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An asymptotic computational method for the nonlinear weakly singular integral models in option pricing

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Abstract. The integral representation of the optimal exercise boundary problem for generating the continuous-time early exercise boundary for the American put option is a well-known topic in the mathematical finance community. The main focus of this paper is to provide an efficient asymptotically computational method to improve the accuracy of American put options and their optimal exercise boundary. Initially, we reformulate the nonlinear singular integral model of the early exercise premium problem given in [Kim et al., A simple iterative method for the valuation of American options, Quant. Finance. 13 (2013) 885–895] to an equivalent form which is more tractable from a numerical point of view. We then obtain the existence and uniqueness results with verifiable conditions on the functions and parameters in the resulting operator equation. The asymptotic behavior for the early exercise boundary is also analyzed which is mostly compatible with some realistic financial models.

Keywords: Non-standard Volterra integral equation, weakly singular kernel, numerical treatments, asymptotic representation, option pricing.
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1 Introduction

It is known that American option allows the holder the privilege of early exercise during the term of the derivative contract. It differs from European ones in that the holder can select to exercise at any time before the expiry date. As the optimal exercise boundary is a free boundary, its determination is combined with the computation of the option price. The valuation of the American options as well as the behavior of the early exercise boundary near expiry is computationally challenging due to the fact that in order to proceed an optimal exercise boundary must be calculated as part of the solution. Generally,

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all the methods for pricing American options, request to find out the optimal exercise boundary first in each time step. Performing this procedure is an absolute necessity, and accurate location of this optimal exercise boundary is crucial to the overall accuracy. That is why we should follow the high order numerical algorithms to derive an appropriate approximation of optimal exercise boundary.

One of the main features of integral representation for the American option problem is compromise to balance between maximizing analytical tractability and minimizing the computational time for determination of early exercise boundary. The essence of these particular approaches is to cast the boundary value problem into an integral equation, so that the analytical tractability is preserved in the form of an integral equation. This is due to the fact that, the differentiation issue is an unstable procedure while the integration process is a stable from a numerical point if view.

There is a wealth of literature concerning the integral representation for American option pricing. This issue has been extensively the subject of several work of some researches that have established that proposing this representation is more realistic. They refer to various forms of the integral equation as exact solutions, in the sense that the differential equation as well as all boundary and initial conditions have been exactly satisfied. One of the earliest results was carried out by Kim [14], who has formulated the American option valuation problem associated with the optimal exercise boundary mathematically to derive implicit-form integral equation with respect to the optimal exercise boundary containing a double integral. A very useful feature of Kim's formulation is its quantification of the value of an American option in two parts; a base value that corresponds to its European option and an early exercise premium that is associated exclusively with the early exercise right of an American option. On the other hand, one of its main drawbacks is still the relatively excessive computational time needed for the computation of the two-dimensional integrals involved in finding the unknown optimal exercise boundary. In [9] a different integral equation representation of the early exercise boundary is presented which does not involve the cumulative normal distribution function. We also refer to [1] that the linear splines were considered to solve the integral equations defining the early exercise boundary of an American option. A new integral equation formulation for American put options in form of a nonsingular one-dimensional integral associated with the optimal exercise boundary at the expiry time is proposed in [24]. More recently, the idea of determining the unknown optimal exercise boundary is applied to the pricing of an American-style down-and-out call option with debates in [18]. It is shown that by deriving an integral equation representation for the target option price, only one single nonlinear equation for the optimal exercise boundary needs to be solved numerically.

In this paper, we will focus on an integral equation approach that provides a closed-form formula for the optimal early exercise boundary of the American put option. The challenge in pricing such derivatives is that the optimal exercise policy must be determined all together with the underlying valuation problem.

The rest of the paper is organized as follows, we first review some existing results and state several important properties of our model problem where we will formulate the American option problem as a non-standard weakly singular Volterra integral equation in Section 2. The existence and uniqueness results of the solution are also discussed in this section. Section 3 is devoted to the implementation of the fully discretized spline collocation method on the suitable meshes, which safeguards both the solvability and convergence. In Section 4, the asymptotic behavior for the early exercise boundary for long time expiry is discussed which is mostly in agreement with the realistic cases. Finally, in order to illustrate the theoretical results in the paper some numerical experiments are reported and a comparison with existing results is made to show the accuracy and efficiency of the proposed method in Section 5.

2 The model problem and preceding results

An optimal boundary divides the holding region from the exercise region. The key insight follows from the intuition that the put option was not exercised at earlier dates since stock price was always above the boundary. By graphing the early exercise boundary values, investors are able to observe the shape of this critical stock price, otherwise known as early exercise boundary, and realize when an American put option can be exercised optimally.

Let B be the optimal exercise boundary and as a function on the time interval is well-defined and continuous where the underlying asset price, volatility, the interest rate and exercise price are denoted by S,σ,r and K, respectively. The representation of a new early exercise premium in the context of double integral to the case of single integral was given in Kim et al. [13] and Carr [6]. It can be shown that the optimal exercise boundary is formulated as a solution of the following integral representation (see e.g., Kim et al. [13])

$$B_\tau = \left[\mathscr{N}(d_1(B_\tau,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1(B_\tau,\tau;K)^2\right\}\right]^{-1}\left[\frac{K}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\left(r\tau+\frac{1}{2}d_2(B_\tau,\tau;K)^2\right)\right\}\right.$$
$$\left. + \frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{(\tau-\nu)}}\exp\left\{-\left(r[\tau-\nu]+\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2\right)\right\}d\nu\right], \tag{1}$$

where τ denotes time to expiry and $\mathscr{N}$ is the unit normal distribution function with

$$d_1(S,\tau;S') = \frac{\ln(\frac{S}{S'})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2(S,\tau;S') = d_1(S,\tau;S') - \sigma\sqrt{\tau}. \tag{2}$$

