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Persistence in mean and extinction of a hybrid stochastic delay Gompertz model with Levy jumps*

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Abstract. This paper deals with a stochastic delay Gompertz model under regime switching with Levy jumps. Firstly, the existence of a unique global positive solution has been derived. Secondly, sufficient conditions for extinction and persistence in mean are obtained. Finally, an example is given to illustrate our main results.The results in this paper indicate that Levy jumps noise, the white noise and switching noise have certain effects on the properties of the model.

Keywords: Gompertz model, persistence in mean, extinction, Levy jumps noise, Markov chains.
AMS Subject Classification 2010: 60H10, 92B05.

1 Introduction

To the best of our knowledge, the Gompertz model in mathematical ecology is considered as one of the most suitable models to describe the growth of certain types of tumors. The classical delay Gompertz equation is describe as

$$\mathrm{d}x(t)/\mathrm{d}t = x(t)[a - b\ln x(t-\tau)], \quad t > 0, \tag{1}$$

with the initial date $x_0 = \{\rho(\varsigma), -\tau \le \varsigma \le 0\}$ where $\rho(\varsigma)$ is a continuous function from $[-\tau, 0]$ to R^+. The parameter a is the intrinsic growth rate of the tumour related to the initial mitosis rate and b is the growth deceleration factor.

However, the population dynamics in the real world are often affected by various types of environmental noises, including the white noise [3, 11, 16, 17, 31], switching noise [12, 13, 20–22, 29, 30] and

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jumping noise [1, 2, 7, 14, 18, 19, 27, 28]. M. Liu and K. Wang [16] investigated the population dynamical behaviors of a stochastic systems, then gave some conditions for the extinction and persistence of the cooperative system under the random perturbations. In [13, 21], the authors explored the stochastic models with switching noise, and reveal that the randomness of the environment has a certain effect on the stochastically permanent and extinction of the model. Y. Guo [7] and Q. Liu [19] proposed stochastic systems with Levy jumps and Markov chains, showing that the asymptotic properties of the considered system are closely effected by Levy jumps and Markov switching noise.

If we consider the white noise in model (1), assume that the intrinsic growth rate a are disturbed by the white noise, it will become $a \to a + \sigma \dot{B}(t)$, where $\dot{B}(t)$ is the formal derivative and σ represents the intensity of the white noise. As a result, model (1) can be described as the following Itô equation

$$\mathrm{d}x(t) = x(t)[a - b \ln x(t-\tau)]\mathrm{d}t + \sigma x(t)\mathrm{d}B(t), \qquad t > 0, \tag{2}$$

where $B(t)$ is a standard Brownian motion. Due to the importance both in the theoretical and practical, the stochastic Gompertz model perturbed by Brownian motion has been studied extensively by many authors, and there is a great amount of literature on this topic, see for example [5, 6, 8, 10, 23, 25]. G. Hu [8] discussed the asymptotic behaviors of a stochastic Gompertz model with Markovian switching. M. Jovanovic and M. Krstic [10] considered a stochastic Gompertz model with time delay and gave sufficient conditions for the persistence in mean and extinction.

Now, let us consider another type of environmental noise in model (2), that is the switching noise. Some scholars in [12, 13, 22, 30] pointed that the switching noise may cause a rapid changing in a species from one state to another, this switching is memoryless and the waiting time for the next switch has an exponential distribution. Thus, we can describe this switching with a Markov chain $\xi(t)$, and then the model (2) becomes

$$\mathrm{d}x(t) = x(t)[a(\xi(t)) - b(\xi(t)) \ln x(t-\tau)]\mathrm{d}t + \sigma(\xi(t))x(t)\mathrm{d}B(t), \qquad t > 0, \tag{3}$$

where $a(i),\ b(i),\ \sigma(i)$ are all positive for every $i \in \mathbb{S}$, and $\xi(t)$ is a right-continuous Markov chain, taking values in a finite state space $S = \{1, 2, \ldots, N\}$.

