Journal of Mathematical Modeling
Vol. 11, No. 2, 2023, pp. 245–255. Research Article

JMM

Investigation and solving of initial-boundary value problem including fourth order PDE by contour integral and asymptotic methods

Mohammad Jahanshahi, Reza Danaei*

Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Iran
Email(s): m.jahanshahi1554@gmail.com, reza.danaei@azaruniv.ac.ir

Abstract. In this paper, we consider a fourth order mixed partial differential equation with some initial and boundary conditions which is unsolvable by classical methods such as Fourier, Fourier-Bircove and Laplace Transformation methods. For this problem we will apply the contour integral and asymptotic methods. The convergence of the appeared integrals, existence and uniqueness of solution, satisfying the solution and holding the given initial and boundary conditions are stablished by complex analysis theory and related contour integrals. Finally, the form of analytic and approximate solutions are given due to different cases of eigenvalues distributions in the complex plane.

Keywords: Asymptotic method, Laplace line, eigenvalues, contour integral, spectral problem.
AMS Subject Classification 2010: 35C20, 34E05, 35B40.

1 Introduction

The importance of partial differential equations among the topics of applied mathematics has been recognized for many years. Initial and boundary value problems including partial differential equations (PDEs) are appeared in many fields of physical and engineering problems, especially fourth order partial differential equation see [11, 12, 18]. For example some of fourth order applicable problems that appears in science are bending of elastic plate model, biharmonic equation, etc. The equation that we will consider in this paper is a fourth-order partial differential equation which are both second-order in spatial and second-order in time variables with some initial and boundary conditions. Several methods are applied to investigate and solve this problems such as Fourier methods and Laplace transformation method. In the Fourier method, differential equation and boundary conditions must be separated. The Laplace transform method for solving differential equations is well known and has been applied to many problems in

*Corresponding author.
Received: 7 November 2022/ Revised: 26 December 2022/ Accepted: 4 January 2023
DOI: 10.22124/jmm.2023.23209.2069

applied mathematics. In this method, the integrals in expressions of solution must be converged for all values of λ parameter as long as Laplace line in inverse transformation.

Like other transform methods, Laplace transform has some disadvantages. An approximation of the inverse Laplace transform has investigated in [1]. Another paper that uses new class of inverse Laplace method and give asymptotic results has investigated in [8].

In [3] an asymptotic formula has been obtained for a fourth order boundary value problem. Wazwaz investigated exact solutions for a fourth-order partial differential equation but without mixed term by the Adomian decomposition method in [16, 17]. Khaliq and Twizell used another method to solve fourth-order parabolic partial differential equation without mixed term that can separate [10]. An analytic procedure studied for fourth-order Beam Equation that has no mixed term by using the decomposition method and compared with modal analysis method and gave same results in [4] by author. In [2] for fourth order partial differential equation and also in [5–7] for a mixed problem including third order PDE, for λ-complex parameter dependent have considered and the solutions of the problems was given in the contour integral form. In [13], a numerical method proposed for singularly perturbed fourth order differential equations of convection-diffusion type by combines boundary value technique and asymptotic expansion approximations.

In this paper we consider an initial-boundary value problem in which the partial differential equation has a term of mixed derivative and we can not separate it. Therefore we can not apply the Fourier method. On the other hand, we can not apply the Laplace transformation method, because the integrands in the solution do not converge for all values of the parameter λ. Especially, we can not show satisfication of the initial and boundary conditions.

This paper is organized into four sections. In the first part, the problem statement and its spectral problem are given. In the sequel, the eigenvalues of the spectral problem and their distribution on the complex plane are discussed. In the second part, the asymptotic expansions of the integrands are presented in order to determine the appropriate conditions for the convergence of the integrals and the existence of the analytical solution. In the next section, the forms of analytical and approximate solution are given depending on different cases of the eigenvalues distribution. In the final part, the validity of analytical solution and holding initial and boundary conditions are established.

