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A fitted operator method of line scheme for solving two-parameter singularly perturbed parabolic convection-diffusion problems with time delay

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Abstract. This paper presents a parameter-uniform numerical scheme for the solution of two-parameter singularly perturbed parabolic convection-diffusion problems with a delay in time. The continuous problem is semi-discretized using the Crank-Nicolson finite difference method in the temporal direction. The resulting differential equation is then discretized on a uniform mesh using the fitted operator finite difference method of line scheme. The method is shown to be accurate in $O((\Delta t)^2 + N^{-2})$, where N is the number of mesh points in spatial discretization and Δt is the mesh length in temporal discretization. The parameter-uniform convergence of the method is shown by establishing the theoretical error bounds. Finally, the numerical results of the test problems validate the theoretical error bounds.

Keywords: Singular perturbation, time-delayed parabolic convection-diffusion problems, two small parameters, the method of line, finite difference scheme, uniform convergence.
AMS Subject Classification 2010: 65M06, 65M12, 65L11.

1 Introduction

Singularly perturbed delay partial differential equations have many applications in diverse areas of science and engineering. In these equations, the highest derivative is multiplied by the small parameter ε and involves at least one delay term for the time variable. In such problems, a boundary layer exists in the neighborhood of the boundary as the parameter tends to zero. The results of the preceding investigation show that while the boundary layer locations remain unchanged, small delays have a significantly large effect on the solution. Many real-world models use the mathematical analysis of singularly perturbed time-delay differential equations, where the current state depends on the past state. Several of these

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take place in mass and heat transfer processes where the diffusion coefficients and thermal conductivity are small [27]. However, the rate of reactions is large in many areas of fluid flow [26] and biological modeling [30]. The numerical solution of a singularly perturbed problem has been extensively studied by many authors (see [2, 6] and references therein). Recently, much effort has been placed into the development of numerical methods for the solution of singularly perturbed time delay parabolic problems that are uniformly convergent concerning the perturbation parameter ε. The key idea of these methods is the use of either piecewise uniform meshes that are appropriately condensed in the boundary layer region [5, 9, 11, 12, 29] or the fitted operator method [19–23].

In this study, we consider the following class of two-parameter singularly perturbed time-delay parabolic partial differential equations on the rectangle $D = \Omega_x \times (0,T] = (0,1) \times (0,T]$ in the space-time plane with Dirichlet boundary conditions

$$L_{\varepsilon,\mu} u(x,t) \equiv \varepsilon u_{xx}(x,t) + \mu a(x,t) u_x(x,t) - u_t(x,t) - b(x,t) u(x,t) = r(x,t), \tag{1}$$

subject to the initial and boundary conditions

$$\begin{cases} u(x,t) = \phi_b(x,t), \ (x,t) \in \Gamma_b = [0,1] \times [-\tau, 0], \\ u(0,t) = \phi_l(t), \ \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) = \phi_r(t), \ \Gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \tag{2}$$

where $r(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D$.

Here $\Gamma = \Gamma_b \cup \Gamma_l \cup \Gamma_r$, Γ_l and Γ_r are the left and the right sides of the rectangular domain D corresponding to $x = 0$ and $x = 1$. The ε and μ are small positive parameters such that $0 < \varepsilon \le 1, 0 \le \mu \le 1$ and $\tau > 0$ represents the delay parameter. The functions $a(x,t), b(x,t), c(x,t), f(x,t)$ on $\overline{D} = D \cup \Gamma$ and $\phi_b(x,t), \phi_l(t), \phi_r(t)$ on Γ are sufficiently smooth, and bounded that satisfy, $a(x,t) \ge \alpha > 0, b(x,t) \ge \beta > 0, c(x,t) \ge \vartheta > 0, (x,t) \in \overline{D}$. Under sufficient smoothness and suitable compatibility conditions for the data, problem (1)–(2) admits a unique solution (refer [14]). The terminal time T is assumed to satisfy the condition $T = k\tau$ for some positive integer k.

For the parameter $\mu = 0$, the boundary layers appear on both sides of the boundary points on the domain with an approximate width $O\left(\sqrt{\varepsilon}\right)$, whereas for $\mu = 1$, the boundary layers appear in the neighborhood of either the left or right boundary point of width $O(\varepsilon)$ and mainly depend on the sign of the convection coefficient. Such types of problems are widespread in modeling chemical flow reactor theory [24] as well as in the case of boundary layers controlled by the suction (or blowing) of some fluid [28].

Due to the presence of the boundary layer, the numerical methods on a uniform mesh for the singularly perturbed two-parameter parabolic problems with time delay are inadequate and fail to give good accuracy unless an unexpected number of mesh points are used to make the step size at least as small as the perturbation parameter (which is not easy to implement practically). This was the motivation for the development of the parameter-uniform numerical method. Numerical methods for solving two-parameter singularly perturbed boundary value problems are proposed by [17, 25]. Also, uniformly convergent numerical methods have been studied for singularly perturbed two-parameter parabolic partial differential equations without the time delay term [1, 3, 4, 8, 10, 15].

Recently, Govindarao *et al*. [7] developed a parameter-uniform numerical method for solving two-parameter singularly perturbed parabolic convection-diffusion problems with time delay using an upwind difference scheme on Shishkin-type meshes. For problem (1)−(2), Kumar and Kumar [13] studied numerical methods based on a hybrid monotone finite difference scheme on a layer-adapted Shishkin mesh. Negero [18] constructed a uniformly convergent numerical method based on a uniform mesh for singularly perturbed two-parameter time-delay parabolic convection-diffusion equations. Govindarao *et al*. [7] and Kumar and Kumar [13] designed an adaptive Shishkin mesh discretization method to resolve the layer, which requires priori knowledge of the position and width of the boundary layer of the problem. In this paper, the author aim is to provide a parameter-uniform numerical method for two-parameter singularly perturbed parabolic convection-diffusion problems with time delay. Except for the method in Negero [18], no fitted operator numerical methods for solving problem (1)−(2) have been developed. The advantage of this method is that it does not require a priori information about the position and width of the solution boundary layer.

