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A new approach to solve weakly singular fractional-order delay integro-differential equations using operational matrices

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Abstract. In this paper, we propose a new approach to solve weakly singular fractional delay integro-differential equations. In the proposed approach, we apply the operational matrices of fractional integration and delay function based on the shifted Chebyshev polynomials to approximate the solution of the considered equation. By approximating the fractional derivative of the unknown function as well as the unknown function in terms of the shifted Chebyshev polynomials and substituting these approximations into the original equation, we obtain a system of nonlinear algebraic equations. We present the convergence analysis of the proposed method. Finally, to show the accuracy and validity of the proposed method, we give some numerical examples.

Keywords: Operational matrices, fractional delay integro-differential equation, weakly singular kernel.
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1 Introduction

Fractional calculus (FC) plays an important role in modeling many real phenomena [9,17]. To get a superior approach for the description of memory and hereditary properties of various materials and processes, fractional derivatives have an important and fundamental role. Fractional order integro-differential equations (FOIDEs) has been used to model a lot of practical problems, such as the field of dielectric polarization, electromagnetic waves, viscoelasticity, and diffusion equations. To see the proposed numer-

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ical methods for solving fractional-order differential and integro-differential equations, one can refer to [1, 2, 7, 19].

The pantograph-type weakly singular fractional integro-differential equations (WSFIDEs), weakly singular fractional delay integro-differential equations (WSFDIDEs) and Volterra integro-differential equations have recently been studied by many researchers. WSFIDEs arise in mathematical modeling of various phenomena, such as heat conduction problems, radiative equilibrium, elasticity, and fracture mechanics [10, 20, 26]. The pantograph equations (PEs) plays an important role in different fields of sciences such as astrophysics, economy, control, biology and electrodynamics [8, 24]. Applications of delay differential equations can also be seen in various technical systems, such as automatic control, biology and hydraulic networks, long transmission lines, economy and biology [12, 23]. Therefore, solving WSFDIDEs is of particular importance. On the other hand, providing the analytic methods to solve these equations is either complicated or impossible. So, it is necessary to develop numerical methods for solving these equations.

Recently, the authors of [18] presented an operational approach based on fractional-order Euler polynomials for solving WSFDIDEs. The author of [6] proposed various numerical schemes for solving a partial integro-differential equation with a weakly singular kernel. In [11], by combining the Legendre orthonormal basis with the operational matrix and the Gauss quadrature rule, the authors presented a new approach to solving a class of fractional optimal control problems. In [4], the authors proposed a numerical method based on the shifted Jacobi polynomials to solve WSFIDEs. Solving fractional integro-differential equations with the weakly singular kernel by using fractional-order Euler functions is done in [21]. In [13], the author applied block pulse functions for solving fractional integro-differential equation with a weakly singular kernel. The operational Tau method is applied for solving WSFIDEs in [14]. Also, Yi and Huang in [22] proposed CAS wavelets method for solving these equations. Zhao et al. applied the piecewise polynomial collocation method for solving the WSFIDEs in [25]. In [15], a numerical method based on the Chebyshev polynomials of the second kind is presented to solve WSFIDEs. Pedas et al. proposed the spline collocation method for solving these equations in [16].

The main aim of this paper is to solve the WSFDIDEs

$$\begin{aligned} D^{\gamma}y(t) &= \lambda_1 \int_0^t \frac{k_1(s,t)F(y(s))}{(t-s)^{\mu}}ds + \lambda_2 \int_0^t \frac{k_2(s,t)y(h(s))}{(t-s)^{\nu}}ds \\ &\quad + \lambda_3 \int_0^{r(t)} k_3(s,t)y(s)ds + \lambda_4 \int_0^{r(t)} k_4(s,t)y(h(s))ds \\ &\quad + w(t)y(h(t)) + p(t)G(y(t)) + f(t), \quad t \in [0,1], \qquad (1) \end{aligned}$$

$$y^{(j)}(0) = y_0^{(j)}, \quad j = 0,1,2,\ldots,\lceil\gamma\rceil, \qquad (2)$$

where $\gamma > 0$, $0 \le \mu, \nu < 1$, $w(t)$, $f(t)$, $F(t)$, $G(t)$, $p(t)$ and $r,h : [0,1] \to (0,1]$ are continuous known functions and $k_1(s,t)$, $k_2(s,t)$, $k_3(s,t)$ and $k_4(s,t)$ are known kernels which are defined on $[0,1]\times[0,1]$. Also $y_0^{(j)}$, $j = 0,1,\ldots,n-1$, λ_1, λ_2, λ_3, λ_4 are constants and D^{γ} denotes the Caputo' s fractional derivative of order γ. Here, we suppose that $F(t) = t^l$ and $G(t) = t^n$, where $l,n \in \mathbb{N}$. Also, we use the ceiling function $\lceil\gamma\rceil$ to denote the smallest integer greater than or equal to γ. Eq. (1) could be listed in the pantograph-type Volterra integro-differential equations, WSFIDEs and WSFDIDEs.

To propose a numerical approach for solving Eq. (1), we use the shifted Chebyshev polynomials of the first kind. To do this, we first approximate the fractional derivative of the solution of Eq. (1). Then, by applying operational matrices of fractional integration, product, and delay function, we approximate integral parts of this equation. Finally, we convert Eq. (1) to a non-linear system of algebraic equations and solve this system with the help of *Mathematica* software 11 (Version: 11.3.0.0). Also, we present a convergence analysis.

Some of the strengths of the proposed method in this paper are as follows:

- Given the proposed operational method based on the shifted Chebyshev is a reliable method, this method can be used to solve a wide range of equations.
- By using the proposed method, the original problem is reduced to a system of nonlinear algebraic equations without the use of the collocation method.

The rest of the paper is organized as follows: In Section 2, we present some definitions and results about fractional operators, shifted Chebyshev polynomials of the first kind, and the expansion of functions by and these polynomials. Section 3 provided the operational matrices for giving the proposed method. Solving Eq. (1) with the help of the proposed method is done in Section 4. In Section 5, we provide the error analysis for the introduced method. Some numerical experiments are presented in Section 6.

2 Shifted Chebyshev polynomials and their properties

In this section, we will review some definitions and basic concepts of the shifted Chebyshev polynomials of the first kind that we will use in this paper.