2 Main problem and its spectral problem

We consider the following fourth-order partial differential equation which is second-order in spatial variable and second-order in time variable withsome initial and boundary conditions

$$\frac{\partial^4 u(x,t)}{\partial t^2 \partial x^2} + a\frac{\partial^2 u(x,t)}{\partial t^2} + b\frac{\partial^2 u(x,t)}{\partial x^2} + cu(x,t) = 0, \quad x \in (0,1), \quad t > 0, \tag{1}$$

$$\frac{\partial^k u(x,t)}{\partial t^k}|_{t=0} = \varphi_k(t), \quad k = 0,1, \quad x \in [0,1], \tag{2}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{3}$$

where $a > 0, b > 0, c \in \mathbb{R}$. If $\tilde{u}(x,\lambda)$ is the Laplace transform of $u(x,t)$, then

$$\tilde{u}(x,\lambda) = \int_0^\infty e^{-\lambda t} u(x,t)dt.$$

By applying this transform for Eq. (1) together with initial conditions (2), we obtain

$$\int_0^\infty e^{-\lambda t}\frac{\partial^2 u(x,t)}{\partial t^2}dt = -[\lambda\varphi_0(x)+\varphi_1(x)]+\lambda^2\tilde{u}(x,t).$$

Therefore, we will have the following spectral problem

$$\begin{cases}(\lambda^2+b)\tilde{u}''(x,t)+(a\lambda^2+c)\tilde{u}(x,t)=\lambda\varphi_0''(x)+\varphi_1''(x)+a\lambda\varphi_0(x)+a\varphi_1(x),\\ \tilde{u}(0,\lambda)=\tilde{u}(1,\lambda)=0, \qquad x\in(0,1).\end{cases}$$

We can rewrite the spectral equation in the following form [15]

$$\tilde{u}''(x,\lambda)+\frac{a\lambda^2+c}{\lambda^2+b}\tilde{u}(x,\lambda)=\frac{(\frac{d^2}{dx^2}+a)(\lambda\varphi_0(x)+\varphi_1(x))}{\lambda^2+b}.$$

3 Calculating of eigenvalues of the spectral problem

For this, at first, we consider the general solution of associate nonhomogeneous equation as follows

$$\tilde{u}(x,\lambda)=C_1(x)e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x}+C_2(x)e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x},$$

where $C_1(x)$ and $C_2(x)$ can be writen as follows

$$C_1(x)=c_1-\frac{1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_{x_1}^{x}e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi,$$

and

$$C_2(x)=c_2+\frac{1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_{x_2}^{x}e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi.$$

By imposing the boundary conditions in (2) for arbitary constants c_1 and c_2 we will have the following algebraic system

$$\begin{cases}c_1+c_2=\frac{1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_0^1 e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi,\\ c_1+c_2e^{2i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}=\frac{e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_0^1 e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi.\end{cases}$$

The determinant of the coefficient matrix of this system is

$$\Delta(\lambda)=\begin{vmatrix}1 & 1\\ e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}} & e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}}\end{vmatrix}=e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}}-e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}}.$$

If this determinant is equal zero, then we have the eigenvalues of this problem:

$$\lambda_{k_m}=(-1)^m\sqrt{\frac{c-k^2\pi^2 b}{k^2\pi^2-a}}, \quad m=1,2,k\in\mathbb{Z}. \tag{4}$$

We consider the limit behaviour of eigenvalues as $\lim_{k\longrightarrow\infty}\lambda_{k_m}=\pm i\sqrt{b}$. Therefore, we have infinitely many eigenvalues and they have two limit points in complex plane, as shown by distributed points in Figure 1.

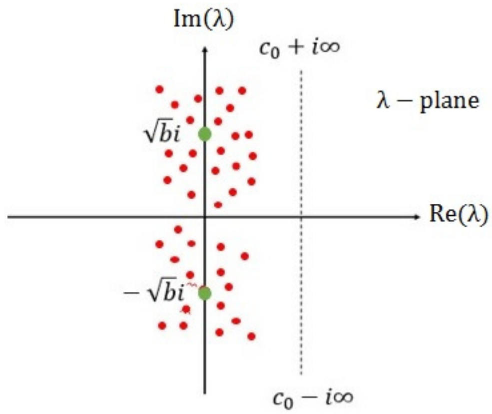

Figure 1: Distribution of eigenvalues in $\lambda - plane$.