2 Solution bounds for continuous problem

In this section, we discuss some properties of the solution $u(x,t)$ of problem (1)−(2), that are essential in later sections for the analysis of the appropriate numerical solution. These properties consist of the continuous minimum principle, classical bounds on the solution and its derivatives, and parameter-uniform bounds to analyze the proposed scheme. For any continuous function $u(x,t)$, we use $\|u(x,t)\|_\infty$, or the continuous maximum norm on the corresponding interval.

The differential operator $L_{\varepsilon,\mu}$ satisfies the following continuous minimum principle.

Lemma 1 (Continuous minimum principle)**.** *Let $\upsilon(x,t) \in C^2(D) \cap C^0(\bar{D})$. If $\upsilon(x,t) \geq 0$, $\forall (x,t) \in \Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$ and $L_{\varepsilon,\mu}\upsilon(x,t) \leq 0$, then $\upsilon(x,t) \geq 0$, $\forall (x,t) \in \bar{D}$.*

Proof. The proof easily follows by contradiction. Let there exists a point $(x^*,t^*) \in \bar{D}$ such that $\upsilon(x^*,t^*) = \min_{(x,t)\in\bar{D}} \upsilon(x,t) < 0$. Therefore, from the given condition on the boundary Γ, it is clear that the point $(x^*,t^*) \notin \Gamma$, which implies that $(x^*,t^*) \in D$. It follows from the definition of the point (x^*,t^*) that

$$\upsilon_{xx}(x^*,t^*) \geq 0, \upsilon_x(x^*,t^*) = \upsilon_t(x^*,t^*) = 0.$$

But then $L_{\varepsilon,\mu}\upsilon(x^*,t^*) > 0$, which is a contradiction. Thus our initial assumption is false and we can conclude that the minimum of υ is non-negative. □

An immediate consequence of the above minimum principle is the following parameter-uniform bound of the solution to problem (1)−(2).

Lemma 2. *The solution $u(x,t)$ to problem (1)−(2) satisfies the following bounds*

$$|u(x,t)| \leq C, \forall (x,t) \in \bar{D}.$$

Proof. The proof is given in [13]. □

The following lemma proves the stability estimate to obtain a unique solution.

Lemma 3 (Uniform stability estimate). *Let $u(x,t)$ be the solution of (1)−(2), then $\forall \varepsilon > 0, \mu \geq 0$ we have the following bound*

$$\|u(x,t)\| \leq \beta^{-1}\|g\| + \max(|\phi_b|, |\phi_l|, |\phi_r|).$$

Proof. For the barrier functions $\Psi(x,t)=\beta^{-1}\|g\|+\max(|\phi_b|,(|\phi_l|+|\phi_r|))\pm u(x,t),(x,t)\in\bar{D}$, we have

$$\Psi(0,t)=\|u\|_{\Gamma}+\beta^{-1}\|g\|+\max(|\phi_b|,(|\phi_l|+|\phi_r|))\pm u(0,t)\geq\|u\|_{\Gamma}\pm u(0,t)\geq 0,$$

$$\Psi(1,t)=\|u\|_{\Gamma}+\beta^{-1}\|g\|+\max(|\phi_b|,(|\phi_l|+|\phi_r|))\pm u(1,t)\geq\|u\|_{\Gamma}\pm u(1,t)\geq 0.$$

Also, for $(x,t)\in\Gamma_b$,

$$\Psi(x,t)=\|u\|_{\Gamma}+\beta^{-1}\|g\|+\max(|\phi_b|,(|\phi_l|+|\phi_r|))\pm u(x,t)\geq\|u\|_{\Gamma}\pm u(x,t)\geq 0.$$

Furthermore, for all $(x,t)\in D$, $L_{\varepsilon,\mu}\Psi(x,t)\geq 0$. Therefore, by using the minimum principle, we obtain the required result. □

Lemma 4. *[13] For all non-negative integers j,k, satisfying $0\leq j+2k\leq 4$, the derivatives of the exact solution $u(x,t)$ of the problem (1) - (2) satisfy the estimate*

$$\left|\frac{\partial^{j+k}u}{\partial x^j\partial t^k}\right|\leq C\begin{cases}\dfrac{1}{(\sqrt{\varepsilon})^j}, & \text{if } \mu^2\leq\dfrac{\varepsilon\eta}{\alpha},\\[2ex] \left(\dfrac{\mu}{\varepsilon}\right)^j\left(\dfrac{\mu^2}{\varepsilon}\right)^k, & \text{if } \mu^2\geq\dfrac{\varepsilon\eta}{\alpha},\end{cases}$$

where $\eta\approx\min_{(x,t)\in\bar{D}}\frac{b(x,t)}{a(x,t)}$, and the constant C is independent of parameters ε and μ.

3 Numerical discretization

3.1 Temporal discretization

The time interval $[0,T]$ is divided into M equal sub-intervals with uniform step size Δt, as defined by

$$\Omega_t^{\Delta t}=\{t_m=m\Delta t, m=0(1)M,\Delta t=T/M,t_M=T\},$$

and

$$\Omega_t^{s}=\{t_m=m\Delta t, m=0(1)s,t_s=\tau,\Delta t=\tau/s\},$$

where s mesh elements are used on the interval $[-\tau,0]$. Then, using the Crank-Nicolson method on the time variable of problem $(1)-(2)$, we obtain the semi-discrete scheme as follows

$$\begin{cases} L^{\Delta t}_{\varepsilon,\mu}U^{m+1}(x)\equiv\varepsilon(U_{xx})^{m+1}(x)+\mu a^{m+1}(x)(U_x)^{m+1}(x)-q^{m+1}(x)U^{m+1}(x)=G(x,t_{m+1}),\\ U^{m+1}(0)=\phi_l(t_{m+1}),\ m=0,...,M,\\ U^{m+1}(1)=\phi_r(t_{m+1}),\ m=0,...,M,\\ U^{m+1}(x)=\phi_b(x,t_{m+1}),\ x\in(0,1),-(s+1)\leq m\leq -1,\end{cases}\tag{3}$$

where $U^{m+1}(x)$ is the approximate solution to the exact solution $u(x,t_{m+1})$,

$$\begin{aligned} G(x,t_{m+1}) &= -c^{m+1}(x)U^{m+1-s}(x)+f^{m+1}(x)-\varepsilon(U_{xx})^m(x)-\mu a^m(x)(U_x)^m(x)\\ &\quad+\left(-\frac{2}{\Delta t}+b^m(x)\right)U^m(x)-c^m(x)U^{m-s}(x)+f^m(x),\\ q^{m+1}(x) &= \frac{2}{\Delta t}+b^{m+1}(x)\geq\bar{\alpha}>0.\end{aligned}$$