Moreover, from the view of the biological point, the population systems are often disturbed by the sudden environmental perturbations, for instance, earthquakes, hurricanes, tsunamis, etc., (see e.g. [1, 2, 7, 14, 18]). These phenomena cannot be accurately described by the white noise and switching noise, so introducing Levy jumps into this model may be a reasonable approach. There are a large number of literatures on this topic, for example, J. Bao et al. [1, 2], Q. Liu et al. [18, 19] and the references therein. Inspired by the above discussions, we will consider the following stochastic delay Gompertz model under regime switching with Levy jump

$$\begin{aligned}\mathrm{d}x(t) &= r(\xi(t))[x(t) - \beta(\xi(t)))x(t) \ln x(t-\tau)]\mathrm{d}t \\ &\quad + \sigma(\xi(t)))x(t)\mathrm{d}B(t) + \int_{\mathbb{Y}} x(t^-)h((\xi(t)), u)\tilde{N}(\mathrm{d}t, \mathrm{d}u), \qquad t > 0,\end{aligned} \tag{4}$$

where $x(t^-)$ represent the left limit of $x(t)$, the parameters $r_i,\ \beta_i,\ \sigma_i$ are all positive constants for each $i \in \mathbb{S}$, N is a Poisson counting measure with compensator $\tilde{N}$ and characteristic measure λ on a measurable subset $\mathbb{Y}$ of $(0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$, $\tilde{N}(\mathrm{d}t, \mathrm{d}u) = N(\mathrm{d}t, \mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$, the functions $h : \mathbb{Y} \times (0, \infty) \longrightarrow \mathbb{R}$ is bounded and continuous with respect to λ and is $\mathfrak{B}(\mathbb{Y}) \times \mathscr{F}_t$ -measurable.

As far as we know, there have been many excellent works on stochastic Gompertz models, but very little work on hybrid stochastic delay Gompertz models with jumps. Next we give the organization of this paper. In Section 2, we proved the existence of a unique global positive solution to Eq. (4). In Section 3, we gave the persistence in mean and extinction of Eq. (4) with Levy jumps and time delays. Finally, we concluded our paper with an example and a brief discussion for the further investigations.

2 The existence of positive solutions

Through out this paper we always assume that Markov chain $\xi(t)$ is irreducible, that is to say the following linear equation (see, [21,22])

$$\pi\Gamma = 0, \qquad \sum_{i=1}^{N}\pi_i = 1, \tag{5}$$

has a unique stationary solution $\pi = (\pi_1,\dots,\pi_N)$ satisfying $\pi_i > 0,\ i \in \mathbb{S}$. For more details of Markov chains, one can refer to [4,24,26] and other relevant references.

In this paper, for the sake of simplicity, let us define the following notation and give some assumptions

$$\check{r} = \max_{1\le i\le N} r(i), \qquad \hat{r} = \min_{1\le i\le N} r(i),$$
$$\check{\beta} = \max_{1\le i\le N} \beta(i), \qquad \hat{\beta} = \min_{1\le i\le N} \beta(i).$$

Assumption 1. *There exists a constant $M \ge 0$ such that*

$$\max_{i\in\mathbb{S}} |c(i) + r(i)\beta(i)| \le M, \tag{6}$$

where

$$c(i) =: r(i) - \frac{\sigma^2}{2} - \int_{\mathbb{Y}} [h(i,u) - \ln(1+h(i,u))]\lambda(\mathrm{d}u).$$

Assumption 2. *There exists a constant $L > 0$ such that, for any $t \ge -\tau$,*

$$\int_{\mathbb{Y}} (\ln(1+h(\xi(t),u)))^2\lambda(\mathrm{d}u) \le L. \tag{7}$$

Theorem 1. *Assume that there exists a positive constant K such that*

$$\max_{i\in\mathbb{S}} |c(i)| \le K. \tag{8}$$

Then there exists a unique global solution $x(t)$ for $t \ge -\tau$ to Eq. (4).