4 Asymptotic expansions of integrands

We are going to investigate the limit behaviour and asymptotic expansion of the solution $u(x,t)$ with respect to the parameter λ. We have [9]

$$u(x,t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda,$$

where

$$\tilde{u}(x,\lambda) = \int_0^1 \frac{(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))}{\Delta(\lambda)2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}} \begin{vmatrix} e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)} & e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x} & e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x} \\ e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\xi} & 1 & 1 \\ e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(1-\xi)} & e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}} & e^{i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}} \end{vmatrix} d\xi.$$

The terms without $|x-\xi|$ are continous functions. The terms containing $|x-\xi|$ may be are discontinuous, so these terms are expanded as follows

$$\begin{aligned}
&\frac{-1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_0^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&\quad+\frac{-1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}(\frac{d^2}{d\xi^2}+a)(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&=\frac{-1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_0^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}\frac{d^2}{d\xi^2}(\lambda\varphi_0(\xi)+\lambda\varphi_1(\xi))d\xi && \text{(I)} \\
&\quad-\frac{a}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_0^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi && \text{(II)} \\
&\quad+\frac{1}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}\frac{d^2}{d\xi^2}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi && \text{(III)} \\
&\quad+\frac{a}{2i\sqrt{(a\lambda^2+c)(\lambda^2+b)}}\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi. && \text{(IV)}
\end{aligned}$$

Now, the last four integrals are calculated using integration by parts. We have

$$\begin{aligned}
\mathrm{I} &= \int_0^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}\frac{d^2}{d\xi^2}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&= \frac{d}{dx}(\lambda\varphi_0(x)+\varphi_1(x)) - e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x}\frac{d}{d\xi}(\lambda\varphi_0(x)+\varphi_1(x))|_{\xi=0}^{x} \\
&\quad - i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}[\lambda\varphi_0(x)+\varphi_1(x)][e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}x}\lambda\varphi_0(0)+\varphi_1(0)] \\
&\quad - (\frac{a\lambda^2+c}{\lambda^2+b})\int_0^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(x-\xi)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi.
\end{aligned}$$

Similarly, if we expand the second and third integrals, we have the following expansions

$$\begin{aligned}
&\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}\frac{d^2}{d\xi^2}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&\qquad = e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}\frac{d}{d\xi}(\lambda\varphi_0(\xi)+\varphi_1(\xi))|_{\xi=1}^{x} \\
&\qquad\quad + i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}\frac{d}{d\xi}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&\qquad = (\lambda\varphi_0(\xi)+\varphi_1(\xi))' - e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(1-x)}\frac{d}{d\xi}(\lambda\varphi_0(\xi)+\varphi_1(\xi))|_{\xi=1}^{x} \\
&\qquad\quad + i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))|_{\xi=1}^{x} \\
&\qquad\quad - (\frac{a\lambda^2+c}{\lambda^2+b})\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi \\
&\qquad = \frac{1}{2i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}}\{(a-\frac{a\lambda^2+c}{\lambda^2+b})\int_1^x e^{-i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\xi-x)}(\lambda\varphi_0(\xi)+\varphi_1(\xi))d\xi\} \\
&\qquad\quad + (\lambda\varphi_0(x)+\varphi_1(x))' + i\sqrt{\frac{a\lambda^2+c}{\lambda^2+b}}(\lambda\varphi_0(x)+\varphi_1(x)).
\end{aligned}$$

Now, if we set the conditions

$$\begin{aligned}
\varphi_0(0) = \varphi_1(0) = \varphi_0'(0) = \varphi_1'(0) = 0, \\
\varphi_0(1) = \varphi_1(1) = \varphi_0'(1) = \varphi_1'(1) = 0,
\end{aligned}$$

