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Application of Green's function and Sinc approximation in the numerical solution of the fractional differential equations

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Abstract. The primary purpose of this paper is the construction of the Green's function and Sinc approximation for a class of Caputo fractional boundary value problems (CFBVPs). By using the inverse derivative of the fractional order, we can derive the equivalent fractional order Volterra integral equations from CFBVPs, which is considered Green's function. It is approximated by the Sinc-Collocation method. A convergence analysis of the presented method is given. Our approach is applied to five examples. We derive that our approach converges to the exact solution rapidly with the order of exponential accuracy.

Keywords: Volterra integral equations, Sinc-Collocation method, Green's function, fractional boundary value problems, fractional integrals.
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1 Introduction

We assume a class of Caputo fractional boundary value problems (CFBVPs) as follows:

$$^{C}D^{\alpha}y(x)+\lambda y(x)=f(x), \qquad 0<x<1, \quad n-1<\alpha<n, \quad n\in\mathbb{N}, \tag{1}$$

subjected to the boundary conditions:

$$y^{(k)}(0)=b_k, \quad y(1)=b, \qquad k=0,1,2,\dots,n-1, \tag{2}$$

where y and f are continuous and bounded in the interval $(0,1)$, and $^{C}D^{\alpha}y$ is an operator of the left Caputo fractional derivative of order α of $y(x)$. By using fractional inverse derivatives from CFBVPs

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(1), we have an equivalent integral equation:

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} (f(t) - \lambda y(t)) dt. \tag{3}$$

By replacing the boundary conditions (2) in Eq.(3), we obtain a system of n equations and n unknowns. Then by solving The resulting system, the unknown coefficients $c_0, c_1, \ldots, c_n$ are obtained. In general, we show that the solution of the integral equation (3) with the boundary conditions (2) is considered as follows:

$$y(x) = \int_0^1 G(x,t) f(t) dt, \tag{4}$$

where $G(x,t)$ is a Green's function. Fractional differential equations play an important role in functional and nonlinear analysis. For different systems, such as the earthquake engineering, biomedical and electrochemistry engineering are modeled by applying fractional differential equations (see some references at the end, e.g., [2, 7, 8, 14]). There are many different numerical solution techniques for approximate solutions of CFBVPs, such as the Variational iteration method [9, 11], Cubic spline method [22], Adomian decomposition method [5, 20], Fractional differential transform method [10], Haar wavelet method [17].

In this paper, we apply a new direction for approximating the nonhomogeneous CFBVPs (1),(2) which can be reduced to a Volterra-Fredholm integral equation with the help of Greens function. These equations are handled by using Sinc approximations. We use the Sinc-Collocation method which introduced by Stenger [19] and Rashidinia [3]. In our approach, the convergence accuracy of the solution is $O(e^{-\zeta\sqrt{N}})$, where $\zeta > 0$, and also converges at an optimal rate. Because the singularity on the boundary of approximation is ignored.

This paper is organized as follows. In Section 2, we give some preliminaries and notations of the Sinc function and Green's function. In Section 3, the Sinc-Collocation method is used for the numerical solution of the Green's function corresponding to the CFBVPs (1)-(2). The convergence of the method is considered in Section 4. In Section 5, we present five examples for comparing the exact and approximate solutions with graphics and tables. Finally, the conclusion is given in Section 6.

2 Preliminaries and notations

In this section, we recall definitions and notations of the left Caputo fractional, Green's function, and the Sinc function, which is useful for this paper (see [6, 18, 21]).

Definition 1. *[6]* Let $y : [a,b] \to \mathbb{R}$ be a function, $\alpha > 0$, and real number, $n \in \mathbb{Z}$ satisfying $n = [\alpha]$, and Gamma function is denoted by Γ. Then ${}^C D_x^\alpha y(x)$ is defined as follows:

$${}^C D_x^\alpha y(x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt = I_x^{n-\alpha}(\frac{d^n}{dx^n} y(x)), & n-1 < \alpha < n, \\ \frac{d^n}{dx^n} y(x), & \alpha = n. \end{cases} \tag{5}$$

Lemma 1. *[21] Let $\alpha > 0$ and $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. If $y(x) \in C[0,1]$, then the homogeneous fractional differential equation ${}^C D^\alpha y(x) = 0$, has a solution of the form*

$$y(x) = \sum_{i=0}^{n} c_i x^i = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n, \tag{6}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$.

Proof. See ([21], Lemma 1-3). □

Lemma 2. *For any* $f \in C[0,1]$, *with* $\int_0^x (x-t)^{\alpha-1} f(t)dt < +\infty$, *the following Caputo fractional differential equation*

$$ {}^C D^\alpha y(x) = f(x), \qquad 0 < x < 1, \quad n-1 < \alpha < n, n \in \mathbb{N}, \tag{7}$$

with the boundary conditions (2) has a unique solution based on Green's function as follows:

$$y(x) = \int_0^1 G(x,t) f(t)dt.$$

Proof. Let $y(x)$ be a solution of CFBVPs (7) and boundary conditions (2). By using fractional inverse derivatives from CFBVPs (7) and using Lemma 1, we have an equivalent integral equation as follows:

$$y(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt. \tag{8}$$

By replacing the boundary conditions (2) in equation (8), we obtain a system of n equations and n unknowns as follows:

$$\begin{cases} y(0) = b_0 \Longrightarrow c_0 = b_0, \\ y'(0) = b_1 \Longrightarrow c_1 = b_1, \\ \qquad\qquad \vdots \\ y^{(n-1)}(0) = b_{n-1} \Longrightarrow c_{n-1} = \frac{b_{n-1}}{(n-1)\times\cdots\times 2\times 1} \\ \\ y(1) = b, \Longrightarrow c_n = b - b_0 - b_1 - \cdots - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(t)dt. \end{cases} \tag{9}$$

By solving the above system, the unknown coefficients $c_0, c_1, \ldots, c_n$ are obtained. Generally, the solution of CFBVPs (7) and boundary conditions (2) are considered as $\int_0^1 G(x,t) f(t)dt$. □

Lemma 3. *For any* $f \in C[0,1]$, *with* $\int_0^x (x-t)^{\alpha-1} f(t)dt < +\infty$, *the following fractional differential equation*

$$ {}^C D^\alpha y(x) + \lambda y(x) = f(x), \qquad 0 < \alpha < 1, \tag{10}$$

with the boundary conditions

$$y(0) = b_0, \quad y(1) = b_1, \tag{11}$$

has a unique solution based on Green's function as follows

$$y(x) = \int_0^1 G(x,t) \left(f(t) - \lambda y(t) \right) dt, \tag{12}$$

where $G(x,t)$ *is equal to*

$$G(x,t) = \begin{cases} \frac{-x(1-t)^{\alpha-1} + (x-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le x, \\ \frac{-x(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & x \le t \le 1. \end{cases} \tag{13}$$