2.1 Shifted Chebyshev polynomials

Definition 1. *Chebyshev polynomials of the first kind of degree n are defined by [5]*

$$T_n(x)=\cos(n\theta),\ \theta=\arccos(x),\ \ n\geq 0,\ x\in[-1,1].$$

The fundamental recurrence relation of these polynomials is as follows:

$$\begin{aligned}&T_0(x)=1, \qquad T_1(x)=x,\\ &T_{n+1}(x)=2xT_n(x)-T_{n-1}(x), \quad n\in\mathbb{N}.\end{aligned}$$

Clearly, the Chebyshev polynomials are orthogonal on [-1,1]:

$$<T_i(x),T_j(x)>=\int_{-1}^{1}T_i(x)T_j(x)\omega(x)\,dx=\begin{cases}\pi, & i=j=0,\\ \frac{\pi}{2}\delta_{ij}, & i,j>0,\end{cases}$$

where $\omega(x)=\frac{1}{\sqrt{1-x^2}}$ is the weight function and

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

By considering $x=2t-1$, $t\in[0,1]$, we get the shifted Chebyshev polynomials (SCPs) defined on $[0,1]$ as follows

$$T_{n+1}(2t-1)=2(2t-1)T_n(2t-1)-T_{n-1}(2t-1), \quad n\in\mathbb{N}.$$

By supposing $T_n^*(t) = T_n(2t-1)$, we have the following recurrence formula for SCPs

$$T_{n+1}^*(t) = 2(2t-1)T_n^*(t) - T_{n-1}^*(t), \quad n \in \mathbb{N}, \tag{3}$$

where $T_0^*(t) = 1$, $T_1^*(t) = 2t-1$. In [3], it is proved that $T_n^*(t)$ has the following representation in terms of power of t

$$T_i^*(t) = i\sum_{k=0}^{i}(-1)^{i-k}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!}t^k. \tag{4}$$

Also, we have the following properties for SCPs

$$\begin{aligned} T_i^*(t)T_j^*(t) &= \tfrac{1}{2}(T_{2i+2j}^*(t) + T_{|2i-2j|}^*(t)), \\ \int T_i^*(t)dt &= \frac{1}{4}\Big(\frac{T_{i+1}^*(t)}{i+1} - \frac{T_{i-1}^*(t)}{i-1}\Big). \end{aligned} \tag{5}$$

The orthogonality of these polynomials is

$$< T_i^*(x), T_j^*(x) >= \int_0^1 T_i^*(t)T_j^*(t)\omega^*(t)\,dt = h_i\delta_{ij}, \tag{6}$$

where $\omega^*(t) = \frac{1}{\sqrt{t-t^2}}$ and $h_i = \frac{a_i}{2}\pi$, $a_0 = 2$, $a_i = 1$, $i \geq 1$.

Clearly, from (4), we can write $T_n^*(t)$ as the follows

$$T_n^*(t) = V^n X(t),$$

where $X(t) = [1, t, t^2, \ldots, t^n]$, $V^n = [v_0^n, v_1^n, \ldots, v_n^n]$, and also v_k^n is as follows

$$v_k^n = (-1)^{n-k} n \frac{(n+k-1)!2^{2k}}{(n-k)!(2k)!}, \quad k = 0, 1, \ldots, n.$$

Suppose that $T^{*,m}(t) = [T_0^*(t), T_1^*(t), \ldots, T_m^*(t)]^T$. Clearly, we conclude that

$$T^{*,m}(t) = A_m X(t), \tag{7}$$

where

$$A_m = \begin{pmatrix} V^0 \\ V^1 \\ \vdots \\ V^m \end{pmatrix},$$

is a matrix of order $(m+1)\times(m+1)$, and

$$\begin{aligned} V^0 &= [1, 0, 0, \ldots, 0], \\ V^1 &= [v_0^1, v_1^1, 0, 0, \ldots, 0], \\ V^2 &= [v_0^2, v_1^2, v_3^2, 0, \ldots, 0], \\ &\vdots \\ V^m &= [v_0^m, v_1^m, \ldots, v_m^m]. \end{aligned}$$

2.2 Expansion of functions in terms of SCPs

2.2.1 One variable function

Let $u(t) \in L^2([0,1])$. Given that SCPs are a complete basis in [0,1], we can expand $u(t)$ in terms of these polynomials as

$$u(t) = \sum_{i=0}^{\infty} u_i T_i^*(t) \approx \sum_{i=0}^{m} u_i T_i^*(t) = U^T T^{*,m}(t) = T^{*,m}(t)^T U, \tag{8}$$

where the coefficients u_i are given by

$$u_i = \frac{1}{h_i} \int_0^1 u(t) T_i^*(t) w^*(t) dt, \quad i = 0, 1, 2, \dots, \tag{9}$$

and also $U = [u_0, u_1, \dots, u_m]^T$.

2.2.2 Two variable function

Similarly, we can expand any continuous two variable function, $f(s,t)$, defined on $[0,1] \times [0,1]$ in terms of the double SCPs as the following form

$$f(s,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} f_{ij} T_i^*(s) T_j^*(t) \approx \sum_{i=0}^{m} \sum_{j=0}^{m} f_{ij} T_i^*(s) T_j^*(t) = T^{*,m}(s)^T \mathbf{F} T^{*,m}(t), \tag{10}$$

where $\mathbf{F} = [f_{ij}]$ is a matrix of order $(m+1) \times (m+1)$, and

$$f_{ij} = \frac{1}{h_i} \frac{1}{h_j} \int_0^1 \int_0^1 f(s,t) T_i^*(s) T_j^*(t) w^*(s) w^*(t) ds dt, \quad i, j = 0, 1, 2, \dots, m. \tag{11}$$

2.2.3 Functions including delay function

Consider $u(h(t))$, where $h : [0,1] \to [0,1]$ be a delay function. By using (8), we can expand this delay function in terms of SCPs. Therefore, we have

$$u(h(t)) = \sum_{i=0}^{\infty} u_i T_i^*(h(t)) \approx \sum_{i=0}^{m} u_i T_i^*(h(t)) = U^T T_i^{*,m}(h(t)) = U^T A_m X(h(t)),$$

where $X(h(t)) = [1, (h(t)), \dots, (h(t))^m]^T$. Now, by expanding $(h(t))^j$ for $j = 0, 1, \dots, m$, in terms of the SCPs, we conclude that

$$(h(t))^j = \sum_{i=0}^{\infty} H_i^j T_i^*(t) \approx \sum_{i=0}^{m} H_i^j T_i^*(t) = (H^j)^T T^{*,m}(t),$$

where $H^j = [H_0^j, H_1^j, \dots, H_m^j]^T$ is coefficient vector, and

$$H_i^j = \frac{1}{h_i} \int_0^1 (h(t))^j T_i^*(t) w^*(t) dt, \quad i, j = 0, 1, 2, \dots, m. \tag{12}$$

So, we have

$$u(h(x)) \approx U^T A_m H T^{*,m}(t), \tag{13}$$

where

$$H = \begin{pmatrix} (H^0)^T \\ (H^1)^T \\ \vdots \\ (H^m)^T \end{pmatrix}. \tag{14}$$

3 Main results

In this section, we will write the matrix forms of all terms in (1) by the help of SCPs.

