)$ *and* $b_1^N U(1) = b_1^N Z(1)$,

$$|U - Z| \le C|T^N(U - Z)|.$$

Proof.

$$
\begin{aligned}
(T^N(U-Z))(t_j) &= -\varepsilon\,\delta^2(U-Z)(t_j) + f(t_j,U(t_j)) - f(t_j,Z(t_j)) \\
&= -\varepsilon\,\delta^2(U-Z)(t_j) + \frac{\partial f}{\partial y}(t_j,M(t_j))(U-Z)(t_j) \\
&= T^{N'}(U-Z)(t_j),
\end{aligned} \tag{32}
$$

where $\frac{\partial f}{\partial y}(t_j,M(t_j))$ is the intermediate value and $T^{N'}$ is the Frechet derivative of T^N. Since $T^{N'}$ is linear which satisfies the discrete maximum principle in [13]. Hence, on Ω^N

$$
|U-Z| \le C|T^{N'}(U-Z)| = C|T^N(U-Z)|, \tag{33}
$$

which completes the proof. □

Theorem 4. *Let y be the solution of (1)-(2) and Y be the solution of (30)-(31). Then for $t_j \in \bar{\Omega}^N$,*

$$
|(Y-y)(t_j)| \le CN^{-1}\ln(N). \tag{34}
$$

Proof. Let $t_j \in \Omega^N$. Since $b_0^N Y(0) = b_0 y(0)$ and $b_1^N Y(1) = b_1 y(1)$ from (33), $|Y-y| \le C|T^N(Y-y)|$. Using (30), we get $|T^N Y(t_j)| = |(T^N y - T^N Y)(t_j)|$. Consider,

$$
\begin{aligned}
|(T^N y - T^N Y)(t_j)| &= |T^N Y(t_j)| = |(T^N y - Ty)(t_j)| = E|(\delta^2 - D^2)Y(t_j)| \\
&\le E(|(\delta^2 - D^2)p(t_j)| + |(\delta^2 - D^2)q(t_j)|),
\end{aligned}
$$

where $D^2 = \frac{d^2}{dt^2}$. Since the bounds for p, q, their derivatives and the bounds for the local truncation error are same as in [13], $|(T^N(y-Y))(t_j)| \le CN^{-1}\ln(N)$. Thus, $|(Y-y)(t_j)| \le CN^{-1}\ln(N)$. □

5 The continuation method

An artificial system of nonlinear PDEs corresponding to the system of nonlinear ODEs in (1) is given by

$$
\begin{aligned}
& y_x(t,x) - \varepsilon\, y_{tt}(t,x) + f(t,y(t,x)) = 0, && (t,x) \in (0,1)\times(0,X], \\
& (y-y')(0,x) = (y-y')(0), && \\
& (y+y')(1,x) = (y+y')(1), && 0 < x \le X, \\
& y(t,0) = y_{init}(t), && 0 < t \le 1.
\end{aligned} \tag{35}
$$

A variant of the continuation method from [3] is used to solve (35) and is given by

$$
\begin{aligned}
& D_x^- Y(t_j,x_k) - \varepsilon\delta_t^2 Y(t_j,x_k) + f(t_j, Y(t_j,x_{k-1})) = 0, \\
& Y(0,x_k) = (y-y')(0), \\
& Y(1,x_k) = (y+y')(1), \\
& Y(t_j,0) = y_{init}(t_j), \qquad t_j \in \bar{\Omega}^N, \quad j = 1,\dots,N, \quad k = 1,\dots K.
\end{aligned} \tag{36}
$$

The initial guess $y_{init}(t)$ is chosen to be a polynomial in $'t'$ such that, together the polynomial and its first derivative satisfy both the given boundary conditions. The choices of $h_x = x_k - x_{k-1}$ and the number of iterations K are determined as follows. Define

$$e(k) = \max_{1 \le j \le N} \left(\frac{|Y(t_j, x_k) - Y(t_j, x_{k-1})|}{h_x} \right), \tag{37}$$

for $k = 1, 2, \ldots, K,$. The step size h_x is chosen sufficiently small so that

$$e(k) \le e(k-1), \qquad 1 < k \le K. \tag{38}$$

The number of iterations K is chosen such that

$$e(K) \le tol, \tag{39}$$

where tol is a suitably prescribed small tolerance. The algorithm similar to the one found in [9] is used to compute the numerical solution.