Proof. Let $y(x)$ be a solution of CFBVPs (10) - (11). By using fractional inverse derivatives from CFBVPs (10) and appling lemma 1, we have an equivalent integral equation as follows:

$$y(x) = c_0 + c_1 x + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}(f(t) - \lambda y(t))dt. \tag{14}$$

According to the boundary conditions (11), the unknown coefficients c_0 and c_1 are

$$\begin{cases} y(0) = b_0 \Longrightarrow c_0 = b_0, \\ y(1) = b_1 \Longrightarrow c_1 = b_1 - b_0 - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1}(f(t) - \lambda y(t))dt. \end{cases} \tag{15}$$

By replacing c_0, c_1 obtained from (15) on the right-hand side of equation (14), we have

$$\begin{aligned} y(x) = \; & b_0 + (b_1 - b_0)x - \frac{x}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1}(f(t) - \lambda y(t))dt \\ & + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}(f(t) - \lambda y(t))dt. \end{aligned}$$

To convert the above solution based on equation (12), we perform the following simplifications and by applying a variety of variables $\eta_1(x) = b_0 + (b_1 - b_0)x$ and $y_1(x) = y(x) - \eta_1(x)$, we obtain the following solution based on the Green's function for CFBVPs (10) and (11).

$$\begin{aligned} y_1(x) = \; & \frac{1}{\Gamma(\alpha)} \int_0^x \left(-x(1-t)^{\alpha-1} + (x-t)^{\alpha-1}\right)(f(t) - \lambda y(t))\, dt - \frac{x}{\Gamma(\alpha)} \int_x^1 (1-t)^{\alpha-1}(f(t) - \lambda y(t))dt \\ = \; & \int_0^1 G(x,t)\,(f(t) - \lambda y(t))\, dt. \end{aligned} \tag{16}$$

where $G(x,t)$ satisfies conditions $G(0,t) = 0$ and $G(1,t) = 0$. Therefore, the proof is completed. □

Lemma 4. *For any* $f \in C[0,1]$, *with* $\int_0^x (x-t)^{\alpha-1} f(t)dt < +\infty$, *CFBVPs (7) for* $(1 < \alpha < 2)$ *with the boundary conditions*

$$y(0) = b_0, \quad y'(0) = b_1, \quad y(1) = b_2, \tag{17}$$

has a unique solution based on the Green's function as follows

$$y(x) = \int_0^1 G(x,t) f(t) dt, \tag{18}$$

where $G(x,t)$ *is equal to*

$$G(x,t) = \begin{cases} \frac{-x^2(1-t)^{\alpha-1} + (x-t)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le x, \\ \frac{-x^2(1-t)^{\alpha-1}}{\Gamma(\alpha)}, & x \le t \le 1. \end{cases} \tag{19}$$

Proof. Let $y(x)$ be a solution of CFBVPs (7) and boundary conditions (17). By using fractional inverse derivatives from CFBVPs (7) and Lemma 1, we have an equivalent integral equation as follows

$$y(x) = c_0 + c_1 x + c_2 x^2 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt. \tag{20}$$

According to the boundary conditions (17), the unknown coefficients c_0, c_1 and c_2 are

$$\begin{cases} y(0) = b_0 \Longrightarrow c_0 = b_0, \\ y'(0) = b_1 \Longrightarrow c_1 = b_1, \\ y(1) = b_2 \Longrightarrow c_2 = b_2 - b_1 - b_0 - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(t)dt. \end{cases} \tag{21}$$

By replacing c_0, c_1, c_2 obtained from (21) on the right hand side of the equation (20), we have

$$y(x) = b_0 + b_1 x + (b_2 - b_1 - b_0)x^2 - \frac{x^2}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(t)dt + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt.$$

To convert the above solution based on integral equation (18), we perform the following simplifications and by applying variation of variables $\eta_2(x) = b_0 + b_1 x + (b_2 - b_1 - b_0)x^2$ and $y_2(x) = y(x) - \eta_2(x)$, we obtain the following solution for CFBVPs (7) and boundary conditions (17)

$$\begin{aligned} y_2(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x \left(-x^2(1-t)^{\alpha-1} + (x-t)^{\alpha-1}\right) f(t)dt - \frac{x^2}{\Gamma(\alpha)} \int_x^1 (1-t)^{\alpha-1} f(t)dt \\ &= \int_0^1 G(x,t) f(t)dt, \end{aligned} \tag{22}$$

where $G(x,t)$ satisfies the following boundary conditions

$$\begin{cases} G(0,t) = 0, \quad G_x(0,t) = 0, \\ G(1,t) = 0. \end{cases}$$

Therefore, the proof of Lemma 4 is completed. □

For approximating $y_1(x)$ and $y_2(x)$ in Eqs. (16) and (22), we need to review some properties of the Sinc function, the Sinc interpolation and the Sinc quadrature [18, 19]. The Sinc function is defined as following

$$Sinc(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0; \\ 1, & z = 0. \end{cases}$$

Definition 2. *[18] We assume that $\mathscr{D} = \{z \in \mathbb{C} : |\mathscr{I}m(z)| < d\}$ is a connected domain in the complex plane($\mathbb{C}$). We consider two different points, a and b of $\partial\mathscr{D}$(boundary of $\mathscr{D}$). In (a,b), we define a conformal map $\varphi_{a,b}(z) = \ln(\frac{z-a}{b-z})$, which has the following inverse*

$$\psi_{a,b}(z) = \varphi_{a,b}^{-1}(z) = \frac{b-a}{2} \tanh(\frac{z}{2}) + \frac{b+a}{2}. \tag{23}$$

For $\varphi_{a,b}$, $\psi_{a,b}$, we consider $z_k = \psi_{a,b}(kh)$, $k \in \mathbb{Z}$ which is the Sinc points.