Lemma 5 (**Semi-discrete minimum principle**)**.** *Let $\psi^{m+1}(x)$ be a continuous function on $\bar{D}$. If $\psi^{m+1}(0) \geq 0$, $\psi^{m+1}(1) \geq 0$ and $L^{\Delta t}_{\varepsilon,\mu}\psi^{m+1}(x) \leq 0\ \forall x \in D$, then $\psi^{m+1}(x) \geq 0$, $\forall x \in \bar{D}$.*

Proof. One can prove this lemma by the same procedure as the proof of Lemma 1. □

Lemma 6 (**Local error estimate**)**.** *Suppose that $\frac{\partial u^l(x,t)}{\partial t^l} \leq C, (x,t) \in \overline{D}, 0 \leq l \leq 2$. In the temporal direction, the local error estimate $e_{m+1} = U^{m+1}(x) - u(x,t_{m+1})$ is given by*

$$\|e_{m+1}\|_\infty \leq C(\Delta t)^3, \textit{ for some constant } C.$$

Proof. Using Taylor's series expansion to $u(x,t_{m+1})$, $e_{m+1}(x)$ satisfies the semi-discrete operator

$$L^M_{\varepsilon,\mu} e_{m+1}(x) = O\left((\Delta t)^3\right).$$

Thus using minimum principle given at Lemma 5 we have $\|e_{m+1}\|_\infty \leq C(\Delta t)^3$. □

Lemma 7 (Estimation of the global error)**.** *Suppose that the assumption of Lemma 6 holds. Then the global truncation error $E_m = U^m(x) - u(x,t_m)$ is estimated as $\|E_m\|_\infty \leq C(\Delta t)^2$.*

Proof. From Lemma 6 it follows that

$$\|E_m\|_\infty = \|\sum_{k=1}^{m} e_k\|_\infty \leq \|e_1\|_\infty + \|e_2\|_\infty + \ldots + \|e_m\|_\infty \leq C(\Delta t)^2,$$

which completes the proof. □

3.2 Spatial discretization

Suppose the domain $[0,1]$ is subdivided into N equal intervals of step size h and forms a uniform mesh as

$$\Omega^N_x = \{x_n = nh, n = 1,2,\ldots,N, x_0 = 0, x_N = 1, h = 1/N\},$$

where x_n is mesh points. Using the method of line fitted operator finite difference method [16], Eq. (3) is written as

$$L^{N,\Delta t}_{\varepsilon,\mu} U^{m+1}_n \equiv \varepsilon \frac{\delta^2_x U^{m+1}_n}{\gamma^2(\varepsilon,\mu)} + \mu a^{m+1}_n D^0_x U^{m+1}_n - q^{m+1}_n U^{m+1}_n = G^{m+1}_n, \tag{4}$$

subject to the following conditions

$$\begin{cases} U^{m+1}_0 = \phi_l(t_{m+1}), & 0 \leq m \leq M, \\ U^{m+1}_N = \phi_r(t_{m+1}), & 0 \leq m \leq M, \\ U_n(x_n,t_{m+1}) = \phi_b(x_n,t_{m+1}), & -(s+1) \leq m \leq -1, \end{cases} \tag{5}$$

where

$$G^{m+1}_n = \begin{cases} -\varepsilon \frac{\delta^2_x U^m_n}{\gamma^2(\varepsilon,\mu)} - \mu a^m_n D^0_x U^m_n + \left(-\frac{2}{\Delta t} + b^m_n\right) U^m_n - c^{m+1}_n \phi^{m+1}_b(x_n) + f^{m+1}_n \\ -c^m_n \phi^m_b(x_n) + f^m_n, \quad \text{if} \quad t_m < s, \\ -\varepsilon \frac{\delta^2_x U^m_n}{\gamma^2(\varepsilon,\mu)} - \mu a^m_n D^0_x U^m_n + \left(-\frac{2}{\Delta t} + b^m_n\right) U^m_n - c^{m+1}_n U^{m+1-s}_n + f^{m+1}_n \\ -c^m_n U^{m-s}_n + f^m_n, \quad \text{if} \quad t_m \geq s, \end{cases}$$

and

$$\gamma^2(\varepsilon,\mu) = \frac{2h\varepsilon}{\mu a_n^{m+1}} \tanh\left(\frac{\mu a_n^{m+1} h}{2\varepsilon}\right), \qquad D_x^0 U_n^m = \frac{U_{n+1}^{m+1} - U_{n-1}^{m+1}}{2h},$$
$$\delta_x^2 U_n^{m+1} = U_{n-1}^{m+1} - 2U_n^{m+1} + U_{n+1}^{m+1}, \qquad q_n^{m+1} = \frac{2}{\Delta t} + b_n^{m+1}.$$

Now, we need to show that the discrete operator $L_{\varepsilon,\mu}^{N,\Delta t}$ in Eq. (4) satisfies the minimum principle. Next, we prove some useful attributes of the discrete problem.