In order to motivate the current study, it will be useful to reformulate the integral representation (1) to an equivalent form which should be more tractable from a numerical point of view. Let us set $\lambda_\tau = \frac{K}{\sigma\sqrt{2\pi\tau}}$ and define an integral operator $\mathscr{V}: C(I)\to C(I)$ by setting

$$(\mathscr{V}B)(\tau) := (\mathscr{V}B_\tau) = \int_0^\tau (\tau-\nu)^{\frac{-1}{2}}\ G(\tau,\nu,B_\tau,B_\nu)\,d\nu, \qquad \tau\in I=[0,T],$$

where

$$G(\tau,\nu,B_\tau,B_\nu) := \left[\mathscr{N}(d_1(B_\tau,\tau;K)) + \frac{\lambda_\tau}{K}\exp\left\{-\frac{1}{2}d_1(B_\tau,\tau;K)^2\right\}\right]^{-1}$$
$$\left[r\lambda_\tau\sqrt{\tau}\exp\left\{-\left(r[\tau-\nu]+\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2\right)\right\}\right]. \tag{3}$$

We also define a supplementary term g by

$$g(\tau,B_\tau) := \left[\mathscr{N}(d_1(B_\tau,\tau;K)) + \frac{\lambda_\tau}{K}\exp\left\{-\frac{1}{2}d_1(B_\tau,\tau;K)^2\right\}\right]^{-1}\left[\lambda_\tau\exp\left\{-\left(r\tau+\frac{1}{2}d_2(B_\tau,\tau;K)^2\right)\right\}\right]. \tag{4}$$

Under the above notations, the equation (1) can be rewritten in an operator form

$$B_\tau = g(\tau,B_\tau) + (\mathscr{V}B_\tau), \qquad \tau\in I=[0,T], \tag{5}$$

where g and G are assumed to be known continuous nonlinear functions in their variables and are dependent to B at time τ, which is a rather complex nonlinear equation so called a non-standard weakly singular Volterra integral equation. (See e.g., Brunner [4, pp. 145]). This type of equations can arise in some investment models in optimal control problems and piecewise deterministic processes [5] that has raised less attention. (see e.g., [15, 21]).

There are also a few works concerned with the numerical treatment for some particular classes as well as quadratic forms of (5). Lower and upper bounds on the prices of American call and put options to provide two option price approximations were developed in [3]. The Richardson extrapolation based on the integral representation method was used in [10]. Another approximation offering the early exercise boundary as an argument to the logarithmic function in the integral was proposed in [11], where the early exercise boundary was approximated as a piecewise exponential function. A simple approximation on a dividend-paying asset by quadrature formulas was demonstrated in [12]. Ma et al. [19] has developed a high-order collocation method for the free boundary early exercise, where they set up a time-dependent artificial boundary to solve Black-Sholes equation. A product integration approach based on linear barycentric rational interpolation has also given in [20] to the nonlinear weakly singular non-standard Volterra integral equation representing the early exercise boundary of American options. Recently, a numerical method to solve non-standard Volterra integral equations of the second kind in terms of the mean-value theorem for integrals has presented in [7], that allows each Volterra integral equation to correspond to a system of non-linear equations that is solved by means of a numerical method.

Although in some cases the non-standard Volterra equations can be viewed as a quadratic form, there still exist some of them which lead to the weakly singular kernels $(\tau-\nu)^{-\alpha}, \quad (0<\alpha<1)$, with several nonlinearities in terms of the unknown functions. The numerical analysis as well as constructing high order approximation methods for such equations have generally serious difficulties due to the singularity acquired at $\nu=\tau$.

In what follows, we apply the fully discretized collocation method in the piecewise polynomials spaces to solve the underlying weakly singular integral equation (1) in order to simplify the nonlinear functions even more which do not depend on the variables. We will then pay special attention to numerical solvability of the resulting Volterra integral equation for obtaining the early exercise boundary from the exact equation.

2.1 The existence and uniqueness result

This section deals with the conditions that guarantee the existence and uniqueness of solutions of (5). A convenient setting for the analysis of (5) is the Banach space $C(I)$ with the supplementary norm

$$|||u||| = \max_{0\le t\le T} e^{-\alpha t}|u(t)|,$$

for some $\alpha \ge 0$, which is equivalent to the uniform norm $\|u\|$ on $C(I)$ over this space. (See e.g [2] pp. 142).

The following theorem which is essentially based on the process being offered in [2], describes the conditions under which the weakly singular equation (5) possesses a unique continuous solution.

Theorem 1. *Let the nonlinear functions g and G in (5) are continuous on $I\times\mathbb{R}$ and $D\times\mathbb{R}\times\mathbb{R}$ with $D=\{(\tau,\nu)|0\le \tau,\nu\le T\}$, and satisfy the following Lipschitz type conditions:*

$$|G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,B'_\tau,B'_\nu)| \le L_1\ |B_\tau-B'_\tau|+L_2\ |B_\nu-B'_\nu|, \tag{6}$$

$$|g(\tau,B_\tau)-g(\tau,B'_\tau)| \le L_3\ |B_\tau-B'_\tau|, \tag{7}$$

where the constants $L_1,L_2>0$ and $0<L_3<1$, such that $L_1<\frac{1-L_3}{2\sqrt{T}}$. Then there exists a unique solution $B\in C(I)$ of (5).