Proof. We obtain the stochastic delay differential equation by transforming $y(t) = \ln x(t)$.

$$\begin{aligned}\mathrm{d}y(t) = {} & [r(\xi(t)) - \frac{1}{2}\sigma^2(\xi(t)) - r(\xi(t))\beta(\xi(t))y(t-\tau)]\mathrm{d}t \\ & - \int_{\mathbb{Y}} [h(\xi(t),u) - \ln(1+h(\xi(t),u))]\lambda(\mathrm{d}u)\mathrm{d}t + \sigma(\xi(t))\mathrm{d}B(t) \\ & + \int_{\mathbb{Y}} \ln(1+h((\xi(t)),u))\tilde{N}(\mathrm{d}t,\mathrm{d}u),\end{aligned} \tag{9}$$

on $t \geq 0$ with the initial date $y_0 = \ln x_0 = \{\rho(\ln\varsigma), -\tau \leq \varsigma \leq 0\}$. It is easy to see that Eq. (9) satisfies the global Lipschitz condition and the linear growth condition under inequality (8). So for any initial date y_0, there is a unique global solution $y(t)$ on $t \geq -\tau$. Therefore, we get that $x(t) = e^{y(t)}$ is the unique global solution to Eq. (4) with initial data x_0, that is

$$\begin{aligned}
\mathrm{d}x(t) = \ & e^{y(t)}[r(\xi(t)) - \frac{1}{2}\sigma^2(\xi(t)) - r(\xi(t))\beta(\xi(t))y(t-\tau)]\mathrm{d}t + \frac{1}{2}e^{y(t)}\sigma^2(\xi(t))\mathrm{d}t \\
& - e^{y(t)}\int_{\mathbb{Y}}[h(\xi(t),u) - \ln(1+h(\xi(t),u))]\lambda(\mathrm{d}u)\mathrm{d}t \\
& + \int_{\mathbb{Y}}[e^{y(t)+\ln(1+h(\xi(t),u))} - e^{y(t)} - e^{y(t)}\ln(1+h(\xi(t),u))]\lambda(\mathrm{d}u)\mathrm{d}t \\
& + e^{y(t)}\sigma(\xi(t))\mathrm{d}B(t) + \int_{\mathbb{Y}}[e^{y(t)+\ln(1+h(\xi(t),u))} - e^{y(t)}]\tilde{N}(\mathrm{d}t,\mathrm{d}u) \\
= \ & x(t)[r(\xi(t)) - r(\xi(t))\beta(\xi(t))\ln x(t-\tau)]\mathrm{d}t + x(t)\sigma(\xi(t))\mathrm{d}B(t) \\
& + \int_{\mathbb{Y}} x(t^-)h(\xi(t),u)\tilde{N}(\mathrm{d}t,\mathrm{d}u).
\end{aligned}$$

□

3 Persistence in mean and extinction

In this section, we are going to study the survival and extinction of Eq. (4). For definitions of persistence in mean and extinction, we refer the reader to [9, 15] and the references therein.

Definition 1. *Suppose that $x(t)$ is a solution of Eq. (4). Then*
(1) $x(t)$ is said to be ***persistence in mean*** *if $\liminf_{t\to\infty} \frac{1}{t}\int_0^t x(s)\mathrm{d}s > 0$ a.s.,*
(2) $x(t)$ is said to be ***extinction*** *if $\lim_{t\to\infty} x(t) = 0$ a.s..*

Theorem 2. *Let Assumption 1 and Assumption 2 hold, and further assume that*

$$h_* = \sum_{i=1}^{N} \pi_i[c(i) + r(i)\beta(i)] > 0. \tag{10}$$

Then Eq. (4) is persistence in mean.