Lemma 8 (Discrete minimum principle)**.** *Let $\Psi^{m+1}(x_n)$ be a mesh function such that $\Psi^{m+1}(x_0) \geq 0$ and $\Psi^{m+1}(x_N) \geq 0$. Then $L_{\varepsilon,\mu}^{N,\Delta t}\Psi^{m+1}(x_n) \leq 0$ for $1 \leq n \leq N-1$, implies that $\Psi^{m+1}(x_n) \geq 0$ for $0 \leq n \leq N$.*

Proof. Let $p^* \in \{0,1,...,N\}$ be such that $\Psi_{p^*}^{m+1} = \min_{1\leq m\leq N} \Psi^{m+1}$ and suppose $\Psi_{p^*}^{n+1} < 0$. It is clear that $p^* \notin \{0,N\}$. Also we have $\Psi_{p^*+1}^{m+1} - \Psi_{p^*}^{m+1} \geq 0$ and $\Psi_{p^*}^{m+1} - \Psi_{p^*-1}^{m+1} \leq 0$. Now from Eq. (4) we have

$$\begin{aligned} L_{\varepsilon,\mu}^{N,\Delta t}\Psi_{p^*}^{m+1} &= \varepsilon\frac{\delta_x^2\Psi_{p^*}^{m+1}}{\gamma^2(\varepsilon,\mu)} + \mu a_{p^*}^{m+1} D_x^0 \Psi_{p*}^{m+1} - q_{p*}^{m+1}\Psi_{p*}^{m+1} \\ &= \varepsilon\frac{\Psi_{p^*-1}^{m+1} - 2\Psi_{p^*}^{m+1} + \Psi_{p^*+1}^{m+1}}{\gamma^2(\varepsilon,\mu)} + \mu a_{p^*}^{m+1}\frac{\Psi_{p^*+1}^{m+1} - \Psi_{p^*-1}^{m+1}}{2h} - q_{p*}^{m+1}\Psi_{p*}^{m+1} > 0, \end{aligned}$$

which contradicts the given hypothesis $L_{\varepsilon,\mu}^{N,\Delta t}\Psi^{m+1}(x_n) \geq 0$ and our supposition $\Psi_{p^*}^{m+1} < 0$. For $p^* = 0,1,...,N$, which gives $\Psi_{p^*}^{m+1} \geq 0$, and hence $\Psi^{m+1}(x_n) \geq 0$, for all $n = 0,1,...,N$. □

Now, we prove the uniform stability analysis of the discrete problem.

Lemma 9 (Uniform stability estimate)**.** *The discrete scheme solution U_m^{n+1} in (4)−(5) satisfies the bound*

$$\left|U_n^{m+1}\right| \leq \frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\{|\phi_l(t_{m+1})|, |\phi_r(t_{m+1})|\}, n = 0,1,2,...N,$$

where $\frac{2}{\Delta t} + b_n^{m+1} \geq \bar{\beta} > 0$.

Proof. By constructing a barrier functions as

$$\Im_{n,m+1}^{\pm} = \frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\{|\phi_l(t_{m+1})|, |\phi_r(t_{m+1})|\} \pm U_n^{m+1},$$

at the boundary points, we obtain

$$\Im_{0,m+1}^{\pm} = \frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\{|\phi_l(t_{m+1})|, |\phi_r(t_{m+1})|\} \pm U_0^{m+1} \geq 0,$$
$$\Im_{1,m+1}^{\pm} = \frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\{|\phi_l(t_{m+1})|, |\phi_r(t_{m+1})|\} \pm U_N^{m+1} \geq 0.$$

Now for $0 < n < N$, we have

$$\begin{aligned}
L_{\varepsilon,\mu}^{N,\Delta t}\mathfrak{I}_{n,m+1}^{\pm} &= \varepsilon\frac{\delta_x^2\mathfrak{I}_{n,m+1}^{\pm}}{\gamma^2} + \mu a_m^{n+1}D_x^0\mathfrak{I}_{n,m+1}^{\pm} - q_n^{m+1}\mathfrak{I}_{n,m+1}^{\pm},\\
&= \varepsilon\frac{\delta_x^2\mathfrak{I}_{n,m+1}^{\pm}}{\gamma^2} + \mu a_m^{n+1}D_x^0\mathfrak{I}_{n,m+1}^{\pm} - q_n^{m+1}\left(\frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\left\{|\phi_l(t_{m+1})|,|\phi_r(t_{m+1})|\right\} \pm U_n^{m+1}\right)\\
&= -q_n^{m+1}\left(\frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\left\{|\phi_l(t_{m+1})|,|\phi_r(t_{m+1})|\right\}\right) \pm L_{\varepsilon,\mu}^{N,\Delta t}U_n^{m+1}\\
&= -q_n^{m+1}\left(\frac{\|G_n^{m+1}\|}{\bar{\beta}} + \max\left\{|\phi_l(t_{m+1})|,|\phi_r(t_{m+1})|\right\}\right) \pm G_n^{m+1}\\
&\leq 0, \text{ since } q_n^{m+1} \geq \bar{\beta}.
\end{aligned}$$

Using the discrete minimum principle given in Lemma 9 yields $\mathfrak{I}_{n,m+1}^{\pm} \geq 0,\ n = 0,1,2,\ldots,N.$ □

4 Convergence analysis

This section presents the bounds of the truncation error and the convergence analysis of the proposed method.

Lemma 10. *Let $U^{m+1}(x)$ and U_n^{m+1} be the solutions of the schemes (3) and (4), respectively. Then, the error estimate in the spatial discretization is given by $\left\|U^{m+1}(x) - U_n^{m+1}\right\| \leq CN^{-2}$.*

Proof. The truncation error of the proposed method is given by

$$\begin{aligned}
&\left|L_{\varepsilon,\mu}^{N,\Delta t}\left(U^{m+1}(x) - U_n^{m+1}\right)\right| = \left|L_{\varepsilon,\mu}^{\Delta t}U^{m+1}(x) - L_{\varepsilon,\mu}^{N,\Delta t}U_n^{m+1}\right|\\
&\leq \left|\varepsilon\left(\frac{d^2}{dx^2} - \frac{\delta_x^2}{\gamma^2(\varepsilon,\mu)}\right)U_n^{m+1} + \mu a_n^{m+1}\left(\frac{d}{dx} - D^0\right)U_n^{m+1}\right| \qquad (6)\\
&\leq \left|\varepsilon\left(1 - \frac{h^2}{\gamma^2(\varepsilon,\mu)}\right)\frac{d^2U^{m+1}(x)}{dx^2} + \frac{N^{-2}\mu a^{m+1}(x)}{6}\frac{d^3U^{m+1}(x)}{dx^3} + \frac{\varepsilon N^{-4}a^{m+1}(x)}{12\gamma^2(\varepsilon,\mu)}\frac{d^4U^{m+1}(x)}{dx^4}\right|.
\end{aligned}$$