Proof. Let us define a nonlinear operator $\mathscr{T}$ differing from $\mathscr{V}$ merely by a supplementary term

$$\mathscr{T}(B)(\tau)=\mathscr{T}(B_\tau):=g(\tau,B_\tau)+(\mathscr{V}B_\tau).$$

Thus, equation (5) can be written as a compact form $B_\tau=\mathscr{T}(B_\tau)$. For the existence result, it is sufficient to show that $\mathscr{T}(B_\tau)$ restricted to $C(I)$ is a contraction operator. For any $B,B'\in C(I)$, we may write

$$\mathscr{T}(B_\tau)-\mathscr{T}(B'_\tau)=g(\tau,B_\tau)-g(\tau,B'_\tau)+\int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\ \Big(G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,B'_\tau,B'_\nu)\Big)\,d\nu.$$

Using conditions (6) and (7), it follows that

$$\begin{aligned}|\mathscr{T}B_\tau-\mathscr{T}B'_\tau| &\le |g(\tau,B_\tau)-g(\tau,B'_\tau)|+\int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\ |G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,B'_\tau,B'_\nu)|d\nu\\ &\le L_3|B_\tau-B'_\tau|+\int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\ \Big(L_1\ |B_\tau-B'_\tau|+L_2\ |B_\nu-B'_\nu|\Big)\ d\nu\\ &\le e^{\alpha\tau}e^{-\alpha\tau}L_3|B_\tau-B'_\tau|+\int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\Big(L_1\ e^{\alpha\tau}e^{-\alpha\tau}|B_\tau-B'_\tau|+L_2\ e^{\alpha\nu}e^{-\alpha\nu}|B_\nu-B'_\nu|\Big)\ d\nu\\ &\le e^{\alpha\tau}\ L_3\ |||B-B'|||+\int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\Big(L_1\ e^{\alpha\tau}|||B-B'|||+L_2\ e^{\alpha\nu}|||B-B'|||\Big)\ d\nu,\end{aligned}$$

or equivalently

$$\begin{aligned}e^{-\alpha\tau}|\mathscr{T}B_\tau-\mathscr{T}B'_\tau| &\le L_3\ |||B-B'|||+|||B-B'|||\ \int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\ e^{-\alpha\tau}\Big(L_1\ e^{\alpha\tau}+L_2\ e^{\alpha\nu}\Big)\ d\nu\\ &=L_3\ |||B-B'|||+|||B-B'|||\ \int_0^\tau(\tau-\nu)^{\frac{-1}{2}}\ \Big(L_1+L_2\ e^{\alpha(\nu-\tau)}\Big)\ d\nu\\ &=L_3\ |||B-B'|||+|||B-B'|||\ \Big(2\sqrt{\tau}L_1+\frac{\sqrt{\pi}\,Erf(\sqrt{\alpha\tau})}{\sqrt{\alpha}}L_2\Big),\end{aligned}$$

where $Erf(.)$ denotes the error functions and we have taken into account the fact that $Erf(.)\le 1$. Therefore,

$$e^{-\alpha\tau}|\mathscr{T}B_\tau-\mathscr{T}B'_\tau|\le\Big(L_3+2\sqrt{T}L_1+\frac{\sqrt{\pi}}{\sqrt{\alpha}}L_2\Big)\ |||B-B'|||,$$

and when we take the maximum for $\tau\in[0,T]$, we arrive at an estimate

$$|||\mathscr{T}B-\mathscr{T}B'|||\le\Big(L_3+2\sqrt{T}L_1+\frac{\sqrt{\pi}}{\sqrt{\alpha}}L_2\Big)|||B-B'|||. \tag{8}$$

Noting that, α can be chosen arbitrarily large such that $\frac{\sqrt{\pi}}{\sqrt{\alpha}}L_2\to 0$. Accordingly, considering the assumption $L_1<\frac{1-L_3}{2\sqrt{T}}$, yields the coefficient of the last term in (8) should be less than 1. This implies that the operator $\mathscr{T}$ is a contraction on the Banach space $(C(I),|||.|||)$ and possesses a unique fixed point $B\in C(I)$, such that $B=\mathscr{T}(B)$ and this completes the proof. □

2.2 Verification of the model

In what follows, we show that for the model problem (1) there exists constants L_1, L_2 and L_3 such that the functions g and G defined in (4) and (3) satisfying the conditions of Theorem 1.

In order to simplify, we just sketch the analysis of the approach and refrain from going into details. Let us set:

$$F(u):=\mathscr{N}(d_1(u,\tau;K))+\frac{\lambda_\tau}{K}\exp\Big\{-\frac{1}{2}d_1(u,\tau;K)^2\Big\}.$$

This simplification yields

$$\begin{aligned}&\big|G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,B'_\tau,B'_\nu)\big|\\ &\qquad\le\left|\frac{e^{-r[\tau-\nu]}r\lambda_\tau\sqrt{\tau}}{F(B_\tau)F(B'_\tau)}\right|\left|F(B'_\tau)\exp\Big\{-\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2\Big\}-F(B_\tau)\exp\Big\{-\frac{1}{2}d_2(B'_\tau,\tau-\nu;B'_\nu)^2\Big\}\right|\\ &\qquad\le C_1\left|\exp\big\{-\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2\big\}-\exp\Big\{-\frac{1}{2}d_2(B'_\tau,\tau-\nu;B'_\nu)^2\Big\}\right|,\end{aligned}$$

for a positive constant C_1 which is obtained from the boundedness of the normal distribution and exponential functions. Recalling the inequalities $e^{-a} \leq 1$, and $1-e^{a} \leq -a$, for all $a \in \mathbb{R}$, we have

$$\begin{aligned}\left|G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,\beta_\tau,B'_\nu)\right| &\leq C_1 \left|\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2-\frac{1}{2}d_2(B'_\tau,\tau-\nu;B'_\nu)^2\right| \\ &\leq \frac{C_1C_2}{2}\left|d_2(B_\tau,\tau-\nu;B_\nu)-d_2(B'_\tau,\tau-\nu;B'_\nu)\right| \\ &\leq \frac{C_1C_2}{C_3}\Big(|\ln(B_\tau)-\ln(B'_\tau)|+|\ln(B_\nu)-\ln(B'_\nu)|\Big),\end{aligned}$$

