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An efficient approach for solving the fractional model of the human T-cell lymphotropic virus I by the spectral method

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Abstract. This paper aims to present a new and efficient numerical method to approximate the solution of the fractional model of human T-cell lymphotropic virus I (HTLV-I) infection $CD4^{+}T$-cells. The approximate solution of the model is obtained using the shifted Chebyshev collocation spectral method. This model relates to the class of nonlinear ordinary differential equations. The proposed algorithm reduces the Caputo sense fractional model to a system of nonlinear algebraic equations that can be solved numerically. The convergence of the proposed method is investigated. The graphical result is compared with existing numerical methods reported in the literature to indicate the efficiency and reliability of the presented method.

Keywords: HTLV-I, the faction differential equation, nonlinear system, collocation method, shifted Chebyshev polynomial.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

The human T-lymphotropic virus type I (HTLV-I) is a retrovirus with a single-stranded RNA virus and has been associated with some life-threatening diseases like HTLV-I associated myelopathy/tropical spastic paralysis (HAM/TSP) and adult T-cell leukemia (ATL) [16]. The virus predominantly targets $CD4^{+}T$-cells. After infection, it spreads to naive cells via cell-to-cell transmission [3]. According to the world human organization (WHO) statics, about 20 million people live with HTLV-I illness around the world. The disease is developed at a low rate in patients' bodies [23]. It is an incurable disease and is considered a lifelong condition. Infected bodily fluids such as breast milk , semen, and blood are

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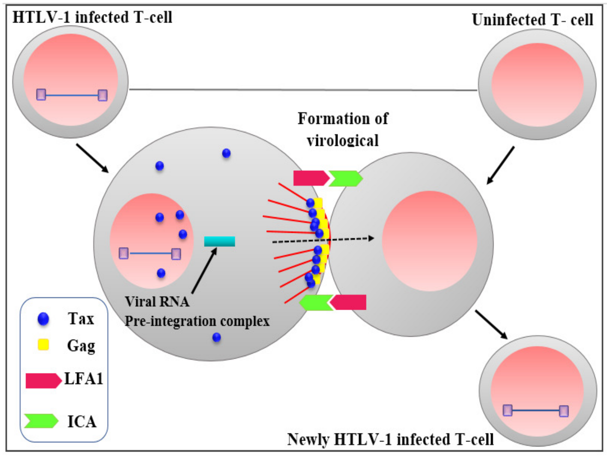

Figure 1: Cell-to-cell transmission of HTLV-I.

known ways to transmit HTLV-I. It is transmitted mainly in three different ways: parenteral transmission [12, 19], sexual contact [18], and mother-infant [15] (primarily through breastfeeding).

Since HTLV-I transmits by cell-to-cell contact, polluted cells must be passed from the infected individual via all three routes. A microtubule organizing center (MTOC) is polarized when an infected cell interacts with an uninfected cell and a virological synapse is constructed. Then, viral genomic RNAs and the Gag complex collect at synapses and exit from the uninfected cells [11]. The engagement of intercellular adhesion molecule 1 (ICAM1) enhances the polarization of the MTOC at the point of contact, demonstrating that the interaction of ICAM1 and lymphocyte function-related antigen 1 (LFA1) is essential for HTLV-I infection. After that, the latency interval may last for a long time. Latently infected cells contain viruses but do not produce DNA. Therefore, they are unable to transmit. Such cells can become active and infect healthy cells when stimulated by antigens (see Figure 1).

Comprehensive research has been done on HTLV-I to understand the dynamics, progression, and interaction of the immune system with HTLV-I. Various models for the human immune system have been offered. Stilianakis and Seydel [21] offered a model consisting of a set of the nonlinear differential equations that separate $CD4^+T$-cells into four compartments: leukemia cells, actively infected cells, latently infected cells, and uninfected $CD4^+T$-cells.

The traditional model of a nonlinear differential equation system was altered by Patricia Katri et al. [10] as follows

$$\begin{cases} \frac{dT_H}{dt} = \mu - \eta_T T_H(t) - kT_V(t)T(t), \\ \frac{dT_I}{dt} = k_1 T_V(t)T_H(t) - (\eta_L + \gamma)T_I(t), \\ \frac{dT_V}{dt} = \gamma T_I(t) - (\eta_A + \zeta)T_V(t), \\ \frac{dT_L}{dt} = \zeta T_V(t) + sT_L(t)(1 - \frac{T_L(t)}{(T_L)_{max}}) - \eta_N T_L(t). \end{cases} \tag{1}$$

This model explains the development of HTLV-I infection and adult T-cell leukemia (ATL). Table 1 shows an explanation of the factors in the HTLV-I model.

Many mathematical models that describe natural phenomena do not have exact solutions. The approximate solution of these models has attracted the attention of many mathematicians [1, 13, 22]. In this

Table 1: The parameters and variables applied in the HTLV-I model.

Parameters	Description
$T_H(t)$	The number of healthy $CD4^+T$-cells at time t
$T_I(t)$	The number of latently infected $CD4^+T$-cells at time t
$T_V(t)$	The number of actively infected $CD4^+T$-cells at time t
$T_L(t)$	The number of leukemic cells at time t
η_T	The natural death rate of healthy $CD4^+T$-cells
η_L	The blanket death term for latently infected $CD4^+T$-cells
η_A	The blanket death term for actively infected $CD4^+T$-cells
η_N	The blanket death term for leukemic infected $CD4^+T$-cells
μ	The origin of $CD4^+T$-cells from precursors
k	The rate of uninfected $CD4^+T$-cells is contacted by actively infected cells
k_1	The infection rate of $CD4^+T$-cells with virus from actively infected cells
γ	The rate of transformation from latently infected $CD4^+T$-cells to actively leukemic cells
ζ	The rate of transformation from actively infected $CD4^+T$-cells to leukemic cells
s	The rate of ATL cells' growth according to a classical logistic growth function
$(T_L)_{max}$	Maximal concentration of leukemic $CD4^+T$-cells
τ	is any positive constant

study, we present a numerical method for the Caputo fractional model of HTLV-I [2] as follows