Now, Eq. (6) becomes

$$\left|L_{\varepsilon,\mu}^{N,\Delta t}\left(U^{m+1}(x) - U_n^{m+1}\right)\right| \leq N^{-2}\mu\frac{d^3U^{m+1}(x)}{dx^3} + \frac{\varepsilon N^{-4}}{12\gamma^2(\varepsilon,\mu)}\frac{d^4U^{m+1}(x)}{dx^4}.$$

Using the bound in Lemma 8, we obtain $\left|L_{\varepsilon,\mu}^{N,\Delta t}\left(U^{m+1}(x) - U_n^{m+1}\right)\right| \leq CN^{-2}$. □

Theorem 1. *Let $u(x_n,t_{m+1})$ and U_n^{m+1} be the solutions of continuous problem (1)–(2) and discrete problem (4)–(5), respectively. Then, the error estimate for the fully discrete scheme is given by*

$$\sup_{n=0(1)N,m=0(1)M}\left|u(x_n,t_{m+1}) - U_n^{m+1}\right| \leq C\left(N^{-2} + (\Delta t)^2\right).$$

Proof. Combining Lemmas 7 and 10, gives the required estimate. □

5 Numerical examples and results

To illustrate the accuracy of the method and the theoretical results of error analysis, we present two numerical examples. For these two test problems, the errors and the corresponding rates of convergence are displayed in several tables. The exact solution to these problems is not known. With the help of double mesh techniques, we compute the maximum point-wise absolute error $E_{\varepsilon,\mu}^{N,M}$ and corresponding rate of convergence $p_{\varepsilon,\mu}^{N,M}$ of the scheme by using the double mesh techniques. We define the maximum point-wise absolute error as

$$E_{\varepsilon,\mu}^{N,M}=\max_{0\le n\le N,0\le m\le M}\left|U_{n,m}^{N,M}-U_{n,m}^{2N,2M}\right|,\quad p_{\varepsilon,\mu}^{N,M}=\log_2\left(\frac{E_{\varepsilon,\mu}^{N,M}}{E_{\varepsilon,\mu}^{2N,2M}}\right),$$

where $U_{n,m}^{N,M}$ and $U_{n,m}^{2N,2M}$ are numerical solutions computed on the mesh, $N\times M$ and $2N\times 2M$, respectively.

Example 1. [13]Consider a singularly perturbed two-parameter parabolic problem with a delay in time

$$-\frac{\partial u}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial x^2}+\mu(1+x)\frac{\partial u}{\partial x}-u(x,t)=-u(x,t-\tau)+16x^2(1-x)^2,(x,t)\in(0,1)\times(0,2],$$

with

$$\begin{cases}u(0,t)=0,\ \ u(1,t)=0,\ \ t\in(0,2],\\ u(x,t)=0,\ \ (x,t)\in[0,1]\times[-\tau,0].\end{cases}$$

Example 2. [13]Consider a singularly perturbed two-parameter parabolic problem with a delay in time

$$-\frac{\partial u}{\partial t}+\varepsilon\frac{\partial^2 u}{\partial x^2}+\mu\left(1+x(1-x)+t^2\right)\frac{\partial u}{\partial x}-(1+5xt)\,u(x,t)=-u(x,t-\tau)+x(1-x)\left(e^t-1\right),$$

with $(x,t)\in(0,1)\times(0,2]$,

$$\begin{cases}u(0,t)=0,u(1,t)=0,t\in(0,2],\\ u(x,t)=0,(x,t)\in[0,1]\times[-\tau,0].\end{cases}$$

The computed maximum point-wise errors ($E_{\varepsilon,\mu}^{N,M}$) and the corresponding order of convergence ($p_{\varepsilon,\mu}^{N,M}$) for Examples 1 and 2 are tabulated in Tables 1–7, for various values of ε,μ, N, and M. We start with $N=32$ and $M=16$, and successively multiply N by 2 and M by 2. We set $\mu=10^{-4}$ and $\varepsilon=10^{-4}$ to get the numerical results given in Tables 1 and 2, respectively. Tabulated results show that the maximum point-wise error has monotonically decreasing behavior with increasing N and M, which confirms the parameter-uniform convergence of proposed schemes (4)–(5) as proved in Theorem 1. Also, tabulated results indicate that for a fixed value of N, M, and (ε or μ), maximum point-wise error going to stabilized as parameter (ε or μ) approaches zero, respectively. Although the numerical results presented in Tables 1–2 reflect that the numerical order of convergence of the proposed schemes (4)–(5) is equal to two, which exhibit the actual theoretical order of convergence both in space and time as proved in Theorem 1. However, the tabulated results in Table 4 show that the maximum point-wise error has a monotonically non-decreasing behavior with increasing M and decreasing N, which contradicts the developed scheme's parameter-uniform convergence proved in Theorem 1. The numerical results in Table

3 clearly show that the proposed schemes (4)−(5) have global $(\varepsilon,\mu)-$uniform convergence of order two and validate the theoretical estimates given in Theorem 1. The numerical results obtained by using the method of line scheme (4)−(5) have also been compared in Tables 5−7 with those obtained by the implicit Euler method and a hybrid scheme consisting of the central difference, upwind, and midpoint finite difference schemes on a Shishkin mesh [13]. A comparison of the numerical results in Tables 5−7 for Examples 1 and 2 also shows that the present scheme gives more accurate results and has a higher order of convergence than the scheme given in [13]. Thus, numerical and theoretical error estimates validate the fact that the proposed method of line scheme (4)−(5) possesses second-order uniform convergence in comparison to the first-order uniform convergence of finite difference schemes discussed in [13]. Numerical solution profiles are given in Figures 1− 2 for different values of (ε,μ) with $N = M = 64$, which validates the physical behavior of the solution. The log −log error plot in Figure 3 also shows that the order of convergence for a method of line schemes (4)−(5) is two, which is opposed to the order one of the discrete schemes in [13], and it supports our theoretical error estimates.

Table 1: Maximum pointwise errors $(E^{N,M}_{\varepsilon,\mu})$ and rate of convergence $(p^{N,M}_{\varepsilon,\mu})$ for Example 1.