Proof. By using the generalised Itô formula to Eq. (4), we can get that

$$\begin{aligned}
\ln x(t) = \ & \ln x(0) + \int_0^t [c(\xi(s)) - r(\xi(s))\beta(\xi(s))\ln x(s-\tau)]\mathrm{d}s \\
& + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) + \int_0^t\int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u).
\end{aligned} \tag{11}$$

Elementary inequality $\ln x \leq x - 1$ for $x > 0$, implies that

$$\begin{aligned}
& \ln x(t) + \int_0^t r(\xi(s))\beta(\xi(s))\ln x(s-\tau)\mathrm{d}s \\
& \leq x(t) - 1 + \int_0^t r(\xi(s))\beta(\xi(s))x(s-\tau)\mathrm{d}s - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s \\
& \leq x(t) + \check{r}\check{\beta}\int_0^t x(s)\mathrm{d}s - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s + \check{r}\check{\beta}\int_{-\tau}^0 \alpha(\xi(s))\mathrm{d}s - 1 \\
& = e^{-\check{r}\check{\beta}t}\frac{d}{dt}\left(e^{\check{r}\check{\beta}t}\int_0^t x(s)\mathrm{d}s\right) - \int_0^t r(\xi(s))\beta(\xi(s))\mathrm{d}s + \check{r}\check{\beta}\int_{-\tau}^0 \alpha(\xi(s))\mathrm{d}s - 1.
\end{aligned} \tag{12}$$

By combining (11) and (12), we have

$$e^{-\check{r\beta}t}\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\check{r\beta}t}\int_0^t x(s)\mathrm{d}s\Big) \geq \ln x(0) + \int_0^t [c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) - \check{r\beta}\int_{-\tau}^0 \alpha(\xi(s))\mathrm{d}s + \int_0^t\int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u) + 1.$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\check{r\beta}t}\int_0^t x(s)\mathrm{d}s\Big) &\geq e^{\check{r\beta}t}\ln x(0) + e^{\check{r\beta}t}\int_0^t [c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s \\ &\quad + e^{\check{r\beta}t}\int_0^t \sigma(\xi(s))\mathrm{d}B(s) - e^{\check{r\beta}t}\check{r\beta}\int_{-\tau}^0 \alpha(\xi(s))\mathrm{d}s \\ &\quad + e^{\check{r\beta}t}\int_0^t\int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u) + e^{\check{r\beta}t}. \end{aligned} \tag{13}$$

Now, by integrating both sides of (13), it yields that

$$\begin{aligned}\int_0^t x(s)\mathrm{d}s &\geq \frac{C}{\check{r\beta}}(1-e^{-\check{r\beta}t}) + \int_0^t \Big(e^{\check{r\beta}(s-t)}\int_0^s [c(\xi(v)) + r(\xi(v))\beta(\xi(v))]\mathrm{d}v\Big)\mathrm{d}s \\ &\quad + \int_0^t \Big(e^{\check{r\beta}(s-t)}\int_0^s \sigma(\xi(v))\mathrm{d}B(v)\Big)ds \\ &\quad + \int_0^t \Big(e^{\check{r\beta}(s-t)}\int_0^s\int_{\mathbb{Y}} (\ln(1+h(\xi(v),u)))\tilde{N}(\mathrm{d}v,\mathrm{d}u)\Big)\mathrm{d}s \\ &= \frac{C}{\check{r\beta}}\big(1-e^{-\check{r\beta}t}\big) + \frac{1}{\check{r\beta}}\int_0^t [c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s \\ &\quad - \frac{1}{\check{r\beta}}\int_0^t e^{\check{r\beta}(s-t)}[c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s \\ &\quad + \frac{1}{\check{r\beta}}\int_0^t \sigma(\xi(s))\mathrm{d}B(s) - \frac{1}{\check{r\beta}}\int_0^t e^{\check{r\beta}(s-t)}\sigma(\xi(s))\mathrm{d}B(s) \\ &\quad + \frac{1}{\check{r\beta}}\int_0^t\int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u) \\ &\quad - \frac{1}{\check{r\beta}}\int_0^t\int_{\mathbb{Y}} e^{\check{r\beta}(s-t)}\ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u), \end{aligned} \tag{14}$$