$$\begin{cases} D^\lambda T_H(t) = \mu - \eta_T T_H(t) - kT_V(t)T_H(t), \\ D^\lambda T_I(t) = k_1 T_V(t)T_H(t) - (\eta_L + \gamma)\, T_I(t), \\ D^\lambda T_V(t) = \gamma T_I(t) - (\eta_A + \zeta)\, T_V(t), \\ D^\lambda T_L(t) = \zeta T_V(t) + sT_L(t)\left(1 - \frac{T_L(t)}{(T_L)_{\max}}\right) - \eta_N T_L(t), \end{cases} \tag{2}$$

with initial conditions

$$T_H(0) = \rho_1, \;\; T_I(0) = \rho_2, \;\; T_V(0) = \rho_3, \;\; T_L(0) = \rho_4, \tag{3}$$

such that $0 \le t \le \tau < \infty$, $0 < \lambda < 1$. This HTLV-I model is considered with many methods, such as the generalized Euler method (GEM) [2], the multi-step generalized differential transform method (MSGDTM) [8] and the natural-Adomian decomposition method (N-ADM) [17]. The spectral method is one of the most well-known numerical methods for solving mathematical models related to natural phenomena due to its rapid rate of convergence and adaptability in application over finite and infinite intervals [9, 24, 25]. Our aim is to use the spectral collocation method based on the shifted Chebyshev polynomials as an orthogonal basis function.

This paper is organized as follows. In Section 2, we go over the fundamentals of fractional calculus and Chebyshev polynomials. In Section 3, we apply the Chebyshev collocation method to obtain the numerical solution of the fractional model of HTLV-I. In Section 4, we discuss the convergence of the presented approach. In Section 5, numerical simulations are illustrated and discussed. In Section 6, the conclusions are presented.

2 Preliminary

2.1 Fractional calculus

In this section, some fundamental concepts of fractional calculus and underlying factors are described [14].

Definition 1. *The determination of the Riemann-Liouville fractional integral of order $\lambda > 0$ is given by*

$$\begin{aligned} &\mathrm{I}^0 g(t) = g(t), \\ &\mathrm{I}^\lambda g(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t-s)^{(\lambda-1)} g(s) ds = \frac{1}{\Gamma(\lambda)} t^{\lambda-1} * g(t), \quad \lambda > 0, t > 0, \end{aligned} \tag{4}$$

where $*$ and $\Gamma(\cdot)$ denote the convolution operator and the gamma function, respectively. Fractional derivative is expressed in a variety of approaches. We will concentrate on the fractional differential operator D^λ in the Caputo sense.

Definition 2. *The fractional derivative of g(t) in the sense of Caputo is defined by*

$$D^\lambda g(t) = I^{m-\lambda} D^m g(t) = \frac{1}{\Gamma(m-\lambda)} \int_0^t (t-s)^{m-\lambda-1} \frac{d^m}{ds^m} g(s) ds, \quad m-1 < \lambda \leq m, t > 0,$$

where $\lambda > 0$ is the order of derivative, m is the smallest integer greater than λ and D^m is the classical differential operator of order m. For the Caputo derivative, we have

$$D^\lambda t^j = \begin{cases} 0, & if\ j \in \mathbb{N}_0\ and\ j < \lceil \lambda \rceil, \\ \frac{\Gamma(j+1)}{\Gamma(j+1-\lambda)} t^{(j-\lambda)}, & if\ j \in \mathbb{N}_0\ and\ j \geq \lceil \lambda \rceil\ or\ j \notin \mathbb{N}\ and\ j > \lfloor \lambda \rfloor. \end{cases} \tag{5}$$

The Caputo derivative is a linear operator, i.e.,

$$D^\lambda (\sum_{j=0}^m b_j g_j(t)) = \sum_{j=0}^m b_j D^\lambda g_j(t),$$

where $b_j, j = 0, 1, \ldots, m$ are constants.

2.2 Chebyshev polynomials

The Chebyshev polynomial of order $j, T_j(t)$ for $j = 0, 1, 2, \ldots$ on the interval $[-1, 1]$ is defined by

$$\begin{aligned} &T_{n+1}(t) = 2t T_n(t) - T_{n-1}(t), \\ &T_0(t) = 1, T_1(t) = t. \end{aligned} \tag{6}$$

We define the shifted Chebyshev polynomials by introducing the change of variable $z = \frac{2t}{\tau} - 1, \tau > 0$, $t \in [0, \tau]$. Just for convenience, $T_j(\frac{2t}{\tau} - 1)$ is denoted by $T_{\tau,j}(t)$. The analytical form of $T_{\tau,j}(t)$ is given by

$$T_{\tau,j}(t) = j \sum_{k=0}^j (-1)^{j-k} \frac{(j+k-1)! 2^{2k}}{(j-k)!(2k)! \tau^k} t^k, \quad j = 1, 2, 3, \ldots, M,$$

where $T_{\tau,j}(0) = (-1)^j$ and $T_{\tau,j}(\tau) = 1$. The set $\{T_{\tau,j}(t)\}_{j=0}^{\infty}$ consists of classical Chebyshev polynomials, specifies a complete orthogonal system on the following space

$$L^2_\omega := \left\{ g \mid g : [-1,1] \to \mathbb{R} \quad s.t. \quad \int_{-1}^{1} \omega_\tau(t) g^2(t) dt < \infty \right\},$$

with respect to $\omega_\tau(t) = \frac{1}{\sqrt{t(\tau - t)}}$ as the weight function, i.e.,

$$\int_0^\tau T_{\tau,j}(t) T_{\tau,k}(t) \omega_\tau dt = h_j,$$

such that

$$h_j = \begin{cases} \frac{\varepsilon_j}{2}\pi, & k = j, \\ 0, & k \neq j, \end{cases}$$

where $\varepsilon_0 = 2,\ \varepsilon_j = 1,\ j \geq 1$. If $g(t) \in L^2_\omega$ for $t \in [0, \tau]$, then it can expanded in terms of the shifted Chebyshev polynomials as

$$g(t) = \sum_{j=0}^{\infty} a_j T_{\tau,j}(t), \tag{7}$$

such that

$$a_j = \frac{4}{\tau\pi\gamma_j} \int_0^\tau g(t) T_{\tau,j}(t) \omega_\tau(t) dt, \quad j = 0, 1, 2, \ldots,$$

where

$$\gamma_j = \begin{cases} 2, & j = 0, \\ 1, & j \geq 1. \end{cases}$$

In practice, the finite number of Eq. (7) are considered as

$$g_M(t) = \sum_{j=0}^{M} a_j T_{\tau,j}(t) = A^T \psi_{\tau,M}(t), \tag{8}$$

where

$$A^T = [a_0, \ldots, a_M], \quad \psi_{\tau,M}(t) = [T_{\tau,0}(t), T_{\tau,1}(t), \ldots, T_{\tau,M}(t)]^T. \tag{9}$$