$\mu = 10^{-4}$ $\varepsilon\downarrow$	$N = 32$ $M = 16$	$N = 64$ $M = 32$	$N = 128$ $M = 64$	$N = 256$ $M = 128$	$N = 512$ $M = 256$
10^{-4}	$7.7542e-04$	$1.9328e-04$	$4.8296e-05$	$1.2074e-05$	$3.0183e-06$
	2.0043	2.0007	2.0000	2.0001	-
10^{-6}	$7.6780e-04$	$1.9093e-04$	$4.7663e-05$	$1.1913e-05$	$2.9780e-06$
	2.0077	2.0021	2.0003	2.0001	-
10^{-8}	$7.6644e-04$	$1.8909e-04$	$4.6268e-05$	$1.1082e-05$	$2.5295e-06$
	2.0191	2.0310	2.0618	2.1313	-
10^{-10}	$7.6644e-04$	$1.8909e-04$	$4.6268e-05$	$1.1082e-05$	$2.5295e-06$
	2.0191	2.0310	2.0618	2.1313	-
10^{-12}	$7.6644e-04$	$1.8909e-04$	$4.6268e-05$	$1.1082e-05$	$2.5295e-06$
	2.0191	2.0310	2.0618	2.1313	-

Table 2: Maximum pointwise errors $(E^{N,M}_{\varepsilon,\mu})$ and rate of convergence $(p^{N,M}_{\varepsilon,\mu})$ for Example 1.

$\varepsilon = 10^{-4}$ $\mu\downarrow$	$N = 32$ $M = 16$	$N = 64$ $M = 32$	$N = 128$ $M = 64$	$N = 256$ $M = 128$	$N = 512$ $M = 256$
10^{-4}	$7.7542e-04$	$1.9328e-04$	$4.8296e-05$	$1.2074e-05$	$3.0183e-06$
	2.0043	2.0007	2.0000	2.0001	-
10^{-6}	$7.7551e-04$	$1.9330e-04$	$4.8302e-05$	$1.2075e-05$	$3.0187e-06$
	2.0043	2.0007	2.0001	2.0000	-
10^{-8}	$7.7551e-04$	$1.9330e-04$	$4.8302e-05$	$1.2075e-05$	$3.0187e-06$
	2.0043	2.0007	2.0001	2.0000	-
10^{-10}	$7.7551e-04$	$1.9330e-04$	$4.8302e-05$	$1.2075e-05$	$3.0187e-06$
	2.0043	2.0007	2.0001	2.0000	-
10^{-12}	$7.7551e-04$	$1.9330e-04$	$4.8302e-05$	$1.2075e-05$	$3.0187e-06$
	2.0043	2.0007	2.0001	2.0000	-

Table 3: Maximum pointwise errors $(E_{\varepsilon}^{N,M})$ and rate of convergence $(p_{\varepsilon}^{N,M})$ before extrapolation for Example 1.

	Number of mesh intervals $N = M$				
	32	64	128	256	512
$\mu \downarrow \varepsilon \rightarrow$	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}
10^{-4}	$1.9394e-04$	$4.6167e-05$	$1.0119e-05$	$2.6598e-06$	$1.3550e-06$
	2.0707	2.1898	1.9277	0.97302	-
10^{-6}	$1.9403e-04$	$4.8223e-05$	$1.2049e-05$	$3.0038e-06$	$7.4851e-07$
	2.0085	2.0008	2.0041	2.0047	-
10^{-8}	$1.9403e-04$	$4.8224e-05$	$1.2055e-05$	$3.0135e-06$	$7.5334e-07$
	2.0085	2.0001	2.0001	2.0001	-
10^{-10}	$1.9403e-04$	$4.8224e-05$	$1.2055e-05$	$3.0135e-06$	$7.5339e-07$
	2.0085	2.0001	2.0001	2.0000	-
10^{-12}	$1.9403e-04$	$4.8224e-05$	$1.2055e-05$	$3.0135e-06$	$7.5339e-07$
	2.0085	2.0001	2.0001	2.0000	-

Table 4: Maximum pointwise errors $(E_{\varepsilon,\mu}^{N,M})$ for Example 1 with $\mu = 10^{-4}$.

	N=512	N=256	N=128	N=64	N=32
$\varepsilon\downarrow$	M=16	M=32	M=64	M=128	M=256
10^{-4}	7.7454e-04	1.9307e-04	4.8296e-05	3.8891e-05	3.9284e-05
	2.0042	1.9991	0.31247	0.014506	-
10^{-6}	7.7412e-04	1.9282e-04	4.7663e-05	1.0044e-05	1.4386e-05
	2.0053	2.0163	2.2465	0.51833	-
10^{-8}	7.7367e-04	1.9199e-04	4.6268e-05	9.4944e-06	1.5956e-05
	2.0107	2.0529	2.2849	0.74895	-
10^{-10}	7.7367e-04	1.9199e-04	4.6268e-05	9.4944e-06	1.5956e-05
	2.0107	2.0529	2.2849	0.74895	-
10^{-12}	7.7367e-04	1.9199e-04	4.6268e-05	9.4944e-06	1.5956e-05
	2.0107	2.0529	2.2849	0.74895	-

Table 5: Maximum pointwise errors $(E^{N,M}_{\varepsilon,\mu})$ and rate of convergence $(p^{N,M}_{\varepsilon,\mu})$ for Example 1.