3.1 Operational matrix of Riemann-Liouville fractional integration

Definition 2. *[17] A real function $f(t)$, $t > 0$ is said to be in the space c_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0,\infty]$, and it is said to be in the space C_μ^m iff $f^{(m)} \in C_\mu$, $m \in N$.*

Definition 3. *[9, 17] The Riemann-Liouville fractional integral operator of order $\gamma \geq 0$ is a function defined as*

$$I^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^x (x-t)^{\gamma-1} f(t)dt, & \gamma > 0, \\ f(x), & \gamma = 0, \end{cases} \tag{15}$$

where $\Gamma(\gamma)$ is the gamma function as

$$\Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t} dt. \tag{16}$$

Clearly, for $\beta_1, \beta_2 \in \mathbb{R}$, we have

$$I^\gamma(\beta_1 f(x) + \beta_2 g(x)) = \beta_1 I^\gamma f(x) + \beta_2 I^\gamma g(x),$$

and also

$$I^\gamma x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{17}$$

Definition 4. *[9, 17] The Caputo fractional derivative of order $\gamma > 0$ is defined as*

$$D^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_0^x (x-t)^{n-\gamma-1} f^{(n)}(t)dt, & \gamma > 0, n-1 < \gamma \leq n, n \in \mathbb{N}, \\ \dfrac{d^{(n)} f(x)}{dx^n}, & \gamma = n, \end{cases} \tag{18}$$

where $x \in [0,\infty)$.

For $x > 0$, $f \in C^n_\gamma, \gamma \geq -1$, the Caputo derivative and Riemann-Liouville integral operators satisfy the following properties

$$D^\gamma I^\gamma f(x) = f(x), \tag{19}$$

$$I^\gamma D^\gamma f(x) = f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{k!} x^k, \quad m-1 < \gamma \leq m. \tag{20}$$

Now, suppose that $\prod^\nu$ be the the operational matrix of Riemann-Liouville fractional integration of the vector $T^{*,m}(t)$. It is clear that

$$I^\nu T^{*,m}(t) = \prod^\nu T^{*,m}(t), \tag{21}$$

where $\prod^\nu$ is the $(m+1)\times(m+1)$ matrix. By using Eq. (7) and the properties of the operator I^μ, we have

$$I^\nu T^{*,m}(t) = I^\nu A_m X(t) = A_m I^\nu X(t). \tag{22}$$

Now, we apply Eq. (17) to compute $I^\nu X(t)$ as the follows:

$$I^\nu X(t) = I^\nu [1, t, t^2, \ldots, t^m]^T = [\frac{\Gamma(1)}{\Gamma(1+\nu)} t^\nu, \frac{\Gamma(2)}{\Gamma(2+\nu)} t^{1+\nu}, \ldots, \frac{\Gamma(m+1)}{\Gamma(m+1+\nu)} t^{m+\nu}]^T.$$

By expanding $\frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} t^{j+\nu}$ for $j = 0, 1, \ldots, m$, in terms of SCPs, we have

$$\frac{\Gamma(j+1)}{\Gamma(j+1+\nu)} t^{j+\nu} \approx \sum_{i=0}^{m} u_i^j T_i^*(t) = (U^j)^T T^{*,m}(t),$$

where $U^j = [u_0^j, u_1^j, \ldots, u_m^j]^T,\ j = 0, 1, \ldots, m$, and u_i^j for $i = 0, 1, \ldots, m$ are defined in Eq. (9). Hence,

$$I^\nu X(t) = \Theta T^{*,m}(t), \tag{23}$$

where

$$\Theta = \begin{pmatrix} U_0^T \\ U_1^T \\ \vdots \\ U_m^T, \end{pmatrix},$$

and also

$$I^\nu T^{*,m}(t) = A_m \Theta T^{*,m}(t). \tag{24}$$

Clearly, $\prod^\nu = A_m \Theta$.

3.2 The product operational matrix based on SCPs

Assume that P is a $(m+1)$-vector. Here, we compute the product operational matrix of SCPs, $\widetilde{P}$, where $T^{*,m}(t)(T^{*,m}(t))^T P = \widetilde{P} T^{*,m}(t)$. By using (7), we have

$$T^{*,m}(t)(T^{*,m}(t))^T P = A_m X(t) X^T(t) A_m^T P = A_m Q A_m^T P,$$

where $Q = [q_{ij}]$, $1 \leq i, j \leq m+1$, and $q_{ij} = t^{i+j-2}$. Now, suppose that $QA_m^T P = R(t)$, where $R(t) = [r_1(t), r_2(t), \ldots, r_{m+1}(t)]^T$. By expanding $r_i(t)$, $i = 1, 2, \ldots, m+1$, in terms of the SCPs, we get the the following equation

$$r_i(t) = F_i^T T^{*,m}(t),$$

where F_i is obtained by using Eq. (9). Therefore, we have

$$T^{*,m}(t)(T^{*,m}(t))^T P = A_m \Delta T^{*,m}(t), \tag{25}$$

and hence $\widetilde{P} = A_m \Delta$, where

$$\Delta = \begin{pmatrix} F_1^T \\ F_2^T \\ \vdots \\ F_{m+1}^T \end{pmatrix}.$$

Similarly, we can write $\int_0^t \frac{T^{*,m}(s)(T^{*,m}(s))^T}{(t-s)^\mu} ds$ in the matrix form by the help of the SCPs as follows:

$$\begin{aligned} \int_0^t \frac{T^{*,m}(s)(T^{*,m}(s))^T}{(t-s)^\mu} ds = \int_0^t \frac{A_m X(s)(X(s))^T (A_m)^T}{(t-s)^\mu} ds = A_m \int_0^t \frac{X(s)(X(s))^T}{(t-s)^\mu} ds (A_m)^T \\ = A_m N (A_m)^T, \end{aligned} \tag{26}$$

where $N = [n_{ij}]$, and

$$n_{ij} = \Gamma(1-\mu) I^{1-\mu} t^{i+j-2} = \Gamma(1-\mu) \frac{\Gamma(i+j-1)}{\Gamma(i+j-\mu)} t^{i+j-1-\mu}, \tag{27}$$

for $1 \leq i, j \leq m+1$.