6 Numerical Illustrations

In this section, three examples are presented. In the first example, Robin boundary conditions without the perturbation parameter ε are considered. Aforesaid, the problem is solved on a uniform mesh and the numerical results are presented under the same example. Whereas in the second example the perturbation parameter ε also occurs in the Robin boundary conditions. Neumann boundary conditions without the perturbation parameter ε are considered in the third example. The continuation method designed in Section 5 is used to solve the examples. The tolerance "tol" in the continuation algorithm for all numerical examples is taken to be 0.00001. Notations p^N, D^N and C_p^N denote the parameter-uniform rate of convergence, parameter-uniform maximum pointwise error and parameter-uniform error constant and respectively and they bear the same meaning as in [3].

Example 1. Consider the nonlinear BVP with Robin boundary conditions

$$-\varepsilon y''(t) + y^5(t) + 3y(t) - 1 = 0, \quad t \in (0,1),$$

with $y(0) - y'(0) = \sin(0.5)$ and $y(1) + y'(1) = e^{-0.7}$.

The maximum pointwise errors and the rate of convergence for the above BVP are presented in Table 2 for a non-uniform mesh and the same are presented in Table 1 for a uniform mesh. Graph of the numerical solution for both $y(t)$ and $y'(t)$ for $N = 256$ and $\varepsilon = 2^{-8}, 2^{-10}, 2^{-12}$ are portrayed in Figure 1 and Figure 2 respectively. The $log - log$ plot for the error in the suggested numerical method of the above BVP is given in Figure 3.

Example 2. Consider the nonlinear BVP with Robin boundary conditions which include $\sqrt{\varepsilon}$

$$-\varepsilon y''(t) + y^5(t) + 3y(t) - 1 = 0, \quad t \in (0,1),$$

with $y(0) - \sqrt{\varepsilon}\, y'(0) = \sin(0.5)$ and $y(1) + \sqrt{\varepsilon}\, y'(1) = e^{-0.7}$.

The maximum pointwise errors and the rate of convergence for the above BVP are presented in Table 3 and graph of the numerical solution for both $y(t)$ and $y'(t)$ for $N = 256$ and $\varepsilon = 2^{-8}, 2^{-10}, 2^{-12}$ are portrayed in Figure 4 and Figure 5 respectively. The $log - log$ plot for the error in the suggested numerical method of the above BVP is given in Figure 6.

Example 3. Consider the nonlinear BVP with Neumann boundary conditions

$$-\varepsilon y''(t) + y^5(t) + 3y(t) - 1 = 0, \quad t \in (0,1),$$

with $y'(0) = \sin(0.5)$ and $y'(1) = e^{-0.7}$.

The maximum pointwise errors and the rate of convergence for the above BVP are presented in Table 4 and graph of the numerical solution for both $y(t)$ and $y'(t)$ for $N = 256$ and $\varepsilon = 2^{-8}, 2^{-10}, 2^{-12}$ are portrayed in Figure 7 and Figure 8 respectively. The $Log - log\, plot$ for the error in the suggested numerical method of the above BVP is given in Figure 9.

Table 1: *Values of* D^N, p^N *and* C_p^N *for* $\alpha = 2.9$ *on a uniform mesh in Example 1.*

ε	Number of mesh points N					
	32	64	128	256	512	1024
2^{-2}	8.7208e-04	4.2972e-04	2.1327e-04	1.0623e-04	5.3017e-05	2.6484e-05
2^{-4}	1.1572e-03	5.5520e-04	2.7150e-04	1.3420e-04	6.6710e-05	3.3257e-05
2^{-6}	1.5298e-03	6.8780e-04	3.2228e-04	1.5554e-04	7.6350e-05	3.7819e-05
2^{-8}	2.0284e-03	8.7619e-04	3.8168e-04	1.7427e-04	8.2791e-05	4.0295e-05
2^{-10}	2.3632e-03	1.1176e-03	4.8762e-04	2.0802e-04	9.2385e-05	4.3050e-05
2^{-12}	2.4426e-03	1.2318e-03	5.9409e-04	2.6660e-04	1.1336e-04	4.9045e-05
2^{-14}	2.4539e-03	1.2531e-03	6.2915e-04	3.0807e-04	1.4297e-04	6.1843e-05
2^{-16}	2.4553e-03	1.2559e-03	6.3467e-04	3.1799e-04	1.5720e-04	7.5080e-05
2^{-18}	2.4555e-03	1.2563e-03	6.3539e-04	3.1940e-04	1.5986e-04	7.9459e-05
2^{-20}	2.4555e-03	1.2563e-03	6.3547e-04	3.1958e-04	1.6022e-04	8.0150e-05
2^{-22}	2.4555e-03	1.2563e-03	6.3549e-04	3.1960e-04	1.6026e-04	8.0239e-05
2^{-24}	2.4555e-03	1.2563e-03	6.3549e-04	3.1960e-04	1.6027e-04	8.0250e-05
2^{-26}	2.4555e-03	1.2563e-03	6.3549e-04	3.1960e-04	1.6027e-04	8.0251e-05
D^N	2.4555e-03	1.2563e-03	6.3549e-04	3.1960e-04	1.6027e-04	8.0251e-05
p^N	9.6683e-01	9.8326e-01	9.9159e-01	9.9578e-01	9.9789e-01	
C_p^N	1.4342e-01	1.4342e-01	1.4180e-01	1.3939e-01	1.3662e-01	1.3371e-01