where

$$C_2=\max_{\tau,\nu\in[0,T]}\{d_2(B_\tau,\tau-\nu;B_\nu)+d_2(B'_\tau,\tau-\nu;B'_\nu)\}, \quad \text{and} \quad C_3=2\sigma\min_{\tau,\nu\in(0,T]}\{\sqrt{\tau-\nu}\}, \text{ (for } \nu<\tau),$$

are positive constants. Consequently, due to the inequality $\ln(1+u)\leq u$, for all $u>-1$, we obtain

$$\left|G(\tau,\nu,B_\tau,B_\nu)-G(\tau,\nu,B'_\tau,B'_\nu)\right| \leq \frac{C_1C_2}{C_3C_4}\Big(|B_\tau-B'_\tau|+|B_\nu-B'_\nu|\Big),$$

where $C_4=\min_{\tau,\nu\in(0,T]}\{B'_\nu,B'_\tau\}$.

In a similar manner, it follows that for the function g defined in (4), there are positive constants C_5, $C_6=\max_{\tau\in[0,T]}\{d_2(B_\tau,\tau;K)+d_2(B'_\tau,\tau;K)\}$ and $C_7=2\sigma\min_{\tau\in(0,T]}\{\sqrt{\tau}B'_\tau\}$, such that

$$\left|g(\tau,B_\tau)-g(\tau,B'_\tau)\right| \leq \frac{C_5C_6}{C_7}\left|B_\tau-B'_\tau\right|.$$

We now turn to (6) and (7). Setting $L_1=L_2=\frac{C_1C_2}{C_3C_4}$ and $L_3=\frac{C_5C_6}{C_7}$, one verifies that for sufficiently large $\alpha>0$, there exist constants L_1,L_2 and L_3 such that the Theorem holds.

3 Numerical approximation of option pricing

The main concern of this section is to analyze the piecewise collocation method for obtaining a high-order continuous solution of (1). Much of our discussion here will make use of the notations in Brunner [5].

Let $I_n:=[\tau_n,\tau_{n+1})$ for $n=0,...,N-1$ be a given uniform mesh on the interval $I=[0,T]$, where $\{\tau_j:=jh,\ j=0,...,N;\ Nh=T\}$. The solution B_τ of (1) will be approximated by the element $\hat{B}_\tau$ of the piecewise polynomial space that each component is a polynomial of degree not exceeding $m-1$.

Let X_n in each subinterval I_n, be given by $X_n:=\{\tau_{n,i}=\tau_n+c_ih: 0<c_1<\ldots<c_m\leq 1\}$. The collocation equation of (1) is written as follows

$$\begin{aligned}\hat{B}_{\tau_{n,i}}=&\left[\mathcal{N}(d_1(\hat{B}_{\tau_{n,i}},\tau_{n,i};K))+\frac{\lambda_\tau}{K}\exp\left\{-\frac{1}{2}d_1(\hat{B}_{\tau_{n,i}},\tau_{n,i};K)^2\right\}\right]^{-1}\left[\lambda_{\tau_{n,i}}\exp\left\{-\left(r\tau_{n,i}+\frac{1}{2}d_2(\hat{B}_{\tau_{n,i}},\tau_{n,i};K)^2\right)\right\}\right.\\&+r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}\,h\sum_{l=0}^{n-1}\int_0^1(\tau_{n,i}-\tau_l-\nu h)^{\frac{-1}{2}}\exp\left\{-\left(r[\tau_{n,i}-\tau_l-\nu h]+\frac{1}{2}d_2(\hat{B}_{\tau_{n,i}},\tau_{n,i}-\tau_l-\nu h;\hat{B}_\nu)^2\right)\right\}d\nu\\&\left.+r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}\,h\int_0^{c_j}(\tau_{n,i}-\tau_n-\nu h)^{\frac{-1}{2}}\exp\left\{-\left(r[\tau_{n,i}-\tau_n-\nu h]+\frac{1}{2}d_2(\hat{B}_{\tau_{n,i}},\tau_{n,i}-\tau_n-\nu h;\hat{B}_\nu)^2\right)\right\}d\nu\right].\end{aligned}$$

Therefore, the fully discrete version of (1) using m-point product quadrature formulas whose abscissas are based on the m collocation parameters c_j and weights depend on $\mathscr{V}B$, takes the form

$$\begin{aligned}\hat{B}_{\tau_{n,i}} = &\left[\mathscr{N}(d_1(\hat{B}_{n,i},\tau_{n,i};K)) + \frac{\lambda_{\tau_{n,i}}}{K}\exp\left\{-\frac{1}{2}d_1(\hat{B}_{n,i},\tau_{n,i};K)^2\right\}\right]^{-1}\left[\lambda_{\tau_{n,i}}\exp\left\{-\left(r\tau_{n,i}+\frac{1}{2}d_2(\hat{B}_{n,i},\tau_{n,i};K)^2\right)\right\}\right.\\ &+ r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}h\sum_{l=0}^{n-1}\left(\sum_{k=1}^{m} w_{n,k}^{(l)}(c_i)\,\exp\left\{-\left(r[\tau_{n,i}-\tau_{l,k}]+\frac{1}{2}d_2(\hat{B}_{\tau_{n,i}},\tau_{n,i}-\tau_{l,k};\hat{B}_{l,t_l+c_kh})^2\right)\right\}\right)\\ &\left.+ r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}\,h\sum_{k=1}^{m} w_{n,k}(c_i)\,\exp\left\{-\left(r[\tau_{n,i}-t_n-c_ic_kh]+\frac{1}{2}d_2(\hat{B}_{\tau_{n,i}},\tau_{n,i}-t_n-c_ic_kh;\hat{B}_{l,t_n+c_ic_kh})^2\right)\right\}\right],\end{aligned} \tag{9}$$