then the most parts of above relations are eliminated and we have only the term

$$\frac{\lambda\varphi_0(x)+\varphi_1(x)}{2(\lambda^2+b)}.$$

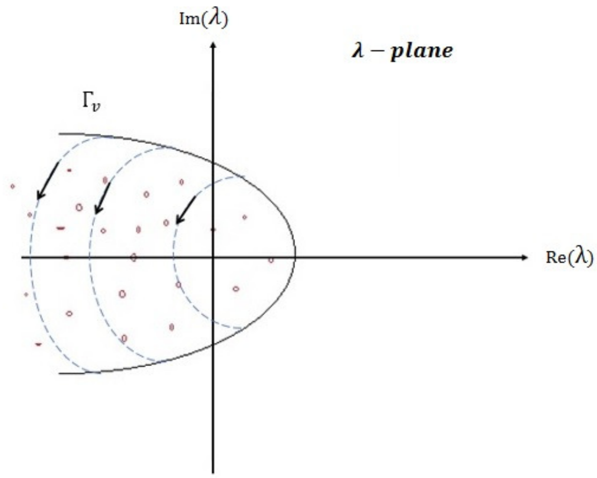

Figure 2: Γ_v closed curves.

By substituting the asymptotic expansion we get

$$\begin{aligned}\frac{\lambda\varphi_0(x)+\varphi_1(x)}{2(\lambda^2+b)} &= \frac{1}{\lambda+\frac{b}{2}}\varphi_0(x)+\frac{\varphi_1(x)}{\lambda^2+b} \\ &= \frac{\varphi_0(x)}{\lambda}-\frac{b\varphi_0(x)}{\lambda^3}+\frac{b^2\varphi_0(x)}{\lambda^5}+O(\lambda^{-7}) \\ &\quad +\frac{\varphi_1(x)}{\lambda^2}-\frac{b\varphi_1(x)}{\lambda^4}+\frac{b^2\varphi_1(x)}{\lambda^6}+O(\lambda^{-8}).\end{aligned}$$

So, the asymptotic expansion of $\tilde{u}(x,\lambda)$ can be written as

$$\tilde{u}(x,\lambda)=\frac{\varphi_0(x)}{\lambda}+\frac{\varphi_1(x)}{\lambda^2}+O(\lambda^{-3}). \tag{5}$$

According to this asymptotic expansion, we can show that the initial and boundary conditions hold. Concerning the initial conditions, we have

$$u(x,t)=\frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\tilde{u}(x,\lambda)d\lambda, \tag{6}$$

where Γ is the union of closed curves Γ_v (contour integrals) shown in the Figure 2. So we have

$$u(x,0)=\frac{1}{2\pi i}\lim_{v\longrightarrow\infty}\int_{\Gamma_v}\tilde{u}(x,\lambda)d\lambda=\frac{\varphi_0(x)}{2\pi i}\lim_{v\longrightarrow\infty}\int_{\Gamma_v}\frac{d\lambda}{\lambda}=\varphi_0(x), \tag{7}$$

$$\frac{\partial u(x,t)}{\partial t}\Big|_{t=0}=\frac{1}{2\pi i}\lim_{v\longrightarrow\infty}\int_{\Gamma_v}(\varphi_0(x)+\varphi_1(x)\frac{1}{\lambda})d\lambda=\varphi_1(x).$$

For satisfying boundary condition (3), we consider

$$u(0,t)-u(1,t)=\frac{1}{2\pi i}\lim_{v\longrightarrow\infty}\int_{\Gamma_v}e^{\lambda t}[\tilde{u}(0,\lambda)-\tilde{u}(1,\lambda)]d\lambda=0. \tag{8}$$

In the next part, we will consider the forms of analytical and numerical solutions of the problem. For this we need to have the following theorem.

Theorem 1. *([14, Page 160]) Suppose that $F(\lambda) = \mathscr{L}\{f(t)\}$ has infinitely many poles at $\lambda_1, \lambda_2, \ldots, \lambda_\infty$ all to the left of the $\textbf{Res}(\lambda) = c_0 > 0$. Choose a sequence of contours $\Gamma_n = C_n \cup [c_0 - i\infty, c_0 + i\infty]$ like Figure 2 enclosing the first n poles $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then by the Cauchy residue theorem,*

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\lambda t} F(\lambda)d\lambda = \sum_{k=1}^{\infty} \textbf{Res}(z_k).$$

5 The form of analytic and approximate solutions of main problem

In this section, we are going to give the general form of the analytic and approximate solutions over some contour integrals. To do this, we need to give and prove the following theorem.