Theorem 1. *[19] We assume $\mathbf{L}_\beta(\mathscr{D})$ is the set of all analytic functions, and $y \in \mathbf{L}_\beta(\mathscr{D})$ for $\beta > 0$. By considering $h = (\pi d/(\beta N))^{1/2}$, there exists a constant $k_1 > 0$, so*

$$\left| y(z) - \sum_{k=-N}^{N} y(z_k)\Theta_k \right| \leq k_1 e^{-(\pi d \beta N)^{1/2}}, \tag{24}$$

where

$$\Theta_k(\varphi(z)) = \begin{cases} \frac{1}{1+e^{\varphi(z)}} - \sum_{k=-N+1}^{N} \frac{1}{1+e^{kh}} S(k,h)\circ\varphi(z), & k=-N, \\ S(k,h)o(\varphi(z)), & k=-N+1,\ldots,N-1, \\ \frac{e^{\varphi(z)}}{1+e^{\varphi(z)}} - \sum_{k=-N}^{N-1} \frac{e^{kh}}{1+e^{kh}} S(k,h)\circ\varphi(z), & k=N, \end{cases} \tag{25}$$

in which

$$S(k,h)\circ\varphi(z) = Sinc([\varphi(z)-kh]/h), \quad k=-N,\ldots,N. \tag{26}$$

The following theorem involves bounding the error of $2N+1$ point Sinc quadrature of y on $\Lambda=[0,1]$.

Theorem 2. *[18] Consider $\frac{y}{\varphi'} \in L_\beta(\mathscr{D})$, $\beta>0$, and $0<d\leq\pi$. Let N be a positive integer, and let h be selected by the formula $h=(\pi d/(\alpha N))^{1/2}$. Then, there exists a constant, $k_2>0$, independent of N, such that*

$$\left| \int_\Lambda y(z)dz - h\sum_{k=-N}^{N} \frac{y(z_k)}{\varphi'(z_k)} \right| \leq k_2 e^{-(\pi d\beta N)^{1/2}}. \tag{27}$$

3 Sinc-Collocation method

We assume an approximate solution of $y_1(x)$ and $y_2(x)$ in the integral equations (16) and (22) by the finite expansion of Sinc basis functions as follows: [1, 12, 16, 19]

$$y_i \approx y_m(x) = c_{-N}\Theta_{-N}(\varphi(x)) + \sum_{k=-N+1}^{N-1} c_k\Theta_k(\varphi(x)) + c_N\Theta_N(\varphi(x)), \quad m=2N+1, \tag{28}$$

where $i=1,2$ and $\Theta_k(\varphi(x))$ is defined in (25). The unknown coefficients c_k in (28) are determined by the Sinc-Collocation method. We try to approximate the integrals in the integral equations (16) and (22). First, we shift interval (a,x) to the interval $(-\infty,\psi^{-1}(s))$ by changing the variable $s=\psi_{a,x}(\tau)$, [13]

$$\psi_{a,x}(\tau) = \frac{x-a}{2}\tanh(\frac{\tau}{2}) + \frac{x+a}{2}, \quad \tau\in(a,x), \tag{29}$$

which has the following derivative and inverse respectively

$$\psi'_{a,x}(\tau) = \frac{(x-a)e^\tau}{(e^\tau+1)^2}, \tag{30}$$

$$\psi^{-1}(s) = \log(\frac{s-a}{x-s}). \tag{31}$$

Now we have

$$\begin{aligned} {}_a^C I_x^{n-\alpha} y(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y(s)}{(x-s)^{1-n+\alpha}} ds \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{\psi^{-1}(x)} \frac{y(\psi_{a,x}(\tau))(\psi'_{a,x}(\tau)}{(x-\psi_{a,x}(\tau))^{1-n+\alpha}} d\tau \\ &= \frac{(x-\alpha)^{n-\alpha}}{\Gamma(n-\alpha)} \int_{-\infty}^{\psi^{-1}(x)} \frac{y(\psi_{a,x}(\tau))}{(1+e^{-\tau})(1+e^\tau)^{n-\alpha}} d\tau. \end{aligned} \tag{32}$$

The Sinc approximation on the whole real line is expressed as

$$F(x) \approx \sum_{k=-N}^{N} F(kh)S(k,h)(x). \tag{33}$$

The truncated Sinc quadrature rule define by

$$\int_{-\infty}^{\infty} F(x)dx \approx \sum_{k=-N}^{N} F(kh) \int_{-\infty}^{\infty} S(k,h)(x)dx = h \sum_{k=-N}^{N} F(kh). \tag{34}$$

By applying the Sinc quadrature rule (34), we obtain the approximation for the fractional integral equation (32) as follows

$$_aI_x^{n-\alpha} y(x) = \frac{(x-\alpha)^{n-\alpha}}{\Gamma(n-\alpha)} h \sum_{k=-N}^{N} \frac{y(\psi_{a,x}(kh))}{(1+e^{-kh})(1+e^{kh})^{n-\alpha}}. \tag{35}$$

By considering $F(x) = f(\psi_{a,b}(x))$, the approximation of $F(x)$ on the interval (a,b) is considered as follows

$$F(x) \approx \sum_{k=-N}^{N} f(\psi_{a,b}(kh))S(k,h)(\psi^{-1}(x)), \qquad x \in (a,b). \tag{36}$$

The SE transformation can be utilized for definite integration (34) as follows

$$\int_a^b F(x)dx \approx \int_{-\infty}^{\infty} f(\psi_{a,b}(\tau))(\psi'_{a,b}(\tau)d\tau \approx h \sum_{k=-N}^{N} f(\psi_{a,b}(kh))\psi'_{a,b}(kh). \tag{37}$$

On the other hand, the improper integral of the relation (34) with respect to the approximation (36) on interval (a,x) is as follows:

$$\int_a^x F(s)ds \approx \int_{-\infty}^{\psi^{-1}(x)} f(\psi_{a,x}(\tau))(\psi'_{a,x}(\tau)d\tau \approx h \sum_{k=-N}^{N} f(\psi_{a,x}(kh))(\psi'_{a,x}(kh)J(k,h)(\psi^{-1}(x)). \tag{38}$$

The basic function $J(k,h)$ is calculated by the Sine integral function $Si(x) = \int_0^x \frac{\sin(\sigma)}{\sigma} d\sigma$ as follows:

$$J(k,h)(x) = (\frac{1}{2} + \frac{1}{2} Si[\pi(\frac{x}{h} - k)]). \tag{39}$$

Now for approximating the integrals in the equations (16) and (22) respectively, we consider two cases.
Case1: Upon replacing each integrals of equation (16) with the integrals given in (37) and (38), we obtain

$$\begin{aligned} y_1(x) &+ \frac{1}{\Gamma(\alpha)} \int_0^1 x(1-t)^{\alpha-1} (f(t) - \lambda y(t))\, dt \\ &- \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\psi^{-1}(x)} \frac{f(\psi_{0,x}(\tau))\psi'_{0,x}(\tau) - \lambda y(\psi_{0,x}(\tau))}{(x-\psi_{0,x}(\tau))^{1-\alpha}} d\tau = 0. \end{aligned} \tag{40}$$