7 Conclusion

From Figures 1 and 7, we observe that due to the absence of singular perturbation parameter in the boundary conditions, the solution y has weak boundary layers and its derivative y' has strong boundary layers. Whereas, from Figure 3, we note that the the solution y itself has strong boundary layers due to the presence of perturbation parameter in the boundary conditions.

Thus from the figures, it is evident that in the absence of the perturbation parameter ε in the boundary conditions at $t = 0$ and $t = 1$, the boundary layers are weak. Hence in this case one may obtain parameter

Table 2: *Values of* D^N, p^N *and* C_p^N *for* $\alpha = 2.9$ *on a non-uniform mesh in Example 1.*

ε	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-2}	4.2972e-04	2.1327e-04	1.0623e-04	5.3017e-05	2.6484e-05	1.3236e-05
2^{-4}	5.0890e-04	2.5071e-04	1.2441e-04	6.1964e-05	3.0922e-05	1.5446e-05
2^{-6}	5.9907e-04	2.8992e-04	1.4251e-04	7.0636e-05	3.5163e-05	1.7543e-05
2^{-8}	3.3439e-04	1.9331e-04	1.1018e-04	6.3429e-05	3.7819e-05	1.8820e-05
2^{-10}	1.7364e-04	1.0028e-04	5.7105e-05	3.2167e-05	1.8820e-05	9.9234e-06
2^{-12}	8.8396e-05	5.1073e-05	2.9077e-05	1.6375e-05	9.9234e-06	5.0504e-06
2^{-14}	4.4557e-05	2.5764e-05	1.4671e-05	8.2618e-06	5.0504e-06	2.5478e-06
D^N	5.9907e-04	2.8992e-04	1.4251e-04	7.0636e-05	3.7819e-05	1.8820e-05
p^N	1.0471	1.0246	1.0126	0.990131	1.0068	
C_p^N	5.4742e-02	4.9481e-02	4.5428e-02	4.2056e-02	4.2056e-02	3.9090e-02

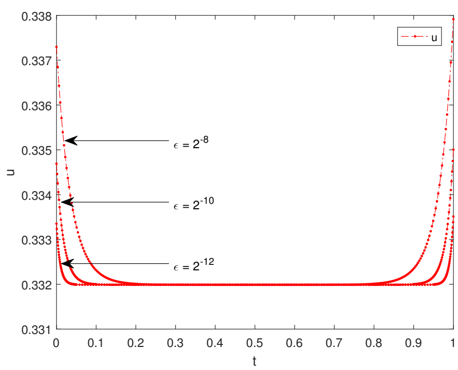

Figure 1: Numerical approximations of $y(t)$ in Example 1.

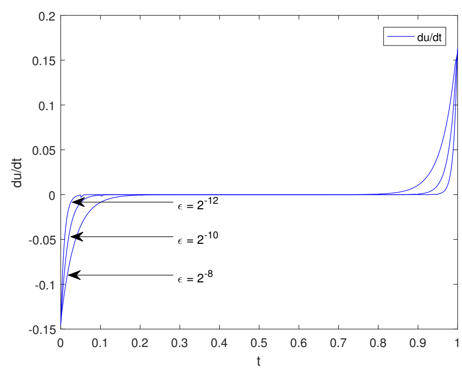

Figure 2: Numerical approximations of $y'(t)$ in Example 1.