Theorem 2. *Let the following conditions hold for the initial-boundary value problem (1)-(3) with $\varphi_0(x)$, $\varphi_1(x) \in \mathbb{C}^2(0,1)$*

$$\begin{cases} \varphi_0(0) = \varphi_1(0) = \varphi_0'(0) = \varphi_1'(0) = 0, \\ \varphi_0(1) = \varphi_1(1) = \varphi_0'(1) = \varphi_1'(1) = 0. \end{cases}$$

Then the initial-boundary value problem (1-3) has a unique analytic solution of the form (6) and its approximate solution takes the following form

$$u(x,t) \approx \sum_{k=0}^{N} e^{\lambda_k t}\tilde{u}(x, \lambda_k).$$

Proof. Regarding that some parts of the proof of this theorem are similar to Theorem 1 in [5, 6], the proof of satisfing the initial conditions and boundary conditions and the form of analytical and numerical solutions are given. Finally, using the asymptotic expansion of $u(x,t)$ given by Eq. (5) and Theorem 1, we show that the solution of $u(x,t)$ satisfies the initial and boundary condition of the main problem (1)-(3). We have

$$u(x,t) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda = \sum_{k=1}^{\infty} \mathbf{Res}(z_k) = \frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\Big(\frac{\varphi_0(x)}{\lambda} + \frac{\varphi_1(x)}{\lambda^2}\Big)d\lambda.$$

So by the use of Theorem 1, we will have

$$u(x,t) = \varphi_0(x) + \varphi_1(x)t.$$

From relations (7)-(8), the inital condition (2) and boundary conditions (3) are hold, that is

$$\begin{cases} u(0,t) = u(1,t) = 0, \\ u(x,0) = \varphi_0(x), \\ \frac{\partial}{\partial t}u(x,0) = \varphi_1(x). \end{cases}$$

For the form of the approximate solution (2), we give the proof of this theorem for different cases. For this, according to the distribution of the eigenvalues (4) in the complex plane, if the eigenvalues lie in the left-hand side of the Laplace line $L = (c_0 - i\infty, c_0 + i\infty)$, where $c_0 > 0$, then we have two cases.

If the eigenvalues are distributed according to Figure 2, then by choosing a suitable closed contour which includes finite eigenvalues in left-hand side of Laplace line, we compute the solution by using rsidual theory in complex analysis. If the closed contour $\Gamma_\nu = [a_\nu, b_\nu] \cup c_\nu$ according to Figure 2, where $[a_\nu, b_\nu] \subset (c_0 - i\infty, c_0 + i\infty)$, $\nu \in \mathbb{N}$, then the analytic solution will be

$$u(x,t) = \int_L e^{\lambda t} \tilde{u}(x,\lambda) d\lambda = \lim_{\nu \to \infty} \int_{\Gamma_\nu} e^{\lambda t} \tilde{u}(x,\lambda) d\lambda = \sum_{k=0}^{\infty} e^{\lambda_k t} \tilde{u}(x,\lambda), \tag{9}$$

and the approximate solution is computed via contour integral method by choosing finite number of

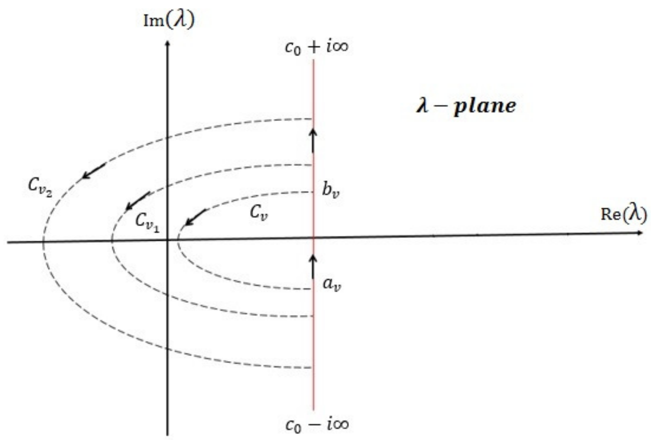

Figure 3: Closed contours in the left-hand of Laplace line.