where on each subinterval I_n, we have

$$\hat{B}_{n,\tau_n+\mu h} = \sum_{j=1}^{m} L_j(\mu)\,\hat{B}_{n,j}, \qquad \mu\in[0,1], \quad \tau_n+\mu h\in I_n, \tag{10}$$

with $\hat{B}_{n,j} := \hat{B}_{\tau_n+c_jh}$, for $j=1,\dots,m$, and $L_j(\mu)$ be a Lagrange polynomial associated with the collocation parameters c_j. Eq. (9) can now be represented as

$$\begin{aligned}&\hat{B}(\tau_{n,i})\\ &= \left[\mathscr{N}(d_1(\hat{B}_{n,i},\tau_{n,i};K)) + \frac{\lambda_{\tau_{n,i}}}{K}\exp\left\{-\frac{1}{2}d_1(\hat{B}_{n,i},\tau_{n,i};K)^2\right\}\right]^{-1}\left[\lambda_{\tau_{n,i}}\exp\left\{-\left(r\tau_{n,i}+\frac{1}{2}d_2(\hat{B}_{n,i},\tau_{n,i};K)^2\right)\right\}\right.\\ &+ r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}\,h\sum_{l=0}^{n-1}\left(\sum_{k=1}^{m} w_{n,k}^{(l)}(c_i)\,\exp\left\{-\left(r[\tau_{n,i}-\tau_{l,k}]+\frac{1}{2}d_2(\hat{B}_{n,i},\tau_{n,i}-\tau_{l,k};\sum_{j=1}^{m}L_j(c_k)\hat{B}_{l,j})^2\right)\right\}\right)\\ &\left.+ r\,\lambda_{\tau_{n,i}}\sqrt{\tau_{n,i}}h\sum_{k=1}^{m} w_{n,k}(c_i)\,\exp\left\{-\left(r[\tau_{n,i}-t_n-c_ic_kh]+\frac{1}{2}d_2(\hat{B}_{n,i},\tau_{n,i}-t_n-c_ic_kh;\sum_{j=1}^{m}L_j(c_kc_j)\hat{B}_{n,j})^2\right)\right\}\right],\end{aligned} \tag{11}$$

where the weights are obtained by

$$w_{n,k}(\nu) = h^{\frac{-1}{2}}\int_0^{\nu}(\nu-s)^{\frac{-1}{2}}L_k(\frac{s}{\nu})\,d\nu, \qquad (\nu>0),$$

and

$$w_{n,k}^{(l)}(\nu) = h^{\frac{-1}{2}}\int_0^{1}(\frac{t_n+\nu h_n-t_l}{h}-\nu)^{\frac{-1}{2}}L_k(\nu)\,d\nu, \qquad (l<n).$$

Using (5), the matrix representation of (11) can be written as

$$\hat{\mathbf{B}}_n - h\hat{\mathbf{A}}_n = \hat{\mathbf{g}}_n + \hat{\mathbf{G}}_n, \qquad (n=0,\dots,N-1) \tag{12}$$

where $\hat{\mathbf{B}}_n := (\hat{B}_{n,1},\dots,\hat{B}_{n,m})^T \in \mathbb{R}^m$ and $\hat{\mathbf{g}}_n := (\hat{g}(t_{n,1},\hat{B}_{n,1}),\dots,\hat{g}(t_{n,m},\hat{B}_{n,m}))$ with

$$\hat{\mathbf{G}}_n := \sum_{l=0}^{n-1} h\,\hat{\mathbf{A}}_n^{(l)},$$

and the matrices $\hat{\mathbf{A}}_n$ and $\hat{\mathbf{A}}_n^{(l)}$ are defined by

$$\hat{\mathbf{A}}_n := \begin{pmatrix} \sum_{k=1}^{m} w_{n,k}(c_i)\; G(\tau_{n,i}, t_n + c_i c_k h, \hat{B}_{n,i}, \hat{B}_{n,t_n+c_i c_k h}) \\ (i = 1,\ldots,m) \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{k=1}^{m} w_{n,k}(c_i)\; G(\tau_{n,i}, t_n + c_i c_k h, \hat{B}_{n,i}, \sum_{j=1}^{m} L_j(c_i c_j)\hat{B}_{n,j}) \\ (i = 1,\ldots,m) \end{pmatrix}$$

and

$$\hat{\mathbf{A}}_n^{(l)} := \begin{pmatrix} \sum_{j=1}^{m} w_{n,j}^{(l)}(c_i)\; G(\tau_{n,i}, \tau_{l,j}, \hat{B}_{n,i}, \hat{B}_{l,j}) \\ (i = 1,\ldots,m) \end{pmatrix} \quad (l < n).$$

The resulting nonlinear system for $\hat{\mathbf{B}}_n$ can be solved by a suitable numerical iterative method. Finding $\hat{\mathbf{B}}_n$ from (12), we make use of (10) to obtain an analytical form of the early exercise boundary curve.

Finally, the American put option can be computed by substituting the obtained closed form solution $B_{n,j}$, $(j = 1,\ldots,m;\ n = 1,\ldots,N)$, into the following integral representation which is due to Kim et al. [13] and Carr [6]:

$$P(S,\tau) = p(S,\tau) + \int_0^\tau rKe^{-r(\tau-\nu)} \mathcal{N}(-d_2(S,\tau-\nu;B_\nu))\, d\nu,$$

where $p(S,\tau)$ represents the Black-Scholes European put pricing formula and the integral can be computed analytically or numerically by suitable quadrature formulas. Because of the convergence of the proposed collocation method is a well-known topic in the literature, we refrain from going into details of the convergence analysis and refer the interested readers to the Monograph [5].