4 The proposed method

To give the proposed method for solving WSFDIDEs (1) and (2), we substitute the approximations of all terms of these equations by the help of matrices forms. So, by using SCPs and the previous section, we approximate the following functions

$$\begin{aligned} D^\gamma y(t) &\approx Y^T T^{*,m}(t), \qquad p(t) \approx P^T T^{*,m}(t), \qquad w(t) \approx W^T T^{*,m}(t), \\ f(t) &\approx F^T T^{*,m}(t), \qquad k_i(s,t) \approx (T^{*,m}(s))^T \mathbf{K}_i T^{*,m}(t), \quad i = 1, 2, 3, 4. \end{aligned} \tag{28}$$

By using (20) and (28), we conclude that

$$y(t) \approx Y^T \prod^{\gamma} T^{*,m}(t) + \sum_{m=0}^{n-1} \frac{y^{(m)}(0^+)}{m!} t^m = (Y^T \prod^{\gamma} + Y_0^T) T^{*,m}(t) = b^T T^{*,m}(t), \tag{29}$$

where $\sum_{m=0}^{n-1} \frac{y^{(m)}(0^+)}{m!} t^m \approx Y_0^T T^{*,m}(t)$. To approximate $F(y(t)) = y^l(t)$, we apply (25). Hence

$$\begin{aligned} y^2(t) &\approx (b^T T^{*,m}(t))((T^{*,m}(t))^T b) = b^T \widetilde{b} T^{*,m}(t), \\ y^3(t) &= y^2(t) y(t) \approx b^T \widetilde{b} T^{*,m}(t)(T^{*,m}(t))^T b = b^T \widetilde{b}^2 T^{*,m}(t), \\ y^l(t) &= y^{l-1}(t) y(t) \approx b^T \widetilde{b}^{l-2} T^{*,m}(t)(T^{*,m}(t))^T b = b^T \widetilde{b}^{l-1} T^{*,m}(t). \end{aligned}$$

Now, we approximate $\int_0^t \frac{k_1(s,t)y^p(s)}{(t-s)^\mu}ds$. To do this, we have

$$\int_0^t \frac{k_1(s,t)y^l(s)}{(t-s)^\mu}ds \approx \int_0^t \frac{b^T\widetilde{b}^{l-1}T^{*,m}(s)(T^{*,m}(s))^T\mathbf{K}_1T^{*,m}(t)}{(t-s)^\mu}ds$$
$$= b^T\widetilde{b}^{l-1}\int_0^t \frac{T^{*,m}(s)(T^{*,m}(s))^T}{(t-s)^\mu}ds\mathbf{K}_1T^{*,m}(t).$$

By applying (26), we get

$$\int_0^t \frac{k_1(s,t)y^l(s)}{(t-s)^\mu}ds \approx b^T\widetilde{b}^{l-1}A_mNA_m^T\mathbf{k}_1T^{*,m}(t), \tag{30}$$

where $N=[n_{ij}]$ is a matrix of $(m+1)\times(m+1)$ defined as (27). Suppose that $V^1=[v_1^1,v_2^1,\dots,v_{m+1}^1]^T = NA_m^T\mathbf{k}_1T^{*,m}(t)$ (as a $(m+1)$-vector). By expanding v_j^1 in terms of SCPs, we have

$$v_j^1=\phi_{1,j,\mu}^TT^{*,m}(t), \quad j=1,2,\dots,m+1,$$

where $\phi_{1,j,\mu}$ is a $(n+1)-$vector. So,

$$\int_0^t \frac{k_1(s,t)y^l(s)}{(t-s)^\mu}ds \approx b^T\widetilde{b}^{p-1}A_m\Phi^1T^{*,m}(t), \tag{31}$$

where

$$\Phi^1=\begin{pmatrix}\phi_{1,1,\mu}^T\\ \phi_{1,2,\mu}^T\\ \vdots\\ \phi_{1,m+1,\mu}^T\end{pmatrix}.$$

The following result can be obtained by a process similar to the above process

$$\int_0^t \frac{k_2(s,t)y(h(s))}{(t-s)^\nu}ds \approx b^TA_mHA_m\mathbf{M}_2A_m^T\mathbf{K}_2T^{*,m}(t), \tag{32}$$

where $\mathbf{M}_2=[m2_{ij}]$, $1\le i,j\le m+1$, and

$$m2_{ij}=\Gamma(1-\nu)\frac{\Gamma(i+j-1)}{\Gamma(i-1+j-\nu)}t^{i+j-1-\nu},$$

and also H is defined in (14). By considering $V^2=[v_1^2,v_2^2,\dots,v_{m+1}^2]^T=\mathbf{M}_2(A_m)^T\mathbf{K}_2T^{*,m}(t)$, and

$$v_j^2=\phi_{2,j,\nu}^TT^{*,m}(t), \quad j=1,2,\dots,m+1,$$

where $\phi_{2,j,\nu}$ is a $(m+1)-$vector, we get the following result

$$\int_0^t \frac{k_2(s,t)y(h(s))}{(t-s)^\nu}ds \approx b^TA_mHA_m\Phi^2T^{*,m}(t), \tag{33}$$

where

$$\Phi^2 = \begin{pmatrix} \phi_{2,1,\nu}^T \\ \phi_{2,2,\nu}^T \\ \vdots \\ \phi_{2,m+1,\nu}^T \end{pmatrix}.$$