where $C = \ln x(0) - \check{r\beta}\int_{-\tau}^0 \alpha(\xi(s))\mathrm{d}s + 1$. On the other hand, let

$$\begin{aligned} M_1(t) &= \int_0^t \sigma(\xi(s))\mathrm{d}B(s), M_2(t) = \int_0^t e^{\check{r\beta}(s-t)}\sigma(\xi(s))\mathrm{d}B(s), \\ M_3(t) &= \int_0^t\int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u), \\ M_4(t) &= \int_0^t\int_{\mathbb{Y}} e^{\check{r\beta}(s-t)}\ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u). \end{aligned}$$

Note that $M_i(t)$ $(i = 1,2,3,4)$ is a martingale with quadratic variation.

$$\begin{aligned}
\langle M_1(t), M_1(t)\rangle &= \int_0^t \sigma^2(\xi(s))\mathrm{d}s \le \check{\sigma}^2 t,\\
\langle M_2(t), M_2(t)\rangle &= \int_0^t \int_{\mathbb{Y}} e^{2r\check{\beta}(s-t)}\sigma^2(\xi(s))\mathrm{d}s \le \check{\sigma}^2 t,\\
\langle M_3(t), M_3(t)\rangle &= \int_0^t \int_{\mathbb{Y}} (\ln(1+h(\xi(s),u)))^2\lambda(\mathrm{d}u)\mathrm{d}s \le Lt,\\
\langle M_4(t), M_4(t)\rangle &= \int_0^t \int_{\mathbb{Y}} e^{2r\check{\beta}(s-t)}(\ln(1+h(\xi(s),u)))^2\lambda(\mathrm{d}u)\mathrm{d}s \le Lt.
\end{aligned}$$

Using the strong law of large numbers for local martingales (see, e.g., [24]), we have

$$\lim_{t\to\infty}\frac{M_i(t)}{t} = 0, \quad \text{a.s., } i = 1,2,3,4.$$

From this, we see that

$$\begin{aligned}
\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s \ge{}& \liminf_{t\to\infty}\frac{1}{tr\check{\beta}}\int_0^t [c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s\\
&- \liminf_{t\to\infty}\frac{1}{tr\check{\beta}}\int_0^t e^{r\check{\beta}(s-t)}[c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s. \qquad (15)
\end{aligned}$$

Since

$$\lim_{t\to\infty}\frac{\left|\int_0^t e^{r\check{\beta}(s-t)}[c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s\right|}{t} \le \lim_{t\to\infty}\frac{M(1-e^{-r\check{\beta}t})}{r\check{\beta}t} = 0,$$

holds a.s., then from (15)

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s \ge \liminf_{t\to\infty}\frac{1}{t}\int_0^t [c(\xi(s)) + r(\xi(s))\beta(\xi(s))]\mathrm{d}s = \sum_{i=1}^N \pi_i[c(i) + r(i)\beta(i)] = h_*.$$

Now, by condition (10), we have

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s > 0 \ \text{ a.s..}$$

This completes the proof.

□

Theorem 3. *Let the partial conditions of Theorem 2 and*

$$\rho = \lim_{t\to\infty}\int_0^t r(\xi(u))\beta(\xi(u))\mathrm{d}s < 1,$$

hold, moreover assume that

$$c(i) \le -\theta < 0,$$

where θ is a constant such that $\theta > \frac{M\rho}{1-\rho}$. Then Eq. (4) is extinction with probability 1.