4 The asymptotic behaviour of the solution

It is interesting to know how does the optimal exercise boundary B_τ as well as the American put option $P(S,\tau)$ behave asymptotically as infinite time to expiry. The determination of $\lim_{\tau\to\infty} B_\tau$ is related to the analysis of the price function of corresponding perpetual American option i.e. the option with infinite time to expiration.

The asymptotic behavior of the optimal exercise boundary near expiration has been examined by Kim [14] and Ma et al. [19]. Following [16], since B_τ is a monotonic decreasing function of τ, the higher bound for the optimal exercise boundary B_τ for $\tau \geq 0$ is given by $\lim_{\tau\to 0^+} B_\tau$. Note that the asymptotic solution of B_τ near expiry is comparatively easy to derive and can be found in [14].

Here, we give a slightly different approach to obtain an equivalent result for the asymptotic behavior of B_τ when $\tau \to \infty$. This asymptotic approach is based on the problem statement as it was reformulated to an integral representation which should be more tractable from a numerical point of view.

Theorem 2. *Let B_τ be the optimal exercise boundary and σ, r, K are the volatility, the interest rate and exercise price, respectively. If $r > \frac{1}{2}\sigma^2$, then the following asymptotic result holds*

$$\lim_{\tau\to\infty} B_\tau = \frac{rK}{r + \frac{1}{2}\sigma^2},$$

where τ denotes time to expiry.

Proof. We can manipulate the non-standard integral equation (1) in order to explicitly investigate the asymptotic behavior. This would simplify and give more detailed about the behavior of the optimal exercise boundary at the time to expiry.

To start, let us consider the equation (1) and set

$$B_\tau = \Bigg[\underbrace{\mathscr{N}(d_1(B_\tau,\tau;K)) + \frac{1}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\frac{1}{2}d_1(B_\tau,\tau;K)^2\right\}}_{I_1(\tau)}\Bigg]^{-1}\Bigg[\underbrace{\frac{K}{\sigma\sqrt{2\pi\tau}}\exp\left\{-\left(r\tau+\frac{1}{2}d_2(B_\tau,\tau;K)^2\right)\right\}}_{I_2(\tau)}$$

$$+\underbrace{\frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{(\tau-\nu)}}\exp\left\{-\left(r[\tau-\nu]+\frac{1}{2}d_2(B_\tau,\tau-\nu;B_\nu)^2\right)\right\}d\nu}_{I_3(\tau)}\Bigg]. \tag{13}$$

We analyze the behaviour of each term of (13) independently when $\tau\to\infty$. In view of (2), we get $\lim_{\tau\to\infty} d_1=\infty$, and for $r>\frac{1}{2}\sigma^2$, we may have $\lim_{\tau\to\infty} d_2=\infty$, and hence

$$\lim_{\tau\to\infty}\mathscr{N}(d_1(B_\tau,\tau;K))=1,$$

which yields $\lim_{\tau\to\infty} I_1(\tau)=1$ and $\lim_{\tau\to\infty} I_2(\tau)=0$.

Consequently for evaluating $\lim_{\tau\to\infty} I_3(\tau)$ by applying change of variable $u=\tau-\nu$, we obtain

$$\lim_{\tau\to\infty} I_3(\tau) = \lim_{\tau\to\infty}\frac{rK}{\sigma\sqrt{2\pi}}\int_0^\tau \frac{1}{\sqrt{u}}\exp\left\{-\left(ru+\frac{1}{2}\left(\frac{r-\frac{1}{2}\sigma^2}{\sigma}\sqrt{u}\right)^2\right)\right\}du.$$

One can easily shows that $\int_0^\infty \frac{e^{-au}}{\sqrt{u}}du=\frac{\sqrt{\pi}}{\sqrt{a}}$. We temporarily introduce $\rho=r-\frac{1}{2}\sigma^2$, and arrive at

$$\lim_{\tau\to\infty} I_3(\tau)=\frac{rK}{\sigma\sqrt{2\pi}}.\frac{\sqrt{\pi}}{\sqrt{r+\frac{1}{2}(\frac{\rho}{\sigma})^2}}=\frac{rK}{\sqrt{2r\sigma^2+\rho^2}},$$

then

$$\lim_{\tau\to\infty} B_\tau=\frac{rK}{r+\frac{1}{2}\sigma^2}.$$

This completes the proof.

□

It should be noted that, the obtained result confirms the asymptotic behavior presented in [14] and [19]. This issue will be further discussed experimentally in the next section even to some realistic models.

5 Numerical results and discussions

In this section, we provide results of some numerical experiments to illustrate the accuracy of the proposed collocation scheme and to validate the theoretical results.

We consider the cases of an American put option for a wide range of parameters. We also compare our numerical results with those obtained by previous work in some typical models . The numerical results are obtained for $m=2$ and $m=3$ and the collocation parameters are considered as the Radau II points on $(0,T]$ that are the zeros of $P_{m-1}(2x-1)-P_m(2x-1)$ where P_m is the Legendre polynomials of order m. All computations obtained by MATLAB® code. Our aim is to collect a variety of test problems with different viewpoints to show the efficiency of the proposed method.

Example 1. (From [13]) As a first numerical test, let us consider the equation (5) and set the parameters for the baseline case ($K = 45,\ \sigma = 0.2,\ T = 1$) with the risk-free interest rate $r = 0.05$, volatility $\sigma = 0.15$, expiration time $T = 3$ and also strike $K = 47$, when the number of nodes N doubles. Of particular interest is the approximation of the optimal exercise boundary which is the most complicated part in pricing American options.

Table 1 gives a comparison between the obtained optimal exercise boundary B_τ using the proposed spline collocation method for $m = 2$ with those obtained in [13] with various number of subintervals N.

Table 1: A comparison between the optimal exercise boundary B_τ and the results of [13] for different parameters K, σ and T.