In the similar way, we can approximate $\int_0^{r(t)} k_3(s,t)y(s)ds$, and present it in the matrix form. Therefore, we have

$$\int_0^{r(t)} k_3(s,t)y(s)ds \approx b^T A_m \mathbf{M}_3 A_m^T \mathbf{K}_3 T^{*,m}(t), \tag{34}$$

where $\mathbf{M}_3 = [m3_{ij}]$, $1 \leq i,j \leq m+1$, and

$$m3_{ij} = \frac{r(t)^{i+j-1}}{i+j-1},\ i,j = 1,2,\ldots,m+1.$$

By supposing $V^3 = [v_1^3, v_2^3, \ldots, v_{m+1}^3]^T = \mathbf{M}_3 A_m^T \mathbf{K}_3 T^{*,m}(t)$, and $v_j^3 = \phi_{3,j}^T T^{*,m}(t),\ j = 1,2,\ldots,m+1$, we will have

$$\int_0^{r(t)} k_3(s,t)y(s)ds \approx b^T A_m \Phi^3 T^{*,m}(t), \tag{35}$$

where $\phi_{3,j}$ is a $(m+1)-$vector, and

$$\Phi^3 = \begin{pmatrix} \phi_{3,1}^T \\ \phi_{3,2}^T \\ \vdots \\ \phi_{3,m+1}^T \end{pmatrix}.$$

Also, for approximating $\int_0^{r(t)} k_4(s,t)y(h(s))ds$ by the help of SCPs, we have:

$$\int_0^{r(t)} k_4(s,t)y(h(s))ds \approx b^T A_m H A_m \mathbf{M}_3 A_m^T \mathbf{K}_4 T^{*,m}(t). \tag{36}$$

Let $V^4 = [v_1^4, v_2^4, \ldots, v_{m+1}^4]^T = \mathbf{M}_3 A_m^T \mathbf{K}_4 T^{*,m}(t)$. Clearly, we conclude that $v_j^4 = \phi_{4,j}^T T^{*,m}(t),\ j = 1,2,\ldots,$ $m+1$, where $\phi_{4,j}$ is a $(m+1)-$vector. So,

$$\int_0^{r(t)} k_4(s,t)y(h(s))ds \approx b^T A_m H A_m \Phi^4 T^{*,m}(t), \tag{37}$$

where

$$\Phi^4 = \begin{pmatrix} \phi_{4,1}^T \\ \phi_{4,2}^T \\ \vdots \\ \phi_{4,m+1}^T \end{pmatrix}.$$

To present the approximation of $w(t)y(h(t))$ in the matrix form, we have:

$$w(t)y(h(t)) \approx b^T A_m H T^{*,m}(t)(T^{*,m}(t))^T W = b^T A_m H \widetilde{W} T^{*,m}(t), \tag{38}$$

where $\widetilde{W} = A_m\Delta$. Also, to approximate $p(t)G(y(t)) = p(t)y^n(t)$, we act as the follows:

$$p(t)y^n(t) \approx b^T\widetilde{b}^{n-1}T^{*,m}(t)(T^{*,m}(t))^T P = b^T\widetilde{b}^{n-1}\widetilde{P}T^{*,m}(t), \tag{39}$$

where $\widetilde{P} = A_m\Delta$. By substituting Eqs. (31), (33), (35), and (37)-(39) in Eq. (1), we get the following nonlinear system

$$Y^T - b^T(\lambda_1\widetilde{b}^{l-1}A_m\Phi^1 + \lambda_2 A_m H A_m\Phi^2 + \lambda_3 A_m\Phi^3 + \lambda_4 A_m H A_m\Phi^4 + A_m H\widetilde{W} + \widetilde{b}^{n-1}\widetilde{P}) = F^T, \tag{40}$$

where $b^T = Y^T\prod^\gamma + Y_0^T$. After solving this system by known methods such as Newton iteration method, we can present the approximate solution of (1) as the follows: $y(t) \approx b^T T^{*,m}(t)$. Also the Mathematica function NSolve is available to deal with such a nonlinear system of algebraic equations.

5 Convergence analysis

In this section, we prove the convergence analysis for the following fractional integro-differential equation

$$D^\gamma y(t) = \lambda_1\int_0^t \frac{k_1(s,t)F(y(s))}{(t-s)^\mu}ds + \lambda_2\int_0^t \frac{k_2(s,t)y(h(s))}{(t-s)^\nu}ds + \lambda_3\int_0^{r(t)} k_3(s,t)y(s)ds$$
$$+ \lambda_4\int_0^{r(t)} k_4(s,t)y(h(s))ds + w(t)y(h(t)) + p(t)G(y(t)) + f(t), \quad t\in[0,1].$$

Also, for $f\in C[a,b]$, we apply norm $\|f\|_\infty = \max\limits_{x\in[a,b]}|f(x)|$.

Theorem 1. *Let $y(t)$ and $y_m(t)$ are the exact and the approximation solution of Eq. (36) in terms of FEFs. Suppose that $y(t)$, $p(t)$ and $k_1(t,s)$, $(t,s)\in[0,1]\times[0,1]$ are continuous functions, and also there exist positive constants $L_1, L_2, L_3 \in \mathbb{R}$ such that the following conditions hold*

1. $\|F(y(t)) - F(y_m(t))\| \le L_1\|y(t) - y_m(t)\|$,
2. $\|G(y(t)) - G(y_m(t))\| \le L_2\|y(t) - y_m(t)\|$,
3. $\|y(h(t))) - y_m(h(t))\| \le L_3\|y(t) - y_m(t)\|$,
4. $\left(|\lambda_1|M_1\frac{\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} + |\lambda_2|L_3M_2\frac{\Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} + (|\lambda_3|M_3 + |\lambda_4|L_3M_4 + L_3\|w\|_\infty + L_2\parallel p\parallel_\infty)\frac{1}{\Gamma(\gamma+1)}\right) < 1.$

Then $y_m(t) \longrightarrow y(t)$, $m\longrightarrow\infty$, where $M_1 = \|k_1\|_\infty$, $M_2 = \|k_2\|_\infty$, $M_3 = \|k_3\|_\infty$ and $M_4 = \|k_4\|_\infty$.