Then, the derivative of $\psi_{\tau,M}(t)$ are obtained by

$$\frac{d\psi_{\tau,M}(t)}{dt} = D^1 \psi_{\tau,M}(t),$$

such that D^1 represents operational matrix of derivative as

$$D^1 = (d_{ij}) = \begin{cases} \frac{4i}{\varepsilon_j \tau}, & j = 0, 1, \ldots, i = j + q, \\ 0, & \text{otherwise}, \end{cases}$$

and q satisfies in the following relation as

$$\begin{cases} q = 1, 3, 5, \ldots, M - 1, & M \text{ is even}, \\ q = 1, 3, 5, \ldots, M, & M \text{ is odd}. \end{cases}$$

The following lemma and theorem specify the fractional order derivative of the shifted Chebyshev polynomial.

Lemma 1 ([7]). *Assume that $T_{\tau,j}(z)$ is the shifted Chebyshev polynomial. Then, $D^{\lambda}T_{\tau,j}(z)=0,\ j=0,1,\ldots,\lceil\lambda\rceil-1$.*

Note that $\lceil\lambda\rceil$ represents the smallest integer higher than or equal to λ.

Theorem 1 ([7]). *Assume $\psi_{\tau,M}(t)$ is the shifted Chebyshev vector introduced in Eq. (9). If λ is considered as the fractional derivative order then*

$$D^{\lambda}\psi_{\tau,M}(t)\approx \boldsymbol{D}^{\lambda}_{\tau,M}\psi_{\tau,M}(t),$$

such that $\boldsymbol{D}^{\lambda}_{\tau,M}$ denotes the $(M+1)\times(M+1)$ operational matrix of the fractional derivative as

$$D^{\lambda}_{\tau,M}=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & 0\\ d_{\lambda}(\lceil\lambda\rceil,0) & d_{\lambda}(\lceil\lambda\rceil,1) & d_{\lambda}(\lceil\lambda\rceil,2) & \cdots & d_{\lambda}(\lceil\lambda\rceil,M)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ d_{\lambda}(i,0) & d_{\lambda}(i,1) & d_{\lambda}(i,2) & \cdots & d_{\lambda}(i,M)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ d_{\lambda}(M,0) & d_{\lambda}(M,1) & d_{\lambda}(M,2) & \cdots & d_{\lambda}(M,M) \end{pmatrix},$$

where

$$d_{\lambda}(i,j)=\sum_{p=\lceil\lambda\rceil}^{i}\frac{(-1)^{i-p}2i(i+s-1)!\Gamma\left(p-\lambda+\frac{1}{2}\right)}{\varepsilon_j\tau^{\lambda}\Gamma\left(p+\frac{1}{2}\right)(i-p)!\Gamma(p-\lambda-j+1)\Gamma(p+j-\lambda+1)}.$$

Note that in D^{λ}, the first $\lceil\lambda\rceil$ rows are all zero.

3 Numerical algorithm

Here, we present a numerical method to approximate the solution of the HTLV-I model. The Gauss-Radau shifted Chebyshev collocation method is used to approximate the solution of the fractional model of HTLV-I as follows

$$\begin{cases} D^{\lambda}T_H(t)=\mu-\eta_T T_H(t)-kT_V(t)T_H(t),\\ D^{\lambda}T_I(t)=k_1T_V(t)T_H(t)-(\eta_L+\gamma)T_I(t),\\ D^{\lambda}T_V(t)=\gamma T_I(t)-(\eta_A+\zeta)T_V(t),\\ D^{\lambda}T_L(t)=\zeta T_V(t)+sT_L(t)\left(1-\frac{T_L(t)}{(T_L)_{\max}}\right)-\eta_N T_L(t), \end{cases}\tag{10}$$

with initial conditions

$$T_H(0)=\rho_1,\ \ T_I(0)=\rho_2,\ \ T_V(0)=\rho_3,\ \ T_L(0)=\rho_4.\tag{11}$$

Assume that $T_H(t),T_I(t),T_V(t)$, and $T_L(t)$ are smooth functions on $\Lambda=[0,\tau]$. This technique converts nonlinear systems of fractional ordinary differential equations ((10)-(11)) into a nonlinear system of algebraic equations. As a result, with using Eq. (8), we will have functions $T_H(t), T_I(t), T_V(t)$, and $T_L(t)$ as follows:

$$T_H(t) = \sum_{j=0}^{M} c_j T_{\tau,j}(t), \quad T_I(t) = \sum_{j=0}^{M} d_j T_{\tau,j}(t), \quad T_V(t) = \sum_{j=0}^{M} e_j T_{\tau,j}(t), \quad T_L(t) = \sum_{j=0}^{M} f_j T_{\tau,j}(t), \tag{12}$$

where c_j, d_j, e_j and f_j for $j = 0, \ldots, M$ are the unknown coefficients. We substitute Eq. (12) into Eqs. (10-11) to determine $4(M+1)$ unknowns. Therefore, we have