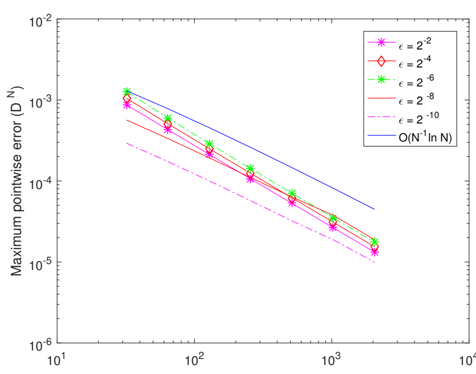

Figure 3: *log-log* plot for the error in Example 1.

Table 3: *Values of* D^N, p^N *and* C_p^N *for* $\alpha = 2.9$ *in Example* 2.

ε	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-2}	5.6375e-04	2.8067e-04	1.4003e-04	6.9937e-05	3.4949e-05	1.7468e-05
2^{-4}	1.0576e-03	5.2622e-04	2.6240e-04	1.3102e-04	6.5462e-05	3.2719e-05
2^{-6}	2.1260e-03	1.0551e-03	5.2506e-04	2.6185e-04	1.3074e-04	6.5327e-05
2^{-8}	2.1701e-03	1.2936e-03	7.5053e-04	4.3653e-04	2.6185e-04	1.3074e-04
2^{-10}	2.1687e-03	1.2935e-03	7.5052e-04	4.2732e-04	2.3984e-04	1.3307e-04
2^{-12}	2.1654e-03	1.2931e-03	7.5050e-04	4.2732e-04	2.3984e-04	1.3307e-04
2^{-14}	2.1618e-03	1.2924e-03	7.5042e-04	4.2731e-04	2.3984e-04	1.3307e-04
D^N	2.1701e-03	1.2936e-03	7.5053e-04	4.3653e-04	2.6185e-04	1.3307e-04
p^N	0.74641	0.78539	0.78180	0.73738	0.97653	
C_p^N	1.1643e-01	1.1571e-01	1.1192e-01	1.0852e-01	1.0852e-01	9.1947e-02

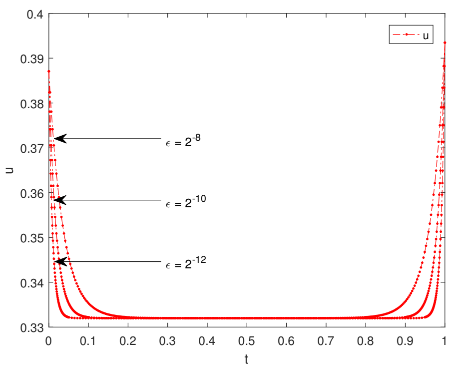

Figure 4: Numerical approximations of $y(t)$ in Example 2.

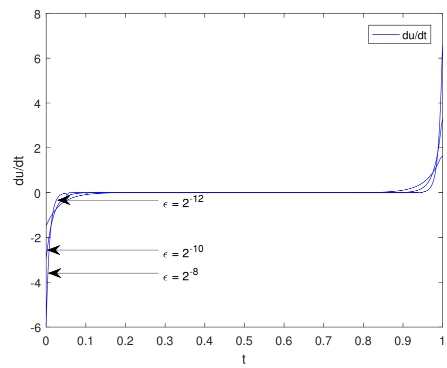

Figure 5: Numerical approximations of $y'(t)$ in Example 2.

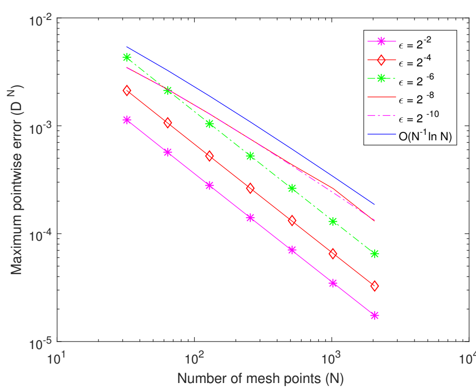

Figure 6: *log-log plot* for the error in Example 2.

Table 4: *Values of* D^N, p^N *and* C_p^N *for* $\alpha = 2.9$ *in Example* 3.