Proof. By using (20), we have:

$$y(t) = \sum_{m=0}^{n-1}\frac{y^{(m)}(0^+)}{m!}t^m + \lambda_1 I^\gamma\int_0^t \frac{k_1(s,t)F(y(s))}{(t-s)^\mu}ds + \lambda_2 I^\gamma\int_0^t \frac{k_2(s,t)y(h(s))}{(t-s)^\nu}ds$$
$$+ \lambda_3 I^\gamma\int_0^{r(t)} k_3(s,t)y(s)ds + I^\gamma p(t)y(t) + \lambda_4 I^\gamma\int_0^{r(t)} k_4(s,t)y(h(s))ds$$
$$+ I^\gamma w(t)y(h(t)) + I^\gamma p(t)G(y(t)) + I^\gamma f(t). \tag{41}$$

Now, suppose that $y_m(t) = Y^T T^{*,m}(t)$ is the approximation solution of Eqs. (1)-(2) in terms of the SCPs, where $Y = [y_0, y_1, \ldots, y_m]^T$. By substituting $y_m(t)$ instead of $y(t)$ in Eq. (41), we get

$$\begin{aligned} y_m(t) = &\sum_{m=0}^{n-1} \frac{y^{(m)}(0^+)}{m!} t^m + \lambda_1 I^\gamma \int_0^t \frac{k_1(s,t)F(y_m(s))}{(t-s)^\mu} ds + \lambda_2 I^\gamma \int_0^t \frac{k_2(s,t) y_m(h(s))}{(t-s)^\nu} ds \\ &+ \lambda_3 I^\gamma \int_0^{r(t)} k_3(s,t) y_m(s) ds + I^\gamma p(t) y_m(t) + \lambda_4 I^\gamma \int_0^{r(t)} k_4(s,t) y_m(h(s)) ds \\ &+ I^\gamma w(t) y_m(h(t)) + I^\gamma p(t) G(y_m(t)) + I^\gamma f(t). \end{aligned} \tag{42}$$

From Eqs. (41)-(42), we can write

$$\begin{aligned} y(t) - y_m(t) = &\lambda_1 I^\gamma \int_0^t \frac{k_1(s,t)(F(y(s)) - F(y_m(s)))}{(t-s)^\mu} ds + \lambda_2 I^\gamma \int_0^t \frac{k_2(s,t)(y(h(s)) - y_m(h(s)))}{(t-s)^\nu} ds \\ &+ \lambda_3 I^\gamma \int_0^{r(t)} k_3(s,t)(y(s) - y_m(s)) ds + \lambda_4 I^\gamma \int_0^{r(t)} k_4(s,t)(y(h(s)) - y_m(h(s))) ds \\ &+ I^\gamma w(t)(y(h(t)) - y_m(h(t))) + I^\gamma p(t)(G(y(t)) - G(y_m(t))), \end{aligned}$$

and also

$$\begin{aligned} |y(t) - y_m(t)| \le &|\lambda_1|.|I^\gamma \int_0^t \frac{k_1(s,t)(F(y(s)) - F(y_m(s)))}{(t-s)^\mu} ds| \\ &+ |\lambda_2|.|I^\gamma \int_0^t \frac{k_2(s,t)(y(h(s)) - y_m(h(s)))}{(t-s)^\nu} ds| \\ &+ |\lambda_3|.|I^\gamma \int_0^{r(t)} k_3(s,t)(y(s) - y_m(s)) ds| \\ &+ |\lambda_4|.|I^\gamma \int_0^{r(t)} k_4(s,t)(y(h(s)) - y_m(h(s))) ds| + |I^\gamma w(t)(y(h(t)) - y_m(h(t)))| \\ &+ |I^\gamma p(t)(G(y(t)) - G(y_m(t)))| \\ \le &|\lambda_1| I^\gamma \int_0^t \frac{|k_1(s,t)|.|F(y(s)) - F(y_m(s))|}{(t-s)^\mu} ds \\ &+ |\lambda_2| I^\gamma \int_0^t \frac{|k_2(s,t)|.|(y(h(s)) - y_m(h(s)))|}{(t-s)^\nu} ds \\ &+ |\lambda_3| I^\gamma \int_0^{r(t)} |k_3(s,t)|.|(y(s) - y_m(s))| ds \\ &+ |\lambda_4| I^\gamma \int_0^{r(t)} |k_4(s,t)|.|(y(h(s)) - y_m(h(s)))| ds \\ &+ I^\gamma |w(t)|.|(y(h(t)) - y_m(h(t)))| + I^\gamma |p(t)|.|(G(y(t)) - G(y_m(t)))||. \end{aligned}$$

By using conditions of theorem, we conclude that

$$\begin{aligned}
|y(t)-y_m(t)| \leq{} & |\lambda_1|L_1 I^\gamma \int_0^t \frac{\|k_1\|_\infty \|y-y_m\|_\infty}{(t-s)^\mu} ds + |\lambda_2| L_3 I^\gamma \int_0^t \frac{\|k_2\|_\infty \|y-y_m\|_\infty}{(t-s)^\nu} ds \\
& + |\lambda_3| I^\gamma \int_0^{r(t)} ||k_3||_\infty ||y-y_m||_\infty ds + |\lambda_4| L_3 I^\gamma \int_0^{r(t)} ||k_4||_\infty ||y-y_m||_\infty ds \\
& + I^\gamma ||w||_\infty L_3 ||y-y_m||_\infty + I^\gamma \parallel p \parallel_\infty L_2 \parallel y-y_m \parallel_\infty \\
\leq{} & \|y-y_m\|_\infty (|\lambda_1| L_1 M_1 I^\gamma \int_0^t \frac{1}{(t-s)^\mu} ds + |\lambda_2| L_3 M_2 I^\gamma \int_0^t \frac{1}{(t-s)^\nu} ds \\
& + |\lambda_3| M_3 I^\gamma \int_0^{r(t)} ds + |\lambda_4| L_3 M_4 I^\gamma \int_0^{r(t)} ds + I^\gamma L_3 \|w\|_\infty + \parallel p \parallel_\infty L_2 I^\gamma 1) \\
={} & \|y-y_m\|_\infty (\Gamma(1-\mu)|\lambda_1| M_1 I^\gamma I^{1-\mu} + |\lambda_2| \Gamma(1-\nu) L_3 M_2 I^\gamma I^{1-\nu} + |\lambda_3| M_3 I^\gamma r(t) \\
& + |\lambda_4| L_3 M_4 I^\gamma r(t) + L_3 \|w\|_\infty I^\gamma 1 + L_2 \parallel p \parallel_\infty I^\gamma 1) \\
\leq{} & \|y-y_m\|_\infty (\Gamma(1-\mu)|\lambda_1| M_1 I^{\gamma+1-\mu} + |\lambda_2| \Gamma(1-\nu) L_3 M_2 I^{\gamma+1-\nu} \\
& + (|\lambda_3| M_3 + |\lambda_4| L_3 M_4 + L_3 \|w\|_\infty + L_2 \parallel p \parallel_\infty) I^\gamma 1) \\
={} & \|y-y_m\|_\infty (|\lambda_1| M_1 \frac{\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} t^{\gamma+1-\mu} + |\lambda_2| L_3 M_2 \frac{\Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} t^{\gamma+1-\nu} \\
& + (|\lambda_3| M_3 + |\lambda_4| L_3 M_4 + L_3 \|w\|_\infty + L_2 \parallel p \parallel_\infty) \frac{1}{\Gamma(\gamma+1)} t^\gamma) \\
\leq{} & \|y-y_m\|_\infty (|\lambda_1| M_1 \frac{\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} + |\lambda_2| L_3 M_2 \frac{\Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} \\
& + (|\lambda_3| M_3 + |\lambda_4| L_3 M_4 + L_3 \|w\|_\infty + L_2 \parallel p \parallel_\infty) \frac{1}{\Gamma(\gamma+1)}).
\end{aligned}$$