$$\begin{cases} \sum_{j=0}^{M} c_j D^\lambda T_{\tau,j}(t) = \mu - \eta_T \sum_{j=0}^{M} c_j T_{\tau,j}(t) - k \sum_{j=0}^{M} c_j T_{\tau,j}(t) \sum_{j=0}^{M} e_j T_{\tau,j}(t) + R_{1,M}(t), \\ \sum_{j=0}^{M} d_j D^\lambda T_{\tau,j}(t) = k_1 \sum_{j=0}^{M} c_j T_{\tau,j}(t) \sum_{j=0}^{M} e_j T_{\tau,j}(t) - (\eta_L + \gamma) \sum_{j=0}^{M} d_j T_{\tau,j}(t) + R_{2,M}(t), \\ \sum_{j=0}^{M} e_j D^\lambda T_{\tau,j}(t) = \gamma \sum_{j=0}^{M} d_j T_{\tau,j}(t) + (\eta_A + \zeta) \sum_{j=0}^{M} e_j T_{\tau,j}(t) + R_{3,M}(t), \\ \sum_{j=0}^{M} f_j D^\lambda T_{\tau,j}(t) = \zeta \sum_{j=0}^{M} e_j T_{\tau,j}(t) + s \sum_{j=0}^{M} f_j T_{\tau,j}(t) \left(1 - \frac{\sum_{j=0}^{M} f_j T_{\tau,j}(t)}{(T_L)_{max}}\right) \\ \qquad\qquad -\eta_N \sum_{j=0}^{M} f_j T_{\tau,j}(t) + R_{4,M}(t), \end{cases} \tag{13}$$

and

$$\sum_{j=0}^{M} c_j T_{\tau,j}(0) = \rho_1, \ \sum_{j=0}^{M} d_j T_{\tau,j}(0) = \rho_2, \ \sum_{j=0}^{M} e_j T_{\tau,j}(0) = \rho_3, \ \sum_{j=0}^{M} f_j T_{\tau,j}(0) = \rho_4. \tag{14}$$

The residual functions are denoted by $R_{i,M}(t), i = 1, \ldots, 4$ on the interval $[0, \tau]$. Suppose $\{t_{4,M,s}\}_{s=0}^{M}$ is the node of the $M+1$-point Gauss-Radau shifted Chebyshev quadrature formula. We will now assume that the residual functions for $\{t_{4,M,s}\}_{s=0}^{M}$ are all zero. As a result, the following $4M$ algebraic equations for $4(M+1)$ unknowns c_j, d_j, e_j and f_j, for $j = 0, \ldots, M$ are obtained by

$$\begin{cases} \sum_{j=0}^{M} c_j D^\lambda T_{\tau,j}(t_{4,M,s}) = \mu - \eta_T \sum_{j=0}^{M} c_j T_{\tau,j}(t_{4,M,s}) - k \sum_{j=0}^{M} c_j T_{\tau,j}(t_{4,M,s}) \sum_{j=0}^{M} e_j T_{\tau,j}(t_{4,M,s}), \\ \sum_{j=0}^{M} d_j D^\lambda T_{\tau,j}(t_{4,M,s}) = k_1 \sum_{j=0}^{M} c_j T_{\tau,j}(t_{4,M,s}) \sum_{j=0}^{M} e_j T_{\tau,j}(t_{4,M,s}) - (\eta_L + \gamma) \sum_{j=0}^{M} d_j T_{\tau,j}(t_{4,M,s}), \\ \sum_{j=0}^{M} e_j D^\lambda T_{\tau,j}(t_{4,M,s}) = \gamma \sum_{j=0}^{M} d_j T_{\tau,j}(t_{4,M,s}) + (\eta_A + \zeta) \sum_{j=0}^{M} e_j T_{\tau,j}(t_{4,M,s}), \\ \sum_{j=0}^{M} f_j D^\lambda T_{\tau,j}(t_{4,M,s}) = \zeta \sum_{j=0}^{M} e_j T_{\tau,j}(t_{4,M,s}) + s \sum_{j=0}^{M} f_j T_{\tau,j}(t_{4,M,s}) \left(1 - \frac{\sum_{j=0}^{M} f_j T_{\tau,j}(t_{4,M,s})}{(T_L)_{max}}\right) \\ \qquad\qquad -\eta_N \sum_{j=0}^{M} f_j T_{\tau,j}(t_{4,M,s}). \end{cases} \tag{15}$$

By Considering $t_{4,M,0} = 0$, Eq. (14) can be written as

$$\sum_{j=0}^{M} c_j T_{\tau,j}(t_{4,M,0}) = \rho_1, \ \sum_{j=0}^{M} d_j T_{\tau,j}(t_{4,M,0}) = \rho_2, \ \sum_{j=0}^{M} e_j T_{\tau,j}(t_{4,M,0}) = \rho_3, \ \sum_{j=0}^{M} f_j T_{\tau,j}(t_{4,M,0}) = \rho_4. \tag{16}$$

Eq. (16) yields additional 4 algebraic equations for the unknowns c_j, d_j, e_j and f_j for $j = 0, \dots, M$. System of Equations (15)-(16) form a nonlinear system of algebraic equations that includes $4(M+1)$ equations and $4(M+1)$ unknown c_j, d_j, e_j and f_j for $j = 0, \dots, M$.

Algorithm 1 Solving problem (10) subject to initial conditions (11).

Step 1: Compute the approximate of $T_H(t), T_I(t), T_V(t)$ and $T_L(t)$ in form Eq. (12).
Step 2: Compute Eqs. (13) and (14) by setting the approximations obtained in step 1 in (10) and (11).
Step 3: Suppose that $R_{i,M}(t), i = 1, \dots, 4$ in Eq. (13) for $\{t_{4,M,s}\}_{s=0}^{M}$ vanish.
Step 4: Get Eq. (15) by placing $\{t_{4,M,s}\}_{s=1}^{M}$ in Eq. (13).
Step 5: Rewrite Eq. (14) in the form Eq. (16).
Step 6: Compute unknown c_j, d_j, e_j and f_j for $j = 0, \dots, M$ by solving system (15)-(16).
Step 7: Compute $T_H(t), T_I(t), T_V(t), T_L(t)$ by placing c_j, d_j, e_j and f_j for $j = 0, \dots, M$ in Eq. (12).