series solution (9)

$$u(x,t) \approx \sum_{k=0}^{N} e^{\lambda_k t} \tilde{u}(x,\lambda).$$

Now, we choose contour M_ν such that there is no eigenvalue between this contour and the Laplace line, Figure 4. If the eigenvalues are distributed in the left-hand side of Laplace line and the contour M_ν, in this case, we can compute the solution over contour M_ν and the approximate solution can be calculated by Laplace transformation method. As B_ν is a closed contour by the Cauchy integral theorem we have As B_ν is a closed contour by the Cauchy integral theorem we have

$$\int_{B_\nu} e^{\lambda t} \tilde{u}(x,\lambda) d\lambda = 0. \tag{10}$$

Consider the following closed contours B_ν

$$\begin{cases} B_{\nu_1} = L_{\nu_1} \cup C_{\nu_1} \cup D_{\nu_1} \cup M_{\nu_1}, \\ B_{\nu_2} = L_{\nu_2} \cup C_{\nu_2} \cup D_{\nu_2} \cup M_{\nu_2}, \\ \quad\vdots \\ B_{\nu_i} = L_{\nu_i} \cup C_{\nu_i} \cup D_{\nu_i} \cup M_{\nu_i}. \end{cases}$$

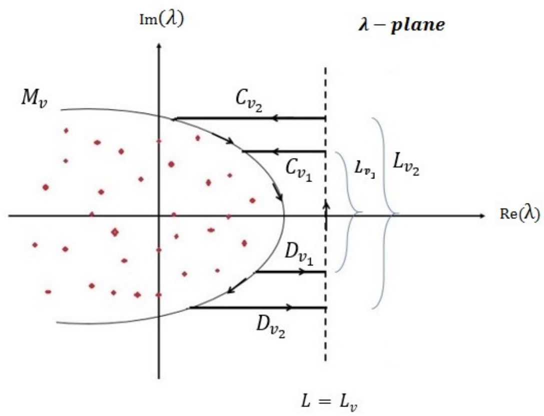

Figure 4: Closed contours in the left-hand of Laplace line.

Now according to the (10), we have

$$0=\int_{B_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=\int_{L_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda+\int_{C_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda+\int_{D_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda$$
$$+\int_{M_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda,$$

note that the curves C_{v_i} and D_{v_i} have opposite directions, so we have

$$\int_{C_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=-\int_{D_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda,$$

therefore

$$\int_{L_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=-\int_{-M_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=\int_{M_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda \Rightarrow \int_{L_v} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=\int_{M_v} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda.$$

So we have

$$u(x,t)=\int_{c_0-i\infty}^{c_0+i\infty} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=\lim_{v_i\to\infty}\int_{L_{v_i}} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda\approx\int_{L_v} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda=\int_{M_v} e^{\lambda t}\tilde{u}(x,\lambda)d\lambda.$$

This completes the proof. □

At the end, some examples are presented according to the place of distribution of eigenvalues.

Example 1. If in the spectral problem (2), we put $c=a=b=1$ with the coefficients of the main problem (1)-(3), $a>0$, $b>0$, $c\in\mathbb{R}$ and $k\in\mathbb{Z}$, then the eigenvalues will be on the imaginary axis, as Figure 5, and we can write the analytic solution and related approximate solution by choosing a closed contour integral, see [6].

Example 2. If we put $a=b=1$ and $c=5$ in the spectral problem (2) with the coefficients of the main problem (1), then the eigenvalues will be on the imaginary and real axis in the complex plane $\mathbb{C}$, see Figure 6. Now, we can write the analytic solution and related approximate solution by choosing a closed contour integral, see [6].

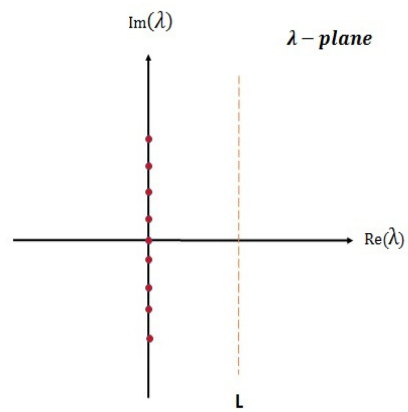

Figure 5: The eigenvalues lie on imaginary axis.