Parameters	N	B_τ	B_τ
		Present method	Method of [13]
Baseline case	4	36.3917	36.3704
$K = 45,\ \sigma = 0.2,\ T = 1$	8	36.3937	36.3881
	16	36.3941	36.3922
	32	36.3949	36.3933
Change in σ	4	39.1079	39.0978
$K = 45,\ \sigma = 0.15,\ T = 1$	8	39.1142	39.1124
	16	39.1162	39.1160
	32	39.1169	39.1170
Change in K	4	38.0092	38.9868
$K = 47,\ \sigma = 0.2,\ T = 1$	8	38.0124	38.0053
	16	38.0124	38.0097
	32	38.0129	38.0108
Change in T	4	34.3029	34.2922
$K = 45,\ \sigma = 0.2,\ T = 3$	8	34.3183	34.3191
	16	34.3239	34.3256
	32	34.3262	34.3274

The results in Table 1 illustrate the performance of the fully discrete spline collocation method applied to equation (1). We observe that the numerical results of the presented scheme versus to the Kim's results in [13] are nearly the same.

In Figures 1 and 2, we have plotted the behaviors of the early exercise boundary for $\tau = T$ and the errors obtained by the presented method with those derived in [13], respectively, which show the better results compared to [13].

Example 2. (From [17, 23]) This problem concerns the efficiency of the proposed method in long time horizons. We work with the same conditions as outlined in [17] and [23]. Consider the equation (5) for the fixed parameters $r = 0.1, \sigma = 0.3, K = 100$. Following [7, 22], the experimental results indicate that in the long term horizon, i.e. $\tau = T - t \geq 1$, most of the methods have been implemented for $0 < \tau < 1$ and the analytical approximation is no longer applicable. A comparison between our results with PSOR method as a bench mark and the numerical results in [17] and [16] is reported in Table 2, for $m = 2$ and $m = 3$. Figure 3 illustrates the early exercise boundary B_τ with various time expiration from $T = 10^{-5}$ to $T = 5$. The numerical results show an improvement in accuracy for this test case even in long time horizons.

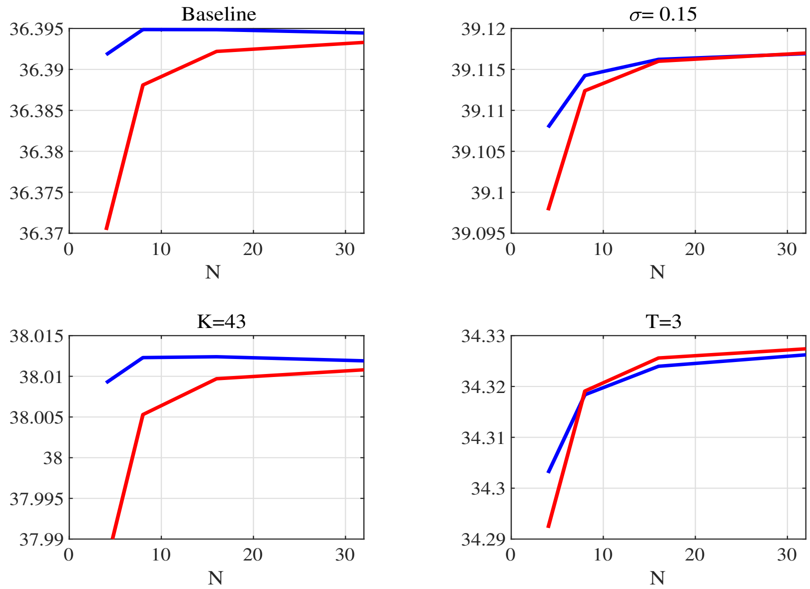

Figure 1: The behavior of the early exercise boundary for $\tau = T$ obtained by the proposed method (Blue) and the method of [13] (Red).

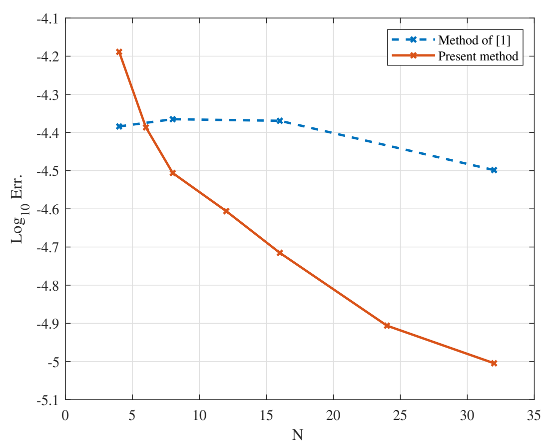

Figure 2: A comparision between the errors of the presented method and the method in [13].

Figure 4 displays the error bahavior of the early exercise boundary from short to long time horizons compared to PSOR method. We observe that by increasing T, the accuracy of the obtained solution is improved.

Example 3. (From [19]) This problem is relevant to the asymptotic behavior of the early exercise boundary. Consider Eq. (5) with parameters $r = 0.1$ and zero dividend $\sigma = 0.2$, $K = 100$ and $T = 10$ years. The asymptotic behavior of the early exercise boundary for $T \to \infty$ is represented in Figures 5 which illustrates the behavior of the boundary B_τ for various time expiration $T = 0.1, 1, 5$ and 10. It is seen that all the obtained results are well consistent and confirm the results those obtained in [14] and [19]. Figure 6 represents the asymptotic behavior of the American option for various T, when S tends to infinity in two different time to maturities.

Table 2: The maximum error for the optimal exercise boundary B_τ in the long time horizons $\tau = T$.