4 Convergence analysis

Let $\Lambda = [0, \tau]$ and $P_M(\Lambda) = \text{span}\{T_{\tau,0}(t), T_{\tau,1}(t), \dots, T_{\tau,M}(t)\}$ for any positive integer number M. We define $\Pi_M f$ from $L^2(\Lambda)$ into $P_M(\Lambda)$ as follows

$$(\Pi_M f - f, u) = 0, \quad \forall u \in P_M(\Lambda),$$

in other words,

$$(\Pi_M f)(t) = \sum_{j=0}^{M} a_j T_{\tau,j}(t).$$

$\Pi_M f$ is considered as the best approximation of f out of $P_M(\Lambda)$ [20]. The Chebyshev weighted Sobolev space $B^m(\Lambda)$ is defined as follows

$$B^m(\Lambda) = \left\{ f : \frac{\partial^k f}{\partial t^k} \in L^2(I), k = 0, 1, \dots, m \right\}.$$

The inner product and norm associated with $B^m(\Lambda)$ are

$$(f, u)_{B^m} = \sum_{i=0}^{m} \left(\frac{\partial^i f}{\partial t^i}, \frac{\partial^i u}{\partial t^i} \right), \quad \|f\|_{B^m} = (f, f)_{B^m}^{\frac{1}{2}}.$$

As pointed by [20], $H^m(I)$ is a subspace of $B^m(I)$, i.e., $\|f\|_{B^m} \le c\|f\|_{H^m}$, $m \ge 0$.

Theorem 2 ([5]). *For all $f \in H^m(\Lambda), m \ge 0$, we have $\|f - \Pi_M f\|_{L^2(\Lambda)} \le CN^{-m}\|f\|_{H^m}$, where C is constant.*

Theorem 3 ([4]). *If $0 \le l < m \le M+1$, for any $f \in B^m(\Lambda)$, we have*

$$\left\| D_t^l (f - \Pi_M f) \right\| \le C_1 \sqrt{\frac{(M-m+1)!}{(M-l+1)!}} (M+m)^{\frac{l-m}{2}} \left\| \partial_t^m f \right\| \le M+1, \quad 0 \le l < m,$$
$$\le C_1 \sqrt{\frac{(M-m+1)!}{(M-l+1)!}} (M+m)^{\frac{l-m}{2}} \|f\|_{B^m}.$$

and in Hilbert space we have

$$\left\|D_t^l\left(f-\Pi_M f\right)\right\|_{L^2(\Lambda)} \leq C_1 \sqrt{\frac{(M-m+1)!}{(M-l+1)!}}(M+m)^{\frac{l-m}{2}}\|f\|_{H^m}.$$

Theorem 4. *Suppose $f \in L^2(\Lambda), n_\gamma < s \leq M+1$ and $n_\gamma - 1 < \gamma \leq n_\gamma = \lceil \gamma \rceil$ and $s \in \mathbb{N}$. Then*

$$\|D^\gamma f - D^\gamma(\Pi_M f)\|_{L^2(\Lambda)} \leq \left(\frac{C_\gamma}{\Gamma(n_\gamma+1-\gamma)}\right)\left(\sqrt{\frac{(M-s+1)!}{(M-n_\gamma+1)!}}(M+s)^{\frac{(n_\gamma-s)}{2}}\right)\|f\|_{H^r(\Lambda)},$$

where C_γ is constant.

Proof. Employing Eq. (4), Theorem 3, and $\|f*g\|_p \leq \|f\|_1\|g\|_p$(see [4]), we obtain

$$\begin{aligned}
\|D^\gamma \mathrm{f} - D^\gamma(\Pi_M f)\|^2_{L^2(\Lambda)} &= \left\|I^{n_\gamma-\gamma}\left(\left(D^n_\gamma f\right) - D^{n_\gamma}(\Pi_M f)\right)\right\|^2_{L^2(\Lambda)} \\
&= \left\|\frac{t^{n_\gamma-\gamma-1}}{\Gamma(n_\gamma-\gamma)} * \left(D^{n_\gamma}\mathrm{f} - D^{n_\gamma}(\Pi_M f)\right)\right\|^2_{L^2(\Lambda)} \\
&\leq \left\|\frac{t^{n_\gamma-\gamma-1}}{\Gamma(n_\gamma-\gamma)}\right\|^2_1 \|D^{n_\gamma} f - D^{n_\gamma}(\Pi_M f)\|^2_{L^2(\Lambda)} \\
&\leq \left(\frac{1}{\Gamma(n_\gamma+1-\gamma)}\right)\left(C_\gamma\sqrt{\frac{(M-s+1)!}{(M-n_\gamma+1)!}}(M+s)^{\frac{(n_\gamma-s)}{2}}\right)^2 \|f\|^2_{H^r(\Lambda)},
\end{aligned}$$

which completes the proof. □

Now, we rewrite the model (2) with initial condition (3) as follows

$$D^\lambda Y(t) = f(t, Y(t)), \quad 0 < \lambda \leq 1, \quad t \in [0, \tau], \tag{17}$$

with initial conditions $Y(0) = Y_0$, where $Y(t) = [y_1(t), y_2(t), y_3(t), y_4(t)]$ is the exact solution and

$$f(t, Y(t)) = [f_1(t, Y(t)), f_2(t, Y(t)), f_3(t, Y(t)), f_4(t, Y(t))].$$

We assume $f(t, Y(t))$ satisfies the global Lipschitz condition, i.e., there exists a constant L such that

$$\|f(t, Y_1(t)) - f(t, Y_2(t))\| \leq L\|Y_2(t) - Y_1(t)\|, \tag{18}$$

for all $t \in [0, \tau]$ and $Y_1, Y_2 \in \mathbb{R}^4$. In order to demonstrate the convergence of the Chebyshev collocation spectral method for solving a Caputo fractional nonlinear ODE system of the HTLV-I model which consists of 4 equations and 4 unknowns, we need to show that the numerical solution obtained using this method approaches the exact solution as the number of collocation points increases.

The numerical solution obtained using this method denotes by $Y_n(t)$, where n is the number of collocation points used. The error between the numerical solution $Y_n(t)$ and the exact solution $Y(t)$ is denoted by $e_n(t) = Y_n(t) - Y(t)$. Substituting $Y_n(t)$ into the differential equation system, we obtain

$$D^\lambda Y_n(t) - f(t, Y_n(t)) - R(t, Y_n(t)) = 0, \tag{19}$$

where $R(t,Y_n(t))$ refers to the residual function. By subtraction Eq. (19) from Eq. (17), we have

$$R(t,Y_n(t)) = D^\lambda e_n(t) - f(t,Y_n(t)) + f(t,Y(t)), \tag{20}$$

with initial conditions $e_n(0) = 0$. Therefore, by using the triangle inequality in Eq. (20) and applying Eq. (18), we can obtain

$$\begin{aligned} \|R(t,Y_n(t))\|_{L^2(\Lambda)} &= \|D^\lambda e_n(t) - f(t,Y_n(t)) + f(t,Y(t))\|_{L^2(\Lambda)} \\ &\le \|D^\lambda e_n(t)\|_{L^2(\Lambda)} + \|f(t,Y(t)) - f(t,Y_n(t))\|_{L^2(\Lambda)} \\ &\le |D^\lambda e_n(t)\|_{L^2(\Lambda)} + L\|Y(t) - Y_n(t)\|_{L^2(\Lambda)}. \end{aligned} \tag{21}$$

According to result of Theorem 2 and Theorem 4, we can conclude if $n \to \infty$ then $R(t,Y_n(t)) \to 0$. Therefore, the numerical solution $Y_n(t)$ approaches the exact solution $Y(t)$ as $n \to \infty$.