$\mu = 10^{-3}$ $\varepsilon\downarrow$	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$
10^{-4}	$3.1277e-03$	$7.7255e-04$	$1.9256e-04$	$4.8115e-05$	$1.2029e-05$
	2.0174	2.0043	2.0007	2.0000	-
10^{-6}	$3.0574e-03$	$7.3557e-04$	$1.7379e-04$	$3.9582e-05$	$9.8836e-06$
	2.0554	2.0815	2.1344	2.0017	-
10^{-8}	$3.0574e-03$	$7.3557e-04$	$1.7368e-04$	$3.8585e-05$	$1.3367e-05$
	2.0554	2.0824	2.1703	1.5294	-
10^{-10}	$3.0574e-03$	$7.3557e-04$	$1.7368e-04$	$3.8585e-05$	$1.3367e-05$
	2.0554	2.0824	2.1703	1.5294	-
10^{-12}	$3.0574e-03$	$7.3557e-04$	$1.7368e-04$	$3.8585e-05$	$1.3367e-05$
	2.0554	2.0824	2.1703	1.5294	-
			Method in [13]		
10^{-4}	$4.3705e-2$	$1.6704e-2$	$7.3802e-3$	$3.7406e-3$	$1.8967e-3$
	1.3876	1.1785	0.9803	0.9797	-
10^{-6}	$4.3471e-2$	$1.6596e-2$	$7.3290e-3$	$3.7218e-3$	$1.8873e-3$
	1.3892	1.1792	0.9776	0.9796	-
10^{-8}	$4.3429e-2$	$1.6573e-2$	$7.3303e-3$	$3.7211e-3$	$1.8870e-3$
	1.3898	1.1769	0.9781	0.9795	-
10^{-10}	$4.4343e-2$	$1.6572e-2$	$7.3303e-3$	$3.7211e-3$	$1.8870e-3$
	1.3898	1.1768	0.9781	0.9795	-
10^{-12}	$4.4343e-2$	$1.6572e-2$	$7.3303e-3$	$3.7211e-3$	$1.8870e-3$
	1.3898	1.1768	0.9781	0.9795	-

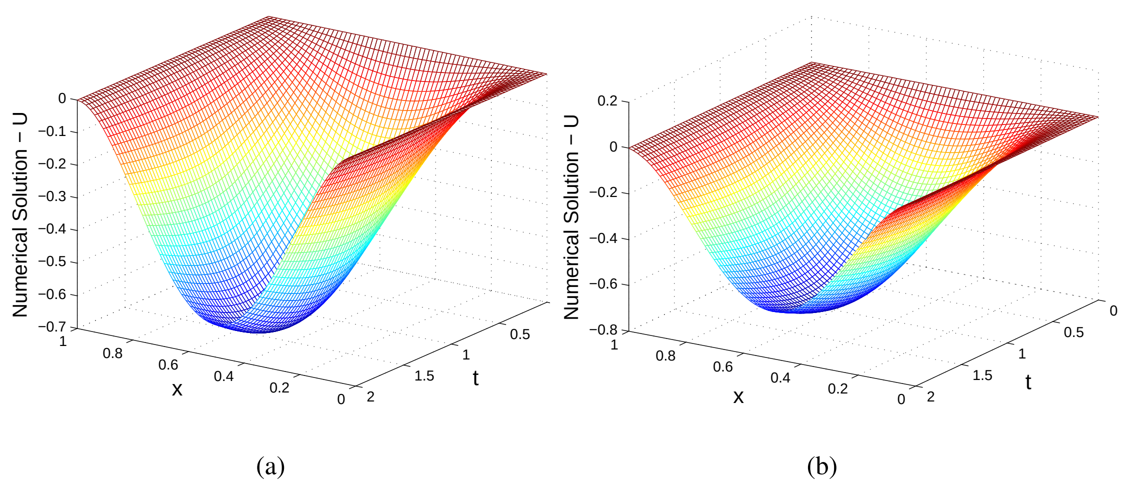
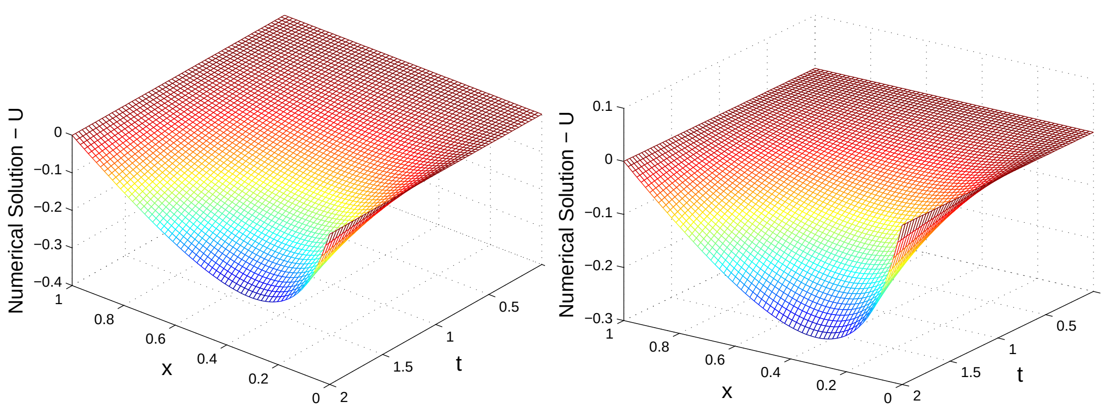


Figure 1: Surface plot of the numerical solution for Example 1 with $N = M = 64$, **(a)** $\varepsilon = 10^{-9},\ \mu = 10^{-3}$, **(b)** $\varepsilon = 10^{-3},\ \mu = 10^{-9}$.

Table 6: Comparison of maximum pointwise errors $(E_{\varepsilon,\mu}^{N,M})$ and rate of convergence $(p_{\varepsilon,\mu}^{N,M})$ for Example 1.

$\mu = 10^{-9}$ $\varepsilon\downarrow$	$N = 32$ $M = 8$	$N = 64$ $M = 16$	$N = 128$ $M = 32$	$N = 256$ $M = 64$	$N = 512$ $M = 128$
10^{-4}	$3.1366e-03$	$7.7478e-04$	$1.9312e-04$	$4.8256e-05$	$1.2064e-05$
	2.0173	2.0043	2.0007	2.0000	-
10^{-6}	$3.1340e-03$	$7.7416e-04$	$1.9296e-04$	$4.8221e-05$	$1.2055e-05$
	2.0173	2.0043	2.0006	2.0000	-
10^{-8}	$3.1339e-03$	$7.7415e-04$	$1.9296e-04$	$4.8221e-05$	$1.2055e-05$
	2.0173	2.0043	2.0006	2.0000	-
10^{-10}	$3.1339e-03$	$7.7415e-04$	$1.9296e-04$	$4.8221e-05$	$1.2055e-05$
	2.0173	2.0043	2.0006	2.0000	-
10^{-12}	$3.1339e-03$	$7.7415e-04$	$1.9296e-04$	$4.8221e-05$	$1.2055e-05$
	2.0173	2.0043	2.0006	2.0000	-
		Method in [13]			
10^{-4}	$4.3708e-2$	$1.6705e-2$	$7.3807e-3$	$3.7407e-3$	$1.8967e-3$
	1.3875	1.1784	0.9804	0.9798	-
10^{-6}	$4.3816e-2$	$1.6749e-2$	$7.4017e-3$	$3.7489e-3$	$1.9008e-3$
	1.3873	1.1781	0.9813	0.9799	-
10^{-8}	$4.3817e-2$	$1.6750e-2$	$7.4019e-3$	$3.7490e-3$	$1.9008e-3$
	1.3873	1.1781	0.9813	0.9799	-
10^{-10}	$4.3817e-2$	$1.6750e-2$	$7.4019e-3$	$3.7490e-3$	$1.9008e-3$
	1.3873	1.1781	0.9813	0.9799	-
10^{-12}	$4.3817e-2$	$1.6750e-2$	$7.4019e-3$	$3.7490e-3$	$1.9008e-3$
	1.3873	1.1781	0.9813	0.9799	-