We replace $y_m(x)$ from (28) with $y_1(x)$ and set points

$$\tau = t_k = kh, t = x_k = \frac{e^{kh}}{1+e^{kh}}, \quad \text{and} \quad x_1 = \psi_{0,x}(kh) = \frac{xe^{kh}}{1+e^{kh}},$$

and step size $h = \frac{\pi}{\sqrt{N}}$. Then we get the following system for determining the unknown coefficients $c_{-N}, \ldots, c_N$

$$\begin{aligned} c_{-N}\Theta_{-N}(\varphi(x_j)) &+ \sum_{k=-N+1}^{N-1} c_k\Theta_k(\varphi(x_j)) + c_N\Theta_N(\varphi(x_j)) \\ &= -\frac{h}{\Gamma(\alpha)} \sum_{k=-N}^{N} \frac{x(f(x_k) - \lambda\Theta_k(\varphi(x))\psi'(x_k)}{(1-x_k)^{1-\alpha}}|_{x=x_j} \\ &+ \frac{hx_j^{\alpha}}{\Gamma(\alpha)} \sum_{k=-N}^{N} \frac{f[x_1]|_{x=x_j} - \lambda c_j}{(1+e^{-kh})(1+e^{kh})^{\alpha}}, \quad j = -N, -N+1, \ldots, N-1, N. \end{aligned} \tag{41}$$

In solving system (41), we apply the Newton's method with $X_{(0)} = \overrightarrow{0}$, which stops iteration whenever $\|X_{(k+1)} - X_{(k)}\| < \varepsilon$.

Case2: Upon replacing each integrals of equation (22) with the integrals given in (37) and (38), we obtain

$$y_2(x) + \frac{1}{\Gamma(\alpha)} \int_0^1 x(1-t)^{\alpha-1} f(t)dt - \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\psi^{-1}(x)} \frac{f(\psi_{0,x}(\tau))\psi'_{0,x}(\tau)}{(x-\psi_{0,x}(\tau))^{1-\alpha}} d\tau = 0. \tag{42}$$

We replace $y_m(x)$ from (28) with $y_2(x)$ and set points

$$\tau = t_k = kh, \ \ t = x_k = \frac{e^{kh}}{1+e^{kh}}, \ \text{and} \ x_1 = \psi_{0,x}(kh) = \frac{xe^{kh}}{1+e^{kh}},$$

and step size $h = \frac{\pi}{\sqrt{N}}$. Then we get the following system for determining the unknown coefficients $c_{-N}, \ldots, c_N$

$$\begin{aligned} c_{-N}\Theta_{-N}(\varphi(x_j)) &+ \sum_{k=-N+1}^{N-1} c_k\Theta_k(\varphi(x_j)) + c_N\Theta_N(\varphi(x_j)) \\ &= -\frac{h}{\Gamma(\alpha)} \sum_{k=-N}^{N} \frac{xf(x_k)}{(1-x_k)^{1-\alpha}\psi'(x_k)}|_{x=x_j} + \frac{x_j^{\alpha}h}{\Gamma(\alpha)} \sum_{k=-N}^{N} \frac{f[x_1]|_{x=x_j}}{(1+e^{-kh})(1+e^{kh})^{\alpha}}, \end{aligned} \tag{43}$$

for $j = -N, -N+1, \ldots, N-1, N$. The matrix form of the system (43) is

$$AX = [|D_{n\times 1}|I_{n\times(n-2)}|E_{n\times 1}|]X = B, \quad n = 2N+1, \tag{44}$$

where

$$\begin{aligned} D &= [\Theta_{-N}(\varphi(x_{-N})), \Theta_{-N}(\varphi(x_{-N+1})), \ldots, \Theta_{-N}(\varphi(x_N))], \\ I &= [\delta_{jk}^{(0)}], \quad j = -N, \ldots, N, \ k = -N, \ldots, N, \\ E &= [\Theta_N(\varphi(x_{-N})), \Theta_N(\varphi(x_{-N+1})), \ldots, \Theta_N(\varphi(x_{N-1})), \Theta_N(\varphi(x_N))]^T \\ X &= [c_{-N}, c_{-N+1}, \ldots, c_{N-1}, c_N]^T. \end{aligned}$$

We replace the right-hand side of the system (43) by $g(x_k)$ as follows

$$B = [g(x_{-N}), g(x_{-N+1}), \ldots, g(x_{N-1}), g(x_N)]^T.$$

In solving system (44), we apply the Newton's method with $X_{(0)} = \overrightarrow{0}$, which terminates the iteration whenever $\|X_{(k+1)} - X_{(k)}\| < \varepsilon$.

4 Error analysis

By the following theorem, we prove that equations (42) and (40) have unique solutions. Suppose the function $y(x)$ in the domain $D_d = \{x \in C : |Im(x)| < d\}$ for $d > 0$, is analytic and finite. First, we define the following definition.

Definition 3. *Let D be a simple continuous domain that holds at $(a,b) \subset D$, and suppose that K, α, and β are positive constants. In this case, the space $L_{k,\alpha,\beta}(D)$ is a family of all functions such as f that are analytic in D, and for every z in D, the following condition holds true*

$$|f(z)| \leq k \left|Q_{\alpha,\beta}(z)\right|, \tag{45}$$

where k is a positive constant and $Q_{\alpha,\beta}(z) = (z-a)^{\alpha}(b-z)^{\beta}$. In the particular case $\alpha = \beta = 1$, we denote $Q_{1,1}(z)$ by $Q(z)$.

Theorem 3. *[18, Theorem 4.2.5] Let for every $0 < d < \pi$, we have $f \in L_{k,\alpha,\beta}(\psi(D_d))$. Also, assume that $\mu = \min\{\alpha, \beta\}$ and N are a positive integers and $h = (\frac{\pi d}{\mu N})^{1/2}$. Then, there exists a number C_1 independent of N such that*

$$\sup_{t \in (a,b)} \left| f(t) - \sum_{k=-N}^{N} f(\psi_{a,b}(kh)) S(k,h)(\psi^{-1}(t) \right| \leq C_1 \sqrt{N} e^{-(\pi d \mu N)^{1/2}}. \tag{46}$$

Theorem 3 successfully reveals the fundamental convergence property.

Theorem 4. *[18] Assume that the assumptions of the previous theorem are fulfilled. In this case, the inequality (46) holds with*

$$C_1 = \frac{2k(b-a)^{\alpha+\beta}}{\mu} \left[\frac{2}{\pi d (1 - e^{-2(\pi d \mu)^{1/2}}) \{\cos(\frac{d}{2})\}^{\alpha+\beta}} + \sqrt{\frac{\mu}{\pi d}} \right]. \tag{47}$$

Note that the constant C_1 here depends only on K, α, β, d, and $(b-a)$, which are all known from the assumptions.