Proof. By using the Itô formula to Eq. (4), we show that

$$\begin{aligned}\ln x(t) = &\ln x(0) + \int_0^t [c(\xi(s)) - r(\xi(s))\beta(\xi(s))\ln x(s-\tau)]\mathrm{d}s \\ &+ \int_0^t \sigma(\xi(s))\mathrm{d}B(s) + \int_0^t \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u). \end{aligned}\tag{16}$$

Consequently,

$$\begin{aligned}|\ln x(t)| \leq &|\ln x(0)| + \int_0^t |c(\xi(s))|\mathrm{d}s + \int_0^t \beta(\xi(s))r(\xi(s))|\ln x(s-\tau)|\mathrm{d}s + |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)| \\ &+ |\int_0^t \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)| \\ \leq &|\ln x(0)| + Mt + \sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} \int_0^t \beta(\xi(s))r(\xi(s))\mathrm{d}s \\ &+ |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)| + |\int_0^t \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)| \\ \leq &|\ln x(0)| + Mt + \rho \sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} + |\int_0^t \sigma(\xi(s))\mathrm{d}B(s)| \\ &+ |\int_0^t \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)|. \end{aligned}\tag{17}$$

It follows from (17) that

$$\begin{aligned}\sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} \leq &\sup_{\varsigma\in[-\tau,0]}\{|\ln x(\varsigma)|\} + \sup_{\varsigma\in[0,t]}\{|\ln x(\varsigma)|\} \\ \leq &\sup_{\varsigma\in[-\tau,0]}\{|\ln x(\varsigma)|\} + |\ln x(0)| + Mt + \rho \sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} \\ &+ \sup_{\varsigma\in[0,t]} |\int_0^\varsigma \sigma(\xi(s))\mathrm{d}B(s)| \\ &+ \sup_{\varsigma\in[0,t]} |\int_0^\varsigma \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)|, \end{aligned}\tag{18}$$

and then

$$\begin{aligned}\sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} \leq &\frac{1}{1-\rho} \sup_{\varsigma\in[-\tau,0]}\{|\ln x(\varsigma)|\} \\ &+ \frac{1}{1-\rho}|\ln x(0)| + \frac{M}{1-\rho}t + \frac{1}{1-\rho}\sup_{u\in[0,t]} |\int_0^u \sigma(\xi(s))\mathrm{d}B(s)| \\ &+ \frac{1}{1-\rho}\sup_{\varsigma\in[0,t]} |\int_0^\varsigma \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)|. \end{aligned}\tag{19}$$

This, together with (16), gives that

$$\begin{aligned}
\ln x(t) \leq\ & \ln x(0) + \int_0^t [c(\xi(s))]\mathrm{d}s + \rho \sup_{\varsigma\in[-\tau,t]}\{|\ln x(\varsigma)|\} + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) \\
& + \int_0^t \int_{\mathbb{Y}} \ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u) \\
\leq\ & \ln x(0) + (-\theta t) + \int_0^t \sigma(\xi(s))\mathrm{d}B(s) + \frac{\rho}{1-\rho}\sup_{\varsigma\in[-\tau,0]}\{|\ln x(\varsigma)|\} \\
& + \frac{\rho}{1-\rho}|\ln x(0)| + \frac{M\rho}{1-\rho}t + \frac{\rho}{1-\rho}\sup_{\varsigma\in[0,t]}|\int_0^{\varsigma}\sigma(\xi(s))\mathrm{d}B(s)| \\
& + \frac{1}{1-\rho}\sup_{\varsigma\in[0,t]}|\int_0^{\varsigma}\int_{\mathbb{Y}}\ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)| \\
\leq\ & \frac{\rho}{1-\rho}\sup_{\varsigma\in[-\tau,0]}\{|\ln x(\varsigma)|\} + (-\theta + \frac{M\rho}{1-\rho})t + \frac{1}{1-\rho}\sup_{\varsigma\in[0,t]}|\int_0^{\varsigma}\sigma(\xi(s))\mathrm{d}B(s)| \\
& + \frac{1}{1-\rho}\ln x(0) + \frac{1}{1-\rho}\sup_{\varsigma\in[0,t]}|\int_0^{\varsigma}\int_{\mathbb{Y}}\ln(1+h(\xi(s),u))\tilde{N}(\mathrm{d}s,\mathrm{d}u)|. \qquad (20)
\end{aligned}$$