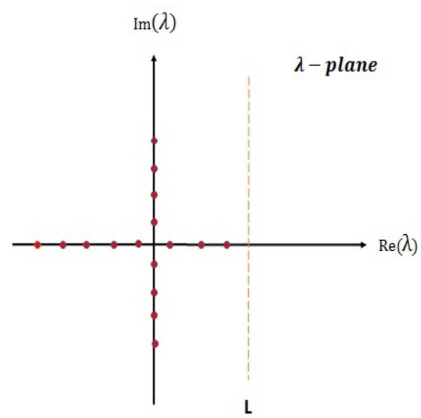

Figure 6: The eigenvalues lie on imaginary and real axes.

6 Conclusion

In this paper, first the eigenvalues of the related spectral problem of the main initial-boundary value problem including a forth order partial differential equation were calculated. Then the existence of analytical solution and approximate solutions were presented by using complex analysis and contour integral method. The presentation of solutions were given due to different cases of eigenvalues distributions on the complex plane with respect to the Laplace line. Finally two examples were illustrated the results of solutions.

References

[1] J.E. Akin, J. Count, *On rational approximations to the inverse Laplace transform*, SIAM J. Appl. Math. **17** (1969) 1035–1040.

[2] S.T. Aleskerova, *Solving a mixed problem with almost regular boundary condition by the contour integral method*, J. Math. Res. **9** (2017) 158–162.

[3] N.M. Aslanova, M. Bayramoglu, K.M. Aslanova, *Eigenvalue problem associated with the fourth order differential operator equation*, Rocky Mt. J. Math. **48** (2018) 1763–1779.

[4] H. Haddadpour, *An exact solution for variable coefficients fourth-order wave equation using the Adomian method*, Math. Comput. Model. **44** (2006) 1144–1152.

[5] M. Jahanshahi, R. Danaei, *Analytical-numerical solution for a third order space-time conformable fractional PDE with mixed derivative by spectral and asymptotic methods*, Sahand Commun. Math. Anal. **20** (2023) 81–93.

[6] M. Jahanshahi, N. Aliyev, F. Jahanshahi, *Solving two initial-boundary value problems including fractional partial differential equations by spectral and contour integral method*, Azarbayjan J. Math. **10** (2020) 31–47.

[7] M. Jahanshahi, N. Aliyev, F. Jahanshahi, *Solving some initial-boundary value problems including non-classical cases of heat equation by spectral and contour integral method*, Int. J. Ind. Math. **9** (2017) 359–364.

[8] P.M. Jordan, P. Puri, G. Boros, *A new class of Laplace inverses and their applications*, Appl. Math. Lett. **13** (2000) 97–104.

[9] M. L. Rasulov, *Methods of Contour Integration*, North-Holland Publishing Company, Amsterdam, 1967.

[10] A.Q.M. Khaliq, E.H. Twizell, *A family of second order methods for variable coefficient fourth order parabolic partial differential equations*, Int. J. Comput. Math. **23** (1987) 63–76.

[11] X.Y. Liu, C.H. Lai, K.A. Pericleous, *A fourth-order partial differential equation denoising model with an adaptive relaxation method*, Int. J. Comput. Math. **92** (2014) 608–622.

[12] K.S. Miller, *Partial Differential Equations In Engineering Problems*, Prentic-Hall, Inc, 1963.

[13] J.S.C. Roja, P. Comminos, A. Tamilselvan, *Numerical method for singularly perturbed fourth order ordinary differential equations of convection-diffusion type*, J. Math. Model. **4** (2016) 79–102.

[14] J.L. Schiff, *The Laplace Transform, Theory and Applications*, Springer-Verlag, 1999.

[15] J. Shen, T. Tang, L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer-Verlag, 2011.

[16] A.M. Wazwaz, *Exact solutions for variable cofficients fourth-order parabolic partial differential equations in higher-dimensional spaces*, Appl. Math. Comput.**130** (2002) 415–424.

[17] A.M. Wazwaz , *On the solution of the fourth order parabolic equation by the decomposition method*, Int. J. Comput. Math. **57** (1995) 213–217.

[18] L.H. You, P. Comminos, J.J. Zhang, *Solid modelling with fourth-order partial differential equation*, Int. J. Comput. **4** (2008) 452–461.