Theorem 5. *[18, Theorem 4.2.6] Let for every $0 < d < \pi$, we have $fQ \in L_{k,\alpha,\beta}(\psi(D_d))$. Also, assume that $\mu = \min\{\alpha, \beta\}$ and N are a positive integer and $h = (\frac{2\pi d}{\mu N})^{1/2}$. There exists a number such as C_2 independent of N such as*

$$\left| \int_a^b f(t)dt - h \sum_{k=-N}^{N} f(\psi_{a,b}(kh))(\psi'_{a,b}(kh) \right| \leq C_2 e^{-(2\pi d \mu N)^{1/2}}. \tag{48}$$

Theorem 6. *[18] Assume that the assumptions of the previous theorem are fulfilled. In this case, the inequality (48) is established by considering the fixed number C_2 as follows*

$$C_2 = \frac{2k(b-a)^{\alpha+\beta-1}}{\mu}\left[\frac{2}{(1-e^{-(2\pi d\mu)^{1/2}})\{\cos(\frac{d}{2})\}^{\alpha+\beta}}+1\right]. \tag{49}$$

Theorem 7. *[4] Let $fQ \in L_{k,\alpha,\beta}(\psi(D_d))$ for d with $0<d<\pi$. Suppose N is a positive integer and $h=(\frac{\pi d}{\mu N})^{1/2}$. In this case, there is a fixed number C_3 independent of N such that*

$$\sup_{t\in(a,b)}\left|\int_a^t f(s)ds - h\sum_{k=-N}^{N} f(\psi_{a,b}(kh))(\psi'_{a,b}(kh)J(k,h)(\psi^{-1}(t)\right| \le C_3\sqrt{N}e^{-(\pi d\mu N)^{1/2}}, \tag{50}$$

where

$$C_3 = \frac{2k(b-a)^{\alpha+\beta-1}}{\mu}\left[\frac{2}{d(1-e^{-2(\pi d\mu)^{1/2}})\{cos(\frac{d}{2})\}^{\alpha+\beta}}\sqrt{\frac{\pi d}{\mu}}+1.1\right]. \tag{51}$$

Theorem 8. *We assume that $y_m(x)$ and $y_1(x)\in L_\alpha(\mathscr{D})$ are the approximate and exact solutions of the integral equation (40), respectively. Suppose that all conditions of Theorems 3-7 are fulfilled. By considering $h=(\frac{\pi d}{\mu N})^{1/2}$ and $\frac{y}{\varphi'}\in L_\alpha(\mathscr{D})$, $(0<\alpha<1)$, there exist a constant $\zeta>0$, so*

$$\max_{0\le x\le 1}|y_1(x)-y_m(x)| \le \zeta_1 e^{-(\pi d\mu N)^{1/2}}.$$

Proof. Using relationships (46), (48), and (50), we have

$$\begin{aligned}
|e_m(x)| &= \max_{0\le x\le 1}|y_1(x)-y_m(x)| \\
&\le \max_{0\le x\le 1}\left|y_1(x)-\left(c_{-N}\Theta_{-N}(\varphi(x))+\sum_{k=-N+1}^{N-1}c_k\Theta_k(\varphi(x))+c_N\Theta_N(\varphi(x))\right)\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\left(\int_0^1 x(1-t)^{\alpha-1}f(t)dt - h\sum_{k=-N}^{N}\frac{xf(\psi_{0,1}(kh))\psi'_{0,1}(kh)}{(1-\psi_{0,1}(kh))^{1-\alpha}}\right)\right| \\
&+\left|\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\psi^{-1}(x)}\frac{f(\psi_{0,x}(\tau))\psi'_{0,x}(\tau)-\lambda y(\psi_{0,x}(\tau))}{(x-\psi_{0,x}(\tau))^{1-\alpha}}d\tau\right| \\
&+\frac{h}{\Gamma(\alpha)}\left|\sum_{k=-N}^{N}\frac{f(\psi_{0,x}(kh))\psi'_{0,x}(kh)J(k,h)\psi^{-1}(x)-\lambda y(\psi_{0,x}(kh))}{(x-\psi_{0,x}(kh))^{1-\alpha}}\right| \\
&\le C_1\sqrt{N}e^{-(\pi d\mu N)^{1/2}}+C'_2e^{-(2\pi d\mu N)^{1/2}}+C'_3\sqrt{N}e^{-(\pi d\mu N)^{1/2}}.
\end{aligned}$$

By considering $\zeta=\max\{C_1,C'_2,C'_3\}$, we have $|e_m(x)|\le\zeta_1 e^{-(\pi d\mu N)^{1/2}}$. □

Theorem 9. *Assume that $y_m(x)$ and $y_2(x)\in L_\alpha(\mathscr{D})$ are the approximate and exact solutions of the integral equation (42), respectively. Suppose that all conditions of theorems 3, 4, 5, 6 and theorem 7 are fulfilled. By considering $h=(\frac{\pi d}{\mu N})^{1/2}$ and $\frac{y}{\varphi'}\in L_\alpha(\mathscr{D})$, $(\alpha>0)$, there exist a constant $\zeta>0$, we have*

$$\max_{0\le x\le 1}|y_2(x)-y_m(x)| \le \zeta_2 e^{-(\pi d\mu N)^{1/2}}.$$

Proof. Using relationships (46), (48), and (50), we have

$$\begin{aligned}|e_m(x)| &= \max_{0\le x\le 1}|y_2(x)-y_m(x)|\\ &\le \max_{0\le x\le 1}\left|y_2(x)-\left(c_{-N}\Theta_{-N}(\varphi(x_j))+\sum_{k=-N+1}^{N-1}c_k\Theta_k(\varphi(x_j))+c_N\Theta_N(\varphi(x_j))\right)\right|\\ &+\left|-\frac{1}{\Gamma(\alpha)}\int_0^1 x(1-t)^{\alpha-1}f(t)dt-h\sum_{k=-N}^{N}f(\psi_{0,1}(kh))(\psi'_{0,1}(kh)\right|\\ &+\left|\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\psi^{-1}(x)}\frac{f(\psi_{0,x}(\tau))\psi'_{0,x}(\tau)}{(x-\psi_{0,x}(\tau))^{1-\alpha}}d\tau-h\sum_{k=-N}^{N}f(\psi_{0,1}(kh))(\psi'_{0,1}(kh)J(k,h)(\psi^{-1}(t)\right|\\ &\le C_1\sqrt{N}e^{-(\pi d\mu N)^{1/2}}+C_2''e^{-(2\pi d\mu N)^{1/2}}+C_3''\sqrt{N}e^{-(\pi d\mu N)^{1/2}}.\end{aligned}$$