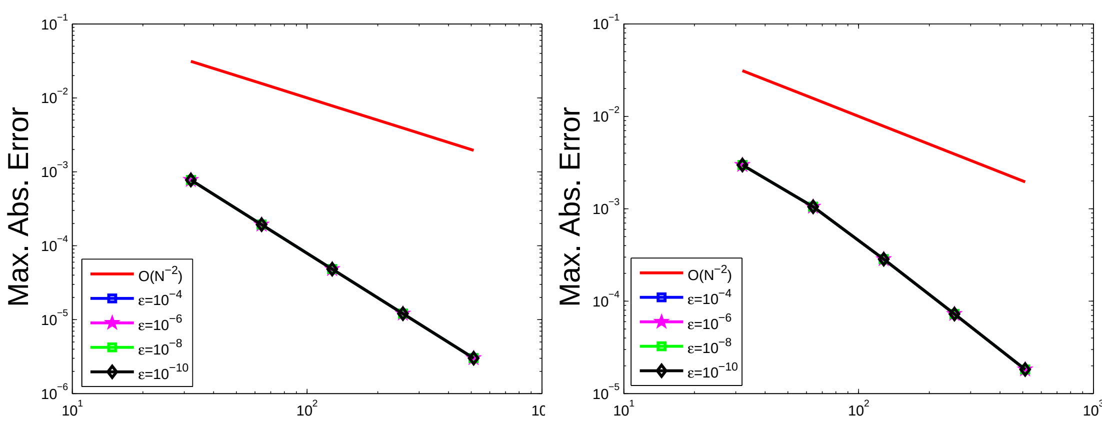


(a) (b)

Figure 2: Surface plot of the numerical solution for Example 2 with $N = M = 64$, **(a)** $\varepsilon = 10^{-9},\ \mu = 10^{-3}$, **(b)** $\varepsilon = 10^{-3},\ \mu = 10^{-9}$.

Table 7: Comparison of maximum pointwise errors $(E^{N,M}_{\varepsilon,\mu})$ for Example 2 with $\mu = 10^{-9}$.

$\mu = 10^{-9}$ $\varepsilon\downarrow$	$N = 32$ $M = 8$	$N = 64$ $M = 32$	$N = 128$ $M = 128$	$N = 256$ $M = 256$	$N = 512$ $M = 2048$
10^{-4}	$2.9753e-03$	$1.0525e-03$	$2.8486e-04$	$7.2614e-05$	$1.8242e-05$
	1.4992	1.8855	1.9719	1.9930	-
10^{-6}	$2.9762e-03$	$1.0528e-03$	$2.8493e-04$	$7.2632e-05$	$1.8246e-05$
	1.4992	1.8856	1.9719	1.9930	-
10^{-8}	$2.9762e-03$	$1.0528e-03$	$2.8493e-04$	$7.2632e-05$	$1.8246e-05$
	1.4992	1.8856	1.9719	1.9930	-
10^{-10}	$2.9762e-03$	$1.0528e-03$	$2.8493e-04$	$7.2632e-05$	$1.8246e-05$
	1.4992	1.8856	1.9719	1.9930	-
10^{-12}	$2.9762e-03$	$1.0528e-03$	$2.8493e-04$	$7.2632e-05$	$1.8246e-05$
	1.4992	1.8856	1.9719	1.9930	-
		Method in [13]			
10^{-4}	$1.1053e-2$	$2.4577e-3$	$6.0306e-4$	$1.5012e-4$	$3.7490e-5$
	2.1691	2.0269	2.0062	2.0015	-
10^{-6}	$1.1046e-2$	$2.4546e-3$	$6.0440e-4$	$1.5048e-4$	$3.7581e-5$
	2.1700	2.0219	2.0059	2.0016	-
10^{-8}	$1.1100e-2$	$2.4588e-3$	$6.0443e-4$	$1.5046e-4$	$3.7582e5$
	2.1745	2.0243	2.0062	2.0013	-
10^{-10}	$1.1093e-2$	$2.4555e-3$	$6.0458e-4$	$1.5049e-4$	$3.7583e-5$
	2.1756	2.0220	2.0062	2.0016	-
10^{-12}	$1.1092e-2$	$2.4551e-3$	$6.0458e-4$	$1.5049e-4$	$3.7583e-5$
	2.1757	2.0218	2.0062	2.0016	-



(a) (b)

Figure 3: Log-Log plot of the maximum error on left (a) for Example 1 and on right (b) for Example 2.

6 Conclusions

A robust numerical method is presented for a class of singularly perturbed two-parameter parabolic convection-diffusion initial boundary value problems with a time delay. On a uniform step size, the developed scheme comprises the Crank-Nicolson method in the time direction and the method of lines in the space direction. This scheme is monotone for various values of parameters ε, μ and N. We examine the proposed scheme for convergence and show that it is convergent in order two in both the temporal and spatial variables. It is also proved that the proposed method is (ε, μ)-uniformly convergent. The performance of the proposed scheme is investigated by comparing the results, and it is observed that the accuracy of the numerical results is comparable to or better than that of existing difference schemes [13], which is verified both theoretically and numerically.

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