Therefore, we obtain the following inequality

$$\begin{aligned}
\|y-y_m\|_\infty \leq \|y-y_m\|_\infty (&|\lambda_1| M_1 \frac{\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} + |\lambda_2| L_3 M_2 \frac{\Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} \\
& + (|\lambda_3| M_3 + |\lambda_4| L_3 M_4 + L_3 \|w\|_\infty + L_2 \parallel p \parallel_\infty) \frac{1}{\Gamma(\gamma+1)}),
\end{aligned}$$

or

$$\begin{aligned}
(1-(|\lambda_1| M_1 \frac{\Gamma(1-\mu)}{\Gamma(\gamma-\mu+2)} + |\lambda_2| L_3 M_2 \frac{\Gamma(1-\nu)}{\Gamma(\gamma-\nu+2)} + (|\lambda_3| M_3 + |\lambda_4| L_3 M_4 \\
+ L_3 \|w\|_\infty + L_2 \parallel p \parallel_\infty) \frac{1}{\Gamma(\gamma+1)})) \|y-y_m\|_\infty \leq 0. \qquad (43)
\end{aligned}$$

By using Condition 4, we conclude that $\lim_{m\to\infty} y_m(t) = y(t)$. □

6 Numerical examples

In this section, some numerical examples are presented to demonstrate the accuracy and the applicability of the proposed method.

Table 1: Absolute errors in Example 1.

	$m=2$	$m=4$	$m=8$
t	$\lvert y(t)-y_m(t)\rvert$		
0.1	1.7479×10^{-3}	1.9660×10^{-4}	7.7083×10^{-6}
0.2	3.3353×10^{-3}	3.1090×10^{-4}	1.2432×10^{-5}
0.3	4.0676×10^{-3}	4.0094×10^{-4}	2.8633×10^{-5}
0.4	3.9449×10^{-3}	4.6192×10^{-4}	2.2303×10^{-5}
0.5	2.9671×10^{-3}	5.4669×10^{-4}	2.3527×10^{-5}
0.6	1.1342×10^{-3}	7.6566×10^{-4}	5.5665×10^{-5}
0.7	1.5536×10^{-3}	1.2869×10^{-3}	9.4956×10^{-5}
0.8	5.0966×10^{-3}	2.3361×10^{-3}	1.2738×10^{-4}
0.9	9.4946×10^{-3}	4.1965×10^{-3}	2.2419×10^{-4}
1	1.4748×10^{-2}	7.2091×10^{-3}	4.1212×10^{-4}

Table 2: CPU time (in seconds) on Example 1 for different values of m.

m	CPU time (in seconds)
2	0.125
4	2.4844
8	0.2344

Example 1. Consider the following nonlinear WSFDIDEs

$$D^{\frac{1}{2}}y(t)=\int_0^t \frac{y^2(x)}{(t-x)^{\frac{1}{2}}}dx+\int_0^t y^2(x)dx+y(t)+y(\frac{t}{2})+f(t),\ \ t\in[0,1],$$

with the exact solution $y(t)=t^2$, $y(0)=0$, and also

$$f(t)=\frac{8}{3\Gamma(0.5)}t^{\frac{3}{2}}-t^2-\frac{t^5}{5}-\frac{t^2}{4}-\Gamma(0.5)\frac{\Gamma(5)}{\Gamma(5.5)}t^{4.5}.$$

By the proposed method, we obtain numerical solution in terms of SCPs. In Table 1, we show the absolute errors for $m=2,4,8$. To compare CPU time (in seconds) for $m=2,4,8$, one can refer to Table 2.

Example 2. Consider the following fractional integro-differential equation with singular kernel [4]

$$D^{\frac{1}{4}}y(t)=\frac{1}{2}\int_0^t \frac{y(x)}{(t-x)^{\frac{1}{2}}}dx+\frac{1}{3}\int_0^1 (t-x)y(x)dx+g(t),\ \ t\in[0,1],$$

with the exact solution $y(t)=t^2+t^3$ and $y(0)=0$. The maximum absolute errors (MAEs) for different values of m are compared in Table 3. Also, in Table 4, MAEs of the proposed method and the methods of [4, 13] are compared for $m=6,12$.

Table 3: Comparing the MAEs for $m = 2, 4, 6$ in Example 2.

m	MAEs
2	4.1712×10^{-2}
4	5.4125×10^{-5}
6	1.4603×10^{-5}
9	4.1754×10^{-6}

Table 4: Comparing the MAEs for $m = 6, 12$ in Example 2.

Method	$m = 6$	$m = 12$
Jacobi collocation method [4] ($\alpha = -1/2, \beta = 1$)	2.4623×10^{-4}	2.7321×10^{-5}
BPFs collocation method [13]	1.5300×10^{-1}	7.2500×10^{-2}
The proposed method	1.5000×10^{-5}	4×10^{-6}

Example 3. In this example, we present the following nonlinear delay integro-differential equation

$$D^{\nu} y(t) = y(\frac{t}{4}) + \frac{1}{2} y(t) + y^2(t) + \int_0^t y^2(x)dx + \int_0^t e^{x+t} y(x)dx + \int_0^{\frac{1}{4}t} xy(x)dx + g(t), \quad t \in [0, 1],$$

where

$$g(t) = \frac{1}{2} - \frac{1}{4} t e^{\frac{1}{4}t} + \frac{t^2}{32} - \frac{1}{2} e^{3t} + e^{2t} - (e^t - 1)^2 - \frac{1}{2} e^{2t} + \frac{1}{2} - t + 2e^t + 2.$$

The exact solution of this example for $\nu = 1$ is $y(t) = e^t - 1$.