5 Results and discussions

In this section, we apply the numerical method described in section (3) to solve the fractional differential equations system (10)-(11) with

$$\begin{array}{lllll} s = 0.0003, & \gamma = 0.0004, & \zeta = 0.00004, & (T_L)_{max} = 2200, & \mu = 6, \\ \eta_T = 0.6, & \eta_L = 0.006, & \eta_A = 0.05, & \eta_N = 0.0005, & \tau = 1 \\ k = \text{varies}, & k_1 = \text{varies}. & & & \end{array} \tag{22}$$

First, the presented method is performed for $\lambda = 1$, and the obtained results are compared with the Natural-Adomian Decomposition method (N-ADM) [17] (see Figures (2)-(3)). According to the graphical results in Figures (2)-(3), the proposed scheme provides accurate results similar to the N-ADM method. Figures (4)-(5) demonstrate the approximated solution of $T_H(t), T_I(t), T_V(t)$ and $T_L(t)$ by using the proposed method for $\lambda = 0.6, 0.7, 0.8, 0.9, 0.98, 1$ and $M = 6, 8$, respectively. As a result, the solution to the fractional system when $\lambda \to 1$, converges to the solution of an integer-order system.

Since we do not have an analytical solution for the problem mentioned, we utilize the formula proposed by Cen et al. [6] to determine the order of convergence. First of all, we choose a sequence of step sizes $h_1, h_2, ..., h_n$ such that $h_{i+1} = \frac{h_i}{2}$, for computing the numerical solution of the fractional HTLV-I model. Compute the error for each solution by comparing it with a reference solution, which can be a numerical solution obtained using a smaller step size as follows: $E_{h,h/2} = \|U_h - U_{h/2}\|$. Since we only know “the exact solution” on mesh points, we use the linear interpolation to get solutions at other points. We define $E_{h,h/2}$ as a difference between the numerical solutions obtained with two different time steps h and $h/2$. The convergence rate is calculated as

$$C_h = \log_2(\frac{E_{h,h/2}}{E_{h/2,h/4}}), \tag{23}$$

where $E_{h,h/2}$ and $E_{h/2,h/4}$ refer to the absolute error between the solutions with step sizes $\{h, h/2\}$ and $\{h/2, h/4\}$, respectively. The results are illustrated in Table 2.

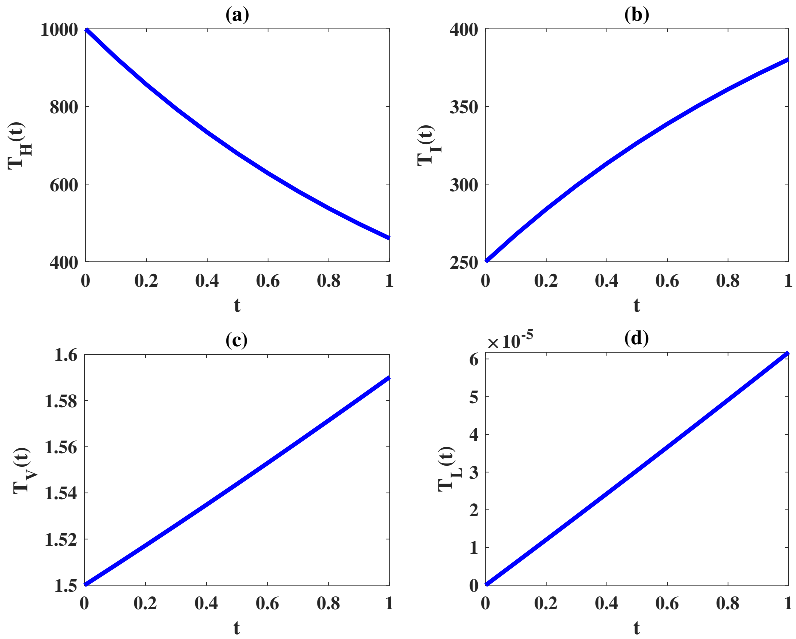

Figure 2: The approximate solutions of (a) $T_H(t), (b)$ $T_I(t), (c)$ $T_V(t)$ and (d) $T_L(t)$ by the Chebyshev collocation method for $M = 6, k = k_1 = 0.1$ and $\lambda = 1$.

Table 2: Numerical errors $E_{h,h/2}$ and rate of convergence C_h for $M = 8$.