By considering $\zeta_2=\max\{C_1,C_2'',C_3''\}$, we have $|e_m(x)|\le\zeta e^{-(\pi d\mu N)^{1/2}}$. □

5 Numerical results

In this section, all the experiments were performed in Mathematica 11.0. Also, to show the errors and the accuracy of the approximation, on the set of Sinc grid points

$$S=\{x_{-N},\ldots,x_N\},\ \ x_k=\frac{e^{kh}}{1+e^{kh}};\quad k=-N,\ldots,N,$$

we apply the following criteria

1. The relative error is defined by
$$E_{rel}=\left|\frac{y_m(x_k)-y(x_k)}{y(x_k)}\right|; \tag{52}$$
2. The maximum absolute the error is defined by
$$E_{abs}=\max_{-N\le k\le N}|y_m(x_k)-y(x_k)|; \tag{53}$$
3. The root mean square (RMS) error is defined for $M=2N+1$ by
$$RMS=\sqrt{\frac{1}{M}\sum_{k=-N}^{N}(y(x_k)-y_m(x_k))^2}; \tag{54}$$
4. The L_2 error norm is defined by
$$\|y_m(x_k)-y(x_k)\|_2=\sqrt{\sum_{k=-N}^{N}(y_m(x_k)-y(x_k))^2}. \tag{55}$$

In our method, we take $d = \frac{\pi}{2}$, $\beta = 1$, and different values of N, and by using $h = (\pi d/(\beta N))^{1/2}$, we can achieve h. For $\alpha \geqslant 0.5$, we consider a step size $h = \frac{\pi}{\sqrt{N}}$, and for $\alpha < 0.5$, we choose $h = \frac{\pi}{\sqrt{0.5N}}$.

Example 1. Consider the following problem with the analytical solution $y(x) = \cos(\pi x)$

$$^{C}D_{0^+}^{\alpha} y(x) = \frac{x^{\alpha-1}}{2\Gamma(\alpha)}((_1F_1(1;1-\alpha;i\pi x) + (_1F_1(1;1-\alpha;-i\pi x) - 2)$$

with the conditions $y(0) = 1$ and $y(1) = -1$ and by using (13) the Green's function is

$$G(x,t) = \begin{cases} \frac{(x-t)^{1-\alpha} - t(1-t)^{1-\alpha} x}{\Gamma(\alpha)}, & 0 \le t \le x, \\ \frac{-t(1-t)^{1-\alpha} x}{\Gamma(\alpha)}, & x \le t \le 1, \end{cases} \tag{56}$$

where $0 < \alpha < 1$ and $_1F_1(b;c;z)$ is a hypergeometric function, which is defined as follows

$$_1F_1(b;c;z) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!},$$

where

$$(q)_k = \begin{cases} 1, & k = 0, \\ q(q+1)(q+2)\cdots(q+k-1), & k > 0. \end{cases} \tag{57}$$

We define $(q)_k$, the (rising) Pochhammer symbol.

We compare relative errors of our results with the Collocation-Sinc single exponential methods (Co-Sinc (SE)), Collocation-Sinc double exponential methods (Co-Sinc (DE)), and the combination of Collocation-Sinc single and double exponential methods (Co-Sinc (SE-DE)) [15] for different values of α in Table 1. In Figure 1, we compare the approximate solution with the analytical solution for $\alpha = \frac{3}{4}$ and $N = 64$. The comparison between the absolute errors for $N = 64$ and $\alpha = \frac{3}{4}$, and $\alpha = \frac{9}{10}$ are in Figure 2. The results show that the Sinc-Collocation method based on Green's function yields relatively more accurate results than methods in [15].

Table 1: Relative errors in the solution of Example 1.

α	*Our method*	$Co-Sinc(SE)$ [15]	$Co-Sinc(DE)$ [15]	$Co-Sinc(SE-DE)$ [15]
$\frac{1}{2}$	1.207×10^{-8}	2.611×10^{-6}	1.743×10^{-5}	1.227×10^{-7}
$\frac{3}{4}$	7.262×10^{-9}	1.180×10^{-4}	1.192×10^{-5}	2.422×10^{-6}
$\frac{9}{10}$	2.002×10^{-10}	3.665×10^{-3}	8.432×10^{-6}	7.576×10^{-5}

Example 2. Consider the Caputo fractional differential equation

$$\frac{1}{x}{}^{C}D_{0^+}^{0.7} y(x) = \frac{120x^{3.3}}{\Gamma(5.7)} - \frac{24}{\Gamma(4.3)}x^{2.3},$$

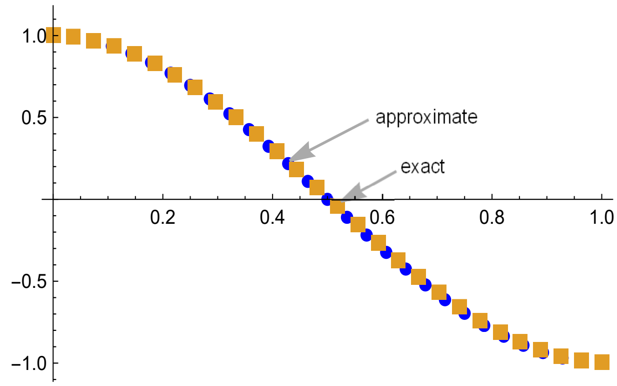

Figure 1: Comparison between the exact and approximate solutions for $N = 64$ and $\alpha = \frac{3}{4}$ in Example 1.

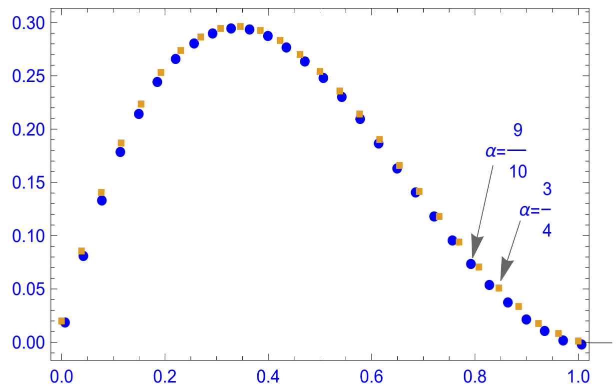

Figure 2: Comparison between the absolute errors for $N = 64$, $\alpha = \frac{3}{4}$, and $\alpha = \frac{9}{10}$ in Example 1.

with the boundary conditions $y(0) = 0$ and $y(1) = 0$, and the analytical solution $y(x) = x^4(x-1)$ and by using (13) the Green's function is

$$G(x,t) = \begin{cases} \frac{(x-t)^{-0.3} - t(1-t)^{-0.3}x}{\Gamma(0.7)}, & 0 \leq t \leq x, \\ \frac{-t(1-t)^{-0.3}x}{\Gamma(0.7)}, & x \leq t \leq 1. \end{cases} \tag{58}$$

We compare absolute errors of our method for $N = 20$ with the Sinc-Collocation and Sinc-Galerkin in [1] in Table 2 and for different values of N in Table 3. In Figure 3, the graph of the approximate and analytical solution for $N = 20$ is compared. The results show that the Sinc-Collocation method based on Green's function yields relatively more accurate results in comparison with the methods of [1].