Then by using the strong law of large numbers for martingales, from (20) we obtain that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \leq -\theta + \frac{M\rho}{1-\rho} < 0.$$

That is $\lim_{t\to\infty} x(t) = 0$ a.s.. The proof is complete. □

4 Numerical experiments

Example: Consider Eq. (4) with the Markov chain $\xi(t)$ taking value in state space $\mathbb{S}=\{1,2\}$, which is regard as the result of the switching between

$$\begin{aligned}
\mathrm{d}x(t) =\ & r(1)[x(t) - \beta(1)x(t)\ln x(t-\tau)]\mathrm{d}t \\
& + \sigma(1)x(t)\mathrm{d}B(t) + \int_{\mathbb{Y}} x(t^-)h(1,u)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \qquad t>0,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}x(t) =\ & r(2)[x(t) - \beta(2)x(t)\ln x(t-\tau)]\mathrm{d}t \\
& + \sigma(2)x(t)\mathrm{d}B(t) + \int_{\mathbb{Y}} x(t^-)h(2,u)\tilde{N}(\mathrm{d}t,\mathrm{d}u), \qquad t>0,
\end{aligned}$$

where $\lambda(\mathbb{Y})=1$, and the coefficients

$$r(1)=0.5,\ \ \beta(1)=0.3,\ \ \sigma(1)=0.7,\ \ h(1,u)=0.5736,$$

$$r(2)=0.4,\ \ \beta(2)=0.2,\ \ \sigma(2)=0.6,\ \ h(2,u)=0.4234.$$

Let the generator of Markov chain $\xi(t)$ be

$$q = \begin{pmatrix} -3 & 3 \\ 4 & -4 \end{pmatrix}.$$

Thus, its stationary distribution is $\pi = (\pi_1, \pi_2) = (\frac{3}{7}, \frac{4}{7})$. A simple computation yields

$$\int_{\mathbb{Y}} [h(1,u) - \ln(1 + h(1,u))]\lambda(\mathrm{d}u) = 0.12,$$

$$\int_{\mathbb{Y}} [h(2,u) - \ln(2 + h(2,u))]\lambda(\mathrm{d}u) = 0.07,$$

and

$$c(1) = r(1) - \frac{1}{2}\sigma^2(1) + r(1)\beta(1) - \int_{\mathbb{Y}} [h(1,u) - \ln(1 + h(1,u))]\lambda(\mathrm{d}u) = 0.285,$$

$$c(2) = r(2) - \frac{1}{2}\sigma^2(2) + r(2)\beta(2) - \int_{\mathbb{Y}} [h(2,u) - \ln(1 + h(2,u))]\lambda(\mathrm{d}u) = 0.23.$$

So, we have

$$h_* = \pi_1[c(1) + r(1)\beta(1)] + \pi_2[c(2) + r(2)\beta(2)] = 0.363 > 0.$$

According to Theorem 2, we can obtain that Eq. (4) is persistence in mean.

5 Conclusion

In this work, we propose a stochastic delay Gompertz model with Levy jumps. We show that the model admits a unique global positive solution. Moreover, we derive sufficient conditions for the extinction and persistence in mean of this model. The results confirm that the intensity of white noise, switching noise and jump noise have a great impact on the properties of this model.

Some interesting topics deserve further investigation. One may propose some more realistic models, for example considering the influences of impulsive perturbations on Eq. (4). It is also interesting to investigate the global asymptotic stability and stochastic permanence of Eq. (4). We leave these for further investigations and look forward to solving them in the future.

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