λ	h	T_H			T_I			T_V			T_L		
		$\|.\|_\infty$	$\|.\|_2$	C_h	$\|.\|_\infty$	$\|.\|_2$	C_h	$\|.\|_\infty$	$\|.\|_2$	C_h	$\|.\|_\infty$	$\|.\|_2$	C_h
0.6	$\frac{1}{10}$	$8.8453e-03$	$4.2345e-02$	–	$1.7345e-03$	$5.3345e-02$	–	$6.7890e-03$	$7.1692e-02$	–	$1.2345e-02$	$2.1365e-01$	–
	$\frac{1}{20}$	$6.9733e-03$	$3.3456e-02$	1.5921	$1.4567e-03$	$3.7561e-02$	1.2311	$4.3210e-03$	$5.1613e-02$	1.0321	$8.7654e-03$	$6.5610e-02$	1.2345
	$\frac{1}{40}$	$4.2114e-04$	$6.4567e-03$	1.6234	$1.2890e-03$	$1.3810e-02$	1.4763	$1.2345e-03$	$6.9025e-03$	1.1234	$6.7890e-03$	$5.3591e-02$	1.4567
	$\frac{1}{80}$	$3.5809e-04$	$7.6789e-03$	1.7142	$2.3456e-04$	$4.1453e-03$	1.6524	$8.7654e-04$	$4.1351e-03$	1.6778	$4.3210e-03$	$3.1270e-02$	1.5221
	$\frac{1}{160}$	$1.5829e-04$	$5.7654e-03$	1.8932	$1.7890e-04$	$2.3810e-03$	1.8747	$5.6789e-04$	$3.1791e-03$	1.4321	$1.2345e-03$	$1.1397e-02$	1.8823
0.7	$\frac{1}{10}$	$9.4322e-03$	$2.1098e-02$	–	$4.1234e-03$	$5.1639e-02$	–	$8.7654e-03$	$6.2604e-02$	–	$9.8765e-03$	$7.1568e-02$	–
	$\frac{1}{20}$	$8.8715e-04$	$7.3210e-03$	1.2567	$3.6789e-03$	$2.4712e-02$	1.2103	$6.7890e-03$	$5.5291e-02$	1.1123	$7.6543e-03$	$6.2503e-02$	1.3254
	$\frac{1}{40}$	$6.3451e-04$	$6.7891e-03$	1.4875	$7.7890e-04$	$6.5581e-03$	1.3654	$4.3210e-03$	$3.1241e-02$	1.4612	$5.4321e-03$	$4.9991e-02$	1.4476
	$\frac{1}{80}$	$4.7896e-04$	$4.6543e-03$	1.8172	$5.9012e-04$	$3.8501e-03$	1.6978	$1.2345e-03$	$2.6390e-02$	1.5998	$3.2198e-03$	$3.1158e-02$	1.5654
	$\frac{1}{160}$	$3.2104e-04$	$2.8765e-03$	1.9486	$3.8798e-05$	$3.6118e-04$	1.8105	$5.6789e-04$	$3.5019e-03$	1.7565	$1.2345e-03$	$2.1343e-02$	1.7312
0.8	$\frac{1}{10}$	$5.7615e-03$	$3.4567e-02$	–	$9.8765e-04$	$5.4715e-03$	–	$1.2345e-02$	$3.1675e-01$	–	$8.7654e-03$	$6.9024e-02$	–
	$\frac{1}{20}$	$3.3201e-03$	$2.1260e-02$	1.2764	$7.6543e-04$	$3.4503e-03$	1.0432	$8.7654e-03$	$6.1684e-02$	1.0123	$6.7890e-03$	$5.1997e-02$	1.0845
	$\frac{1}{40}$	$1.2364e-03$	$1.4321e-02$	1.4279	$5.4321e-04$	$2.1621e-03$	1.1325	$6.7890e-03$	$4.2090e-02$	1.4279	$4.3210e-03$	$3.2917e-02$	1.3256
	$\frac{1}{80}$	$7.6594e-04$	$6.4007e-03$	1.8268	$7.8764e-05$	$4.8842e-04$	1.4756	$4.3210e-03$	$2.1218e-02$	1.7854	$1.2345e-03$	$2.1315e-02$	1.5567
	$\frac{1}{160}$	$4.8476e-04$	$3.6044e-03$	1.9893	$5.8168e-05$	$2.4011e-04$	1.6801	$1.2345e-03$	$1.5941e-02$	1.8889	$5.6789e-04$	$3.7321e-03$	1.7398
0.9	$\frac{1}{10}$	$9.7803e-04$	$8.7315e-03$	–	$8.8765e-03$	$9.5361e-02$	–	$9.8765e-03$	$9.1765e-02$	–	$1.2345e-02$	$4.1365e-01$	–
	$\frac{1}{20}$	$6.2211e-04$	$6.2345e-03$	1.5992	$7.6543e-03$	$6.7511e-02$	1.3054	$7.6543e-03$	$6.7540e-02$	1.3567	$8.7654e-03$	$5.7494e-02$	1.2145
	$\frac{1}{40}$	$4.6766e-04$	$5.7891e-03$	1.6678	$5.4321e-03$	$3.7301e-02$	1.5321	$8.6803e-03$	$5.4833e-02$	1.4654	$6.7890e-03$	$3.1850e-02$	1.4756
	$\frac{1}{80}$	$3.1240e-04$	$4.1098e-03$	1.7337	$3.2198e-03$	$2.6111e-02$	1.6298	$5.4321e-03$	$4.4121e-02$	1.6765	$4.3210e-03$	$2.6280e-02$	1.7890
	$\frac{1}{160}$	$1.7281e-04$	$1.4685e-03$	1.9465	$1.2345e-03$	$1.1095e-02$	1.7801	$3.2198e-03$	$1.1108e-02$	1.7998	$1.2345e-03$	$1.4205e-02$	1.9234
1	$\frac{1}{10}$	$9.9354e-03$	$8.7654e-02$	–	$7.0715e-03$	$6.1185e-02$	–	$1.2345e-02$	$2.1355e-01$	–	$8.7654e-03$	$7.5610e-02$	–
	$\frac{1}{20}$	$6.3452e-04$	$8.4567e-03$	1.3634	$6.6473e-03$	$5.2403e-02$	1.2356	$8.7654e-03$	$5.8094e-02$	1.4067	$6.7890e-03$	$6.1688e-02$	1.1523
	$\frac{1}{40}$	$4.8376e-04$	$6.7654e-03$	1.4076	$4.1128e-03$	$3.1974e-03$	1.3567	$6.7890e-03$	$4.7691e-02$	1.6578	$4.3210e-03$	$3.5507e-02$	1.3765
	$\frac{1}{80}$	$3.6504e-04$	$5.4321e-03$	1.8765	$5.1098e-04$	$2.7099e-03$	1.6678	$4.3210e-03$	$3.0510e-02$	1.7613	$1.2345e-03$	$2.7165e-02$	1.6789
	$\frac{1}{160}$	$1.4382e-04$	$2.2345e-03$	1.9543	$3.2765e-04$	$1.5795e-03$	1.8743	$1.2345e-03$	$2.5385e-02$	1.8765	$5.6789e-04$	$7.2839e-03$	1.8456