Example 3. Consider the Caputo fractional differential equation

$$^{C}D^{\alpha}_{0+}y(x) + y(x) = 0,$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0.442$. The analytical solution is $y(x) = \sum_{k=0}^{\infty} \frac{(-x^{\alpha})^k}{\Gamma(\alpha k + 1)}$

Table 2: Comparison of the absolute errors of Example 2 for $N = 20$.

x	*our method*	*Sinc − Collocation* [1]	*Sinc − Galerkin* [1]
0.1	2.754×10^{-5}	2.470×10^{-4}	2.754×10^{-4}
0.2	2.162×10^{-5}	3.550×10^{-4}	3.777×10^{-4}
0.3	3.059×10^{-5}	3.912×10^{-4}	4.340×10^{-4}
0.4	3.168×10^{-6}	4.380×10^{-4}	4.697×10^{-4}
0.5	1.689×10^{-5}	4.013×10^{-4}	4.125×10^{-4}
0.6	2.241×10^{-5}	3.235×10^{-4}	3.380×10^{-4}
0.7	1.325×10^{-5}	3.344×10^{-4}	2.558×10^{-4}
0.8	1.684×10^{-5}	1.301×10^{-4}	1.402×10^{-4}
0.9	2.700×10^{-5}	2.215×10^{-5}	5.324×10^{-5}

Table 3: Comparison of the absolute errors in Example 2 for different values of N.

N	*our method*	*Sinc − Collocation* [1]	*Sinc − Galerkin* [1]
5	3.237×10^{-3}	6.760×10^{-3}	7.230×10^{-3}
10	2.268×10^{-4}	1.690×10^{-3}	1.519×10^{-3}
20	4.025×10^{-5}	4.118×10^{-4}	4.550×10^{-4}

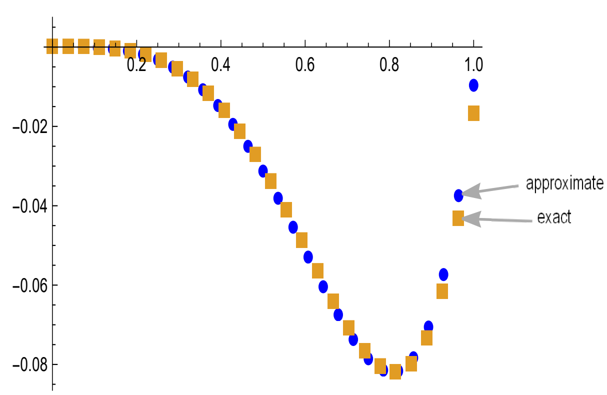

Figure 3: Comparison between the exact and approximate solutions with $N = 20$ for Example 2.

and by using (13) the Green's function is

$$G(x,t) = \begin{cases} \frac{(x-t)^{\alpha} - t(1-t)^{\alpha} x}{\Gamma(\alpha)}, & 0 \le t \le x, \\ \frac{-t(1-t)^{\alpha} x}{\Gamma(\alpha)}, & x \le t \le 1. \end{cases} \tag{59}$$

For different values of α and $N = 32$, the relative errors are in Table 4, figure 4 compares the exact and approximate solutions with $N = 32$ and $\alpha = 0.4$.

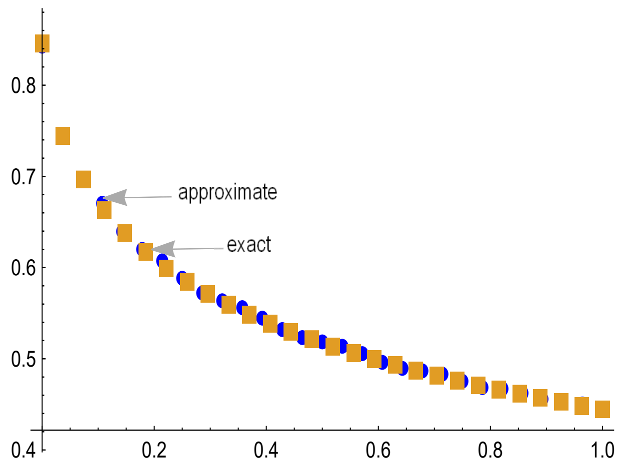

Figure 4: Comparison between the exact and approximate solutions for $N = 32$ in Example 3.

Table 4: Relative errors in the solution of Example 3 for different values of α and $N = 32$.

α	Sinc-Collocation [15]	our method
0.4	5.95×10^{-4}	2.05×10^{-4}
0.5	1.30×10^{-3}	4.04×10^{-4}
0.8	3.24×10^{-4}	2.35×10^{-4}

Example 4. We consider the following CFBVP:

$$^{C}D^{1.5}_{0^+} y(x) = \frac{2x^{0.5}}{\Gamma(1.5)} - \frac{6}{\Gamma(2.5)} x^{1.5},$$

with the boundary conditions $y(0) = 0, y'(0) = 0$, and $y(1) = 0$. The exact solution is $y(x) = x^2 - x^3$, and by using (19) the Green's function is:

$$G(x,t) = \begin{cases} \frac{(x-t)^{1-\alpha} - x^2(1-t)^{1-\alpha}}{\Gamma(\alpha)}, & 0 \le t \le x \\ \frac{-x^2(1-t)^{1-\alpha}}{\Gamma(\alpha)}, & x \le t \le 1. \end{cases} \tag{60}$$

By increasing N, we tabulated the relative errors (52) in Table 5. Table 6 lists the errors (53)-(55), and the results show that solving CFBVPs based on the Green's function have a very high accuracy. Figure

Table 5: Relative errors in the solution of Example 4.

x	$N=30$	$N=40$	$N=50$	$N=60$
0.125	3.76×10^{-5}	9.23×10^{-7}	3.99×10^{-7}	3.63×10^{-8}
0.250	3.74×10^{-6}	8.01×10^{-7}	1.53×10^{-7}	1.73×10^{-8}
0.375	2.76×10^{-6}	6.66×10^{-8}	4.01×10^{-8}	9.19×10^{-9}
0.5	1.10×10^{-11}	2.46×10^{-13}	7.44×10^{-15}	4.44×10^{-16}
0.625	1.56×10^{-6}	3.80×10^{-8}	2.30×10^{-8}	5.29×10^{-9}
0.750	1.10×10^{-6}	2.39×10^{-7}	4.61×10^{-8}	5.27×10^{-9}
0.875	4.26×10^{-6}	1.09×10^{-7}	4.76×10^{-8}	4.40×10^{-9}

Table 6: Comparison of errors in Example 4 for different values of N.