To show the efficiency and the accuracy of the proposed method, MAEs are presented in Table 5 for $m = 2, 4, 6$. Obviously, the MAEs decrease as m increases. Also, comparing the graphs of absolute errors of this example for $m = 2, 4, 6$ are shown in Figure 1. Comparing CPU time (in seconds) for $m = 2, 4, 6$ is done in Table 7.

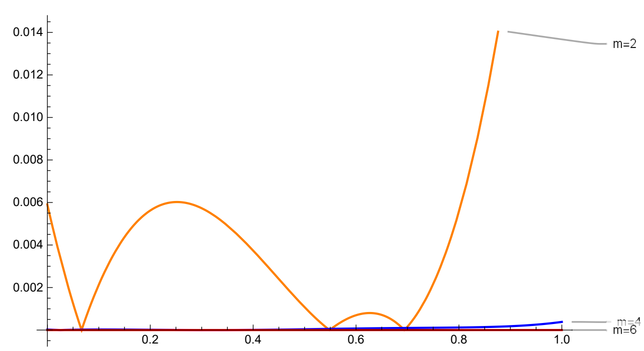

Figure 1: Comparing the graphs of absolute errors in Example 3.

Table 5: Comparing the MAEs for $m = 2, 4, 6$ in Example 3.

m	MAEs
2	3.8709×10^{-2}
4	3.8317×10^{-4}
6	9.7119×10^{-7}

Table 6: Comparing the MAEs of the proposed method and the given methods in [18] and [24] for Example 4.

	The method of [24]		The method of [18]		Our method
N	MAEs	m	MAEs	m	MAEs
10	2.2328×10^{-4}	4	5.1681×10^{-5}	4	5.4496×10^{-5}
20	4.7215×10^{-6}	5	2.6343×10^{-6}	5	4.0071×10^{-7}
60	2.2096×10^{-8}	6	1.9487×10^{-6}	6	4.6035×10^{-8}

Example 4. [8, 18] Consider the following pantograph type Volterra integro-differential equation

$$D^{\nu} y(t) = y(\frac{1}{4}t) + \frac{1}{2}y(t) + \int_0^t e^{x+t} y(x)dx + \int_0^{\frac{1}{4}t} xu(x)dx$$
$$+\frac{1}{2} - \frac{1}{4}te^{\frac{1}{4}t} + \frac{t^2}{32} - \frac{1}{2}e^{3t} + e^{2t}, \ \ y(0) = 0, \ \ t \in [0,1].$$

The exact solution of this equation when $\nu = 1$ is $y(t) = e^t - 1$ [8]. To compare the MAEs of the numerical results with the results of [18] and [24], one can refer to Table 6. In [24], by using $2N+1$ Sinc gride points, the approximate solution of this example obtained by solving $(2N+1)\times(2N+1)$ linear system of algebraic equations. Also, in [18], m denotes the degree of Euler polynomial.

Example 5. Consider the following nonlinear delay fractional integro-differential equation with singular kernel

$$D^{\nu} y(t) = \int_0^t \frac{y^2(s)}{(t-s)^{\frac{1}{2}}} ds + \int_0^t \frac{sy(\frac{1}{2}s)}{(t-s)^{\frac{1}{3}}} ds + y(\frac{1}{2}t) + g(t), \ t \in [0,1],$$

where

$$g(t) = 2t - \Gamma(\frac{1}{2})\frac{24}{\Gamma(\frac{10}{2})} t^{\frac{9}{2}} - \frac{243}{1760} t^{\frac{11}{3}} - \frac{1}{4}t^2.$$

The exact solution of this example for $\nu = 1$ is $y(t) = t^2$. By using the proposed method, we got the $MAE = 6.3 \times 10^{-13}$ for $m = 2$. In Figure 2, we presented the graphs of numerical solutions of this example for different values of ν by the help of proposed method for $m = 2$. Also, comparing CPU time (in seconds) for $m = 2, 3$ is done in Table 8. It was worth to mention when $\nu \to 1$, the numerical solutions converge to the exact solution.

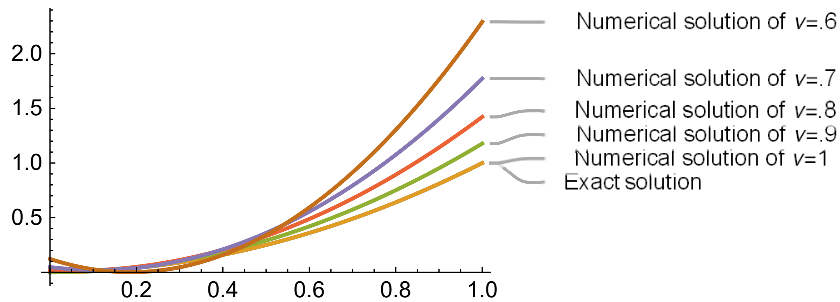

Figure 2: Numerical solutions for different values of ν and $m = 2$.

Table 7: CPU time (in seconds) on Example 3 for different values of m.

m	CPU time (in seconds)
2	0.1093
4	1.8761
6	63.4386

Table 8: CPU time (in seconds) on Example 5 for different values of m.

m	CPU time (in seconds)
2	0.0625
3	1.4848

7 Conclusion

In this paper, an operational method has been proposed to obtain a numerical solution for WSFDIDEs. We have applied operational matrices based on the shifted Chebyshev polynomials. Also, error analysis is investigated and accuracy is shown in 5 numerical experiments. The convergence of the proposed method is confirmed by the results presented in Tables 1, 3, 5, and Figure 1. In Example 2, the accuracy of the obtained results for different values of m are better than those obtained in [4] and [13]. If we compare the results obtained in Example 4 for $m = 4,5,6$ with the results obtained in [18] and [24] for $N = 10,20,60$ and $m = 4,5,6$, respectively, we see that the numerical results of the proposed method are better than the numerical results of these two methods. In Example 5, we presented a nonlinear delay fractional integro-differential equation with a singular kernel. Obviously, we do not have the exact solution for this example for $\nu \neq 1$. But Figure 2 shows the ability to use the proposed method for solving this type of problem. For the next work, the proposed method can be used to solve WSFDIDEs in high dimensions.

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