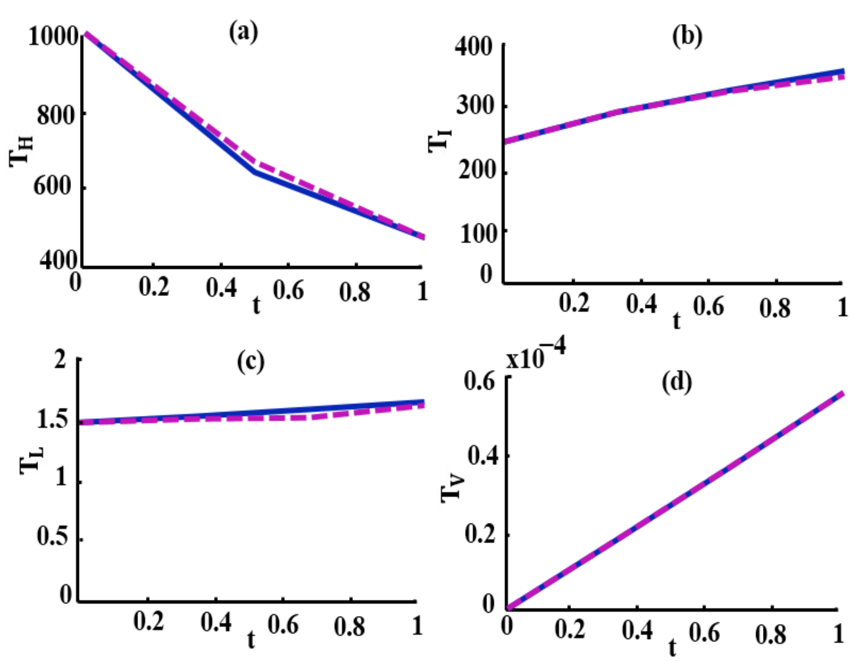

Figure 3: The solutions of $(a)\ T_H(t), (b)\ T_I(t), (c)\ T_V(t)$ and $(d)\ T_L(t)$ by the N-ADM method (Solid line) for $k = k_1 = 0.1, \lambda = 1$ compared with the Runge-Kutta method of order 4 (Dashed line) [17].

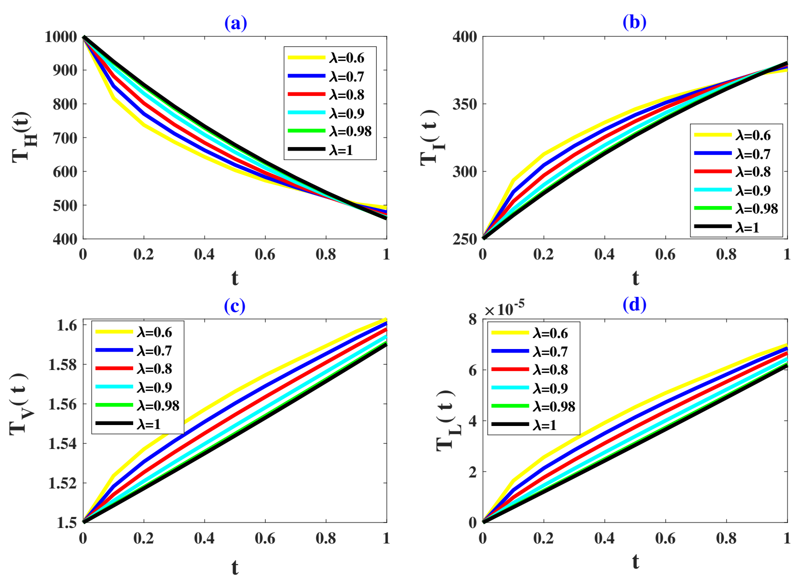

Figure 4: The approximate solutions of $(a)\ T_H(t), (b)\ T_I(t), (c)\ T_V(t)$ and $(d)\ T_L(t)$ by the the Chebyshev collocation method for $M = 6, k = k_1 = 0.1,$ and various values of λ.

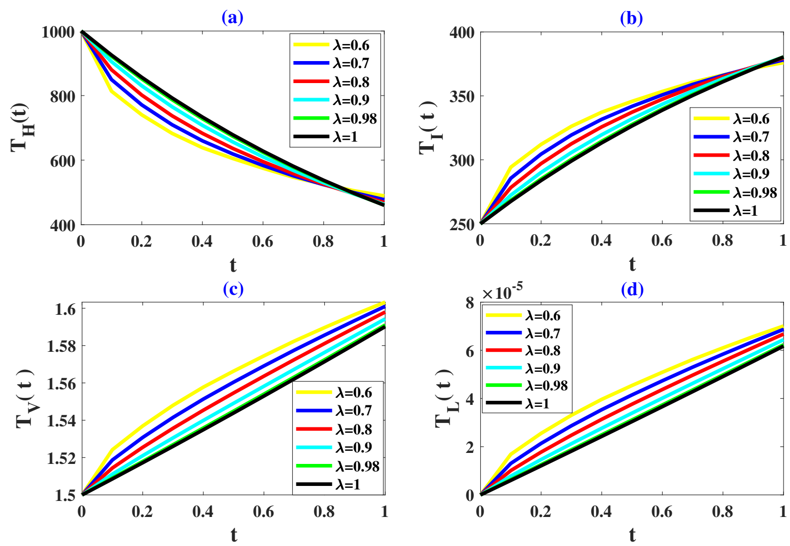

Figure 5: The approximate solutions of (a) $T_H(t), (b)$ $T_I(t), (c)$ $T_V(t)$ and (d) $T_L(t)$ by the Chebyshev collocation method for $M = 8, k = k_1 = 0.1$ and various values of λ.

6 Conclusion

In this paper, we propose an efficient method to approximate the solution of the fractional model of HTLV-I. This technique is based on the Chebyshev collocation method. The error bound and convergence analysis for the proposed method are discussed. The proposed method is performed to solve the model for various values of fractional order derivatives. According to the graphical results, the numerical approximation relies on the order of the time-fractional derivative. When $\lambda \to 1$, the approximate solution of the model in (10) converges that of the standard model for $\lambda = 1$. Since the analytical solution of the model is unknown, the obtained findings are compared with the numerical results of other literature to demonstrate the capability of the suggested method.

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