N	$h=\frac{\pi}{\sqrt{N}}$	$E_{abs}(h)$	$\|\|y(x)-y_m(x)\|\|_2$	RMS
30	0.573573	1.63×10^{-12}	3.26×10^{-12}	4.17×10^{-13}
40	0.496729	3.60×10^{-14}	7.79×10^{-14}	8.65×10^{-15}
50	0.444289	1.18×10^{-15}	2.65×10^{-15}	2.63×10^{-16}
60	0.405578	2.06×10^{-16}	7.22×10^{-16}	6.57×10^{-17}

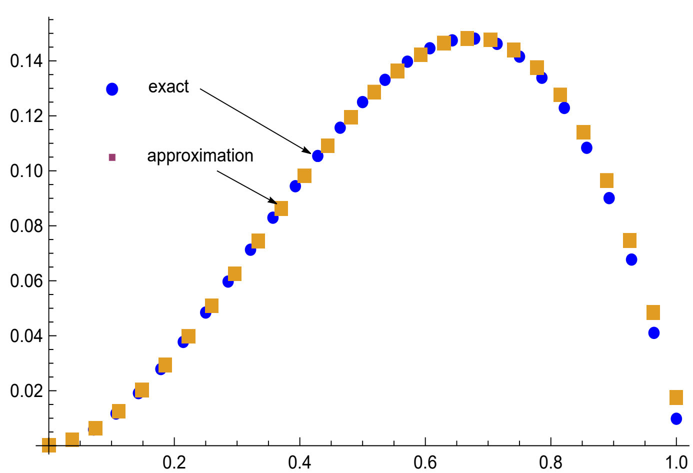

Figure 5: Comparison between the exact and approximat solution with $N=32$ for Example 4.

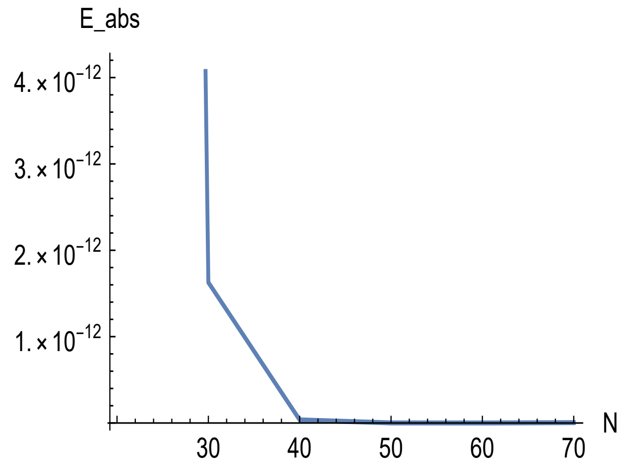

Figure 6: Absolute errors diagram in different values of N for Example 4.

5 compares the approximate and the exact solution with $N = 30$ which are coincide together. Figure 6 shows the absolute errors diagram for different values of N which decreases with increasing N.

Example 5. Consider the Caputo fractional differential equation

$$^{C}D_{0^+}^{0.5}y(x) + y(x) = \frac{1}{2}\sqrt{x} + \frac{\sqrt{\pi}}{2},$$

with the boundary conditions $y(0) = 0$ and $y(1) = 1$. The exact solution is $y(x) = \sqrt{x}$ and by using (13), the Green's function is

$$G(x,t) = \begin{cases} \dfrac{(x-t)^{-0.5} - t(1-t)^{-0.5}x}{\Gamma(0.5)}, & 0 \leq t \leq x, \\ \dfrac{-t(1-t)^{-0.5}x}{\Gamma(0.5)}, & x \leq t \leq 1. \end{cases} \tag{61}$$

For different values of N, the relative errors are in Table 7 that lists the errors (53)-(55), and the results show that solving CFBVPs based on the Greens function have a very high accuracy.

Table 7: Comparison of errors in Example 5 for different values of N.

N	$h = \frac{\pi}{\sqrt{N}}$	$E_{abs}(h)$	$\|\|y(x) - y_m(x)\|\|_2$	RMS
5	1.40496	3.43×10^{-3}	5.73×10^{-3}	1.73×10^{-3}
10	0.99346	8.79×10^{-4}	1.76×10^{-3}	3.85×10^{-4}
25	0.62832	5.42×10^{-5}	1.36×10^{-4}	1.90×10^{-5}
40	0.49672	7.05×10^{-6}	1.97×10^{-5}	2.20×10^{-6}

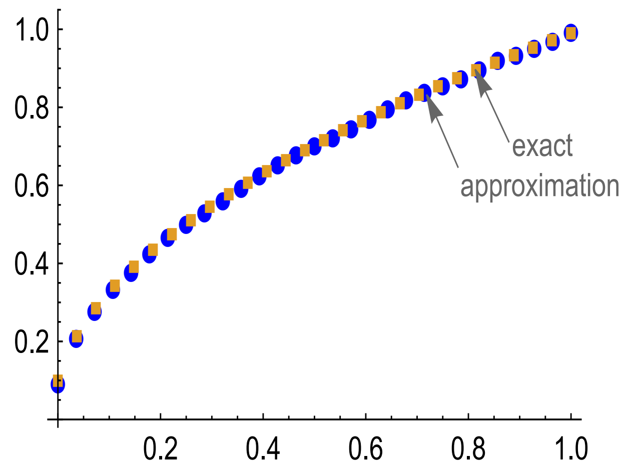

Figure 7: Comparison between the exact and approximate solutions in Example 5 for $N = 30$.

6 Conclusion

In this paper, the Sinc-Collocation method based on Green's function was applied to solve a class of nonhomogeneous CFBVPs. Numerical results indicate that by increasing N, the accuracy increases. In our approach, the convergence accuracy of the solution is $O(e^{-\zeta\sqrt{N}})$.

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