ε	Number of mesh points N					
	64	128	256	512	1024	2048
2^{-2}	1.7585e-03	8.7242e-04	4.3448e-04	2.1681e-04	1.0830e-04	5.4120e-05
2^{-4}	2.0122e-03	9.8632e-04	4.8821e-04	2.4287e-04	1.2112e-04	6.0484e-05
2^{-6}	2.0989e-03	1.0098e-03	4.9493e-04	2.4497e-04	1.2186e-04	6.0774e-05
2^{-8}	1.0907e-03	6.2840e-04	3.5746e-04	2.0556e-04	1.2248e-04	6.0930e-05
2^{-10}	5.4491e-04	3.1416e-04	1.7872e-04	1.0062e-04	6.0930e-05	3.1024e-05
2^{-12}	2.7202e-04	1.5703e-04	8.9356e-05	5.0308e-05	3.1024e-05	1.5512e-05
2^{-14}	1.3577e-04	7.8471e-05	4.4673e-05	2.5153e-05	1.5512e-05	7.7560e-06
D^N	2.0989e-03	1.0098e-03	4.9493e-04	2.4497e-04	1.2248e-04	6.0930e-05
p^N	1.0556	1.0288	1.0146	1.0000	1.0074	
C_p^N	2.6869e-01	2.5854e-01	2.5344e-01	2.5089e-01	2.5089e-01	2.4962e-01

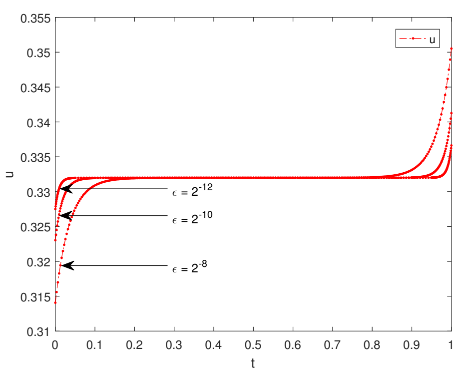

Figure 7: Numerical approximations of $y(t)$ in Example 3.

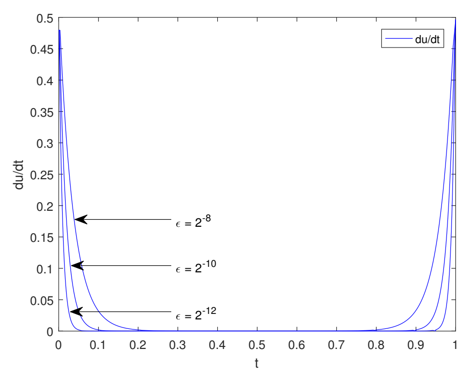

Figure 8: Numerical approximations of $y'(t)$ in Example 3.

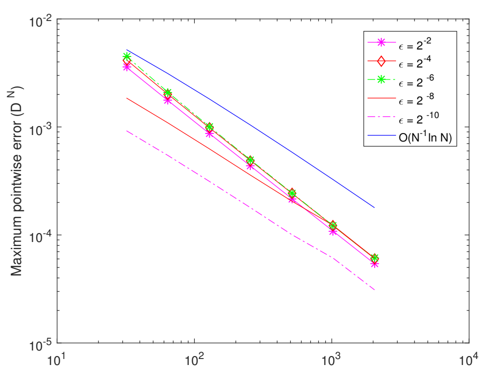

Figure 9: *log-log* plot for the error in Example 3.

uniform convergent numerical solutions even by using classical finite difference operators on uniform meshes. On the other hand, in the same situation, the derivative changes rapidly at both the boundaries $t = 0$ and $t = 1$, as the perturbation parameter ε tends to zero; in such a case classical finite difference operators on uniform meshes render their uselessness. Whereas the presence of the parameter ε at the boundary conditions at $t = 0$ and $t = 1$, increase the significance of the boundary layers at both the boundaries $t = 0$ and $t = 1$. The numerical technique reported in the present article helps to resolve all the above mentioned problems.

From the tables we find that the maximum pointwise errors decrease through the diagonal and the proposed method is almost first order parameter-uniform convergent. Further, from the tables we also observe that the paramter-uniform error constant decreases monotonically.

Moreover from the graphs of the $Log - log$ plot, it is easy to spot that the errors are bounded by $O(N^{-1}\ln(N))$.

It should be noted that the present computational technique is both robust and layer-resolving. Further, it is worth observing that the present computational technique for problems with boundary layers is also applicable to problems with a much wider class of singularities.

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