τ	Present method	Present method	Method of [23]	Method of [17]
	$m=2,\ N=8$	$m=3,\ N=8$		
0.00001	5.0 E -4	5.0 E -4	1.9 E -3	1.0 E -4
0.01	1.0 E -3	3.0 E -4	1.5 E -2	5.5 E -3
0.1	2.1 E -3	0.6 E -4	1.9 E -2	2.1 E -3
1	6.5 E -3	4.6 E -3	1.5 E -2	6.5 E -3
2	7.5 E -3	5.7 E -3	1.2 E -2	3.8 E -2
5	1.8 E -2	1.6 E -2	1.9 E -2	1.8 E -2

In the previous test problems, it was well reported that the main difficulty in pricing American options is to determine the optimal exercise boundary. Once it is computed, the options value can be obtained straightforwardly (see e.g., [23, 24]).

As a final test problem, we intend to show the applicability of the proposed numerical scheme for approximation of American option with respect to those recently introduced in [8], which is a scheme based on the finite difference and the method of lines for solving a free boundary problem as a PDE.

Example 4. (From [8]) Let us focus on Eq. (5) with the parameters $r = 0.1,\ \sigma = 0.4,\ K = 0.2, T = 1$ and $S = 0.2$. The numerical results in Table 3 show that the proposed scheme for $m = 2$ with significantly less subintervals, gives the same accuracy in predicting the American put option in [8]. Eventually, the numerical experiments reported for various test cases confirm that the proposed method can be viewed as a reliable and efficient scheme for valuation of American option.

Table 3: The numerical results of the presented method and the method of lines in [8] for different N.

N	Present method	Method of [8]
6	3.50 E -4	-
9	1.17 E -4	-
12	6.71 E -6	-
100	-	6.74 E -5
200	-	1.68 E -5
400	-	4.24 E -6

6 Conclusion

In this work we have considered a numerical scheme based on fully discretized collocation approximation for construction of the entire early exercise boundary in terms of the solution to a class of non-standard weakly singular pseudo-differential operator equations. We derived asymptotic behavior of approximation for the time close to expiry. In order to show the efficiency and accuracy of the method we presented qualitative and quantitative comparisons of analytical approximations and estimated the model parameters for the real case data. It was also shown that the proposed method is efficient in long term horizons. This methodology can be extended to stochastic volatility models which will be investigated in our future work.

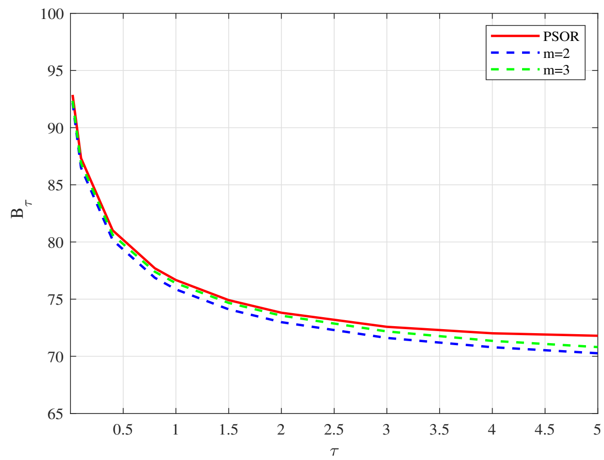

Figure 3: Behavior of the eraly exercise boundary on a long time horizon from $T = 10^{-5}$ to $T = 5$.

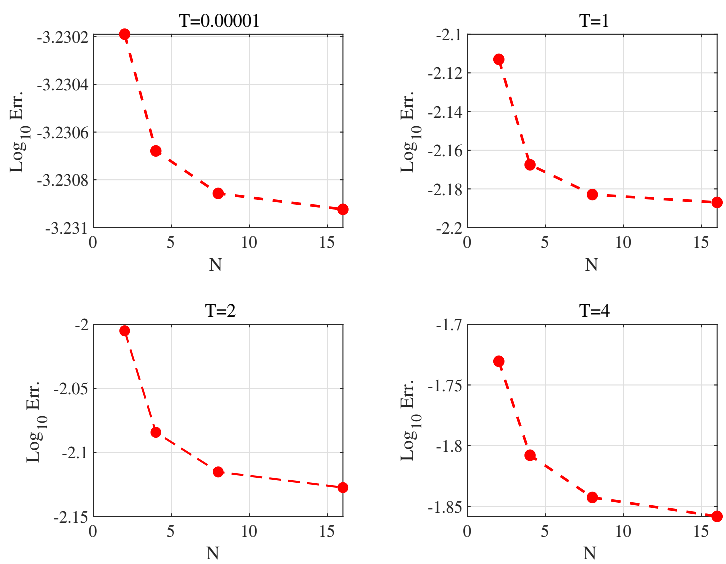

Figure 4: The error behaviors of the early exercise boundary compared to PSOR method for various time expiration.

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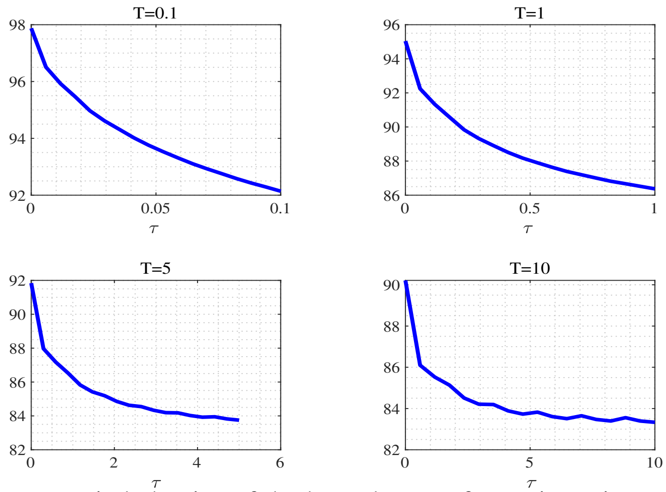

Figure 5: Asymptotic behavior of the boundary B_τ for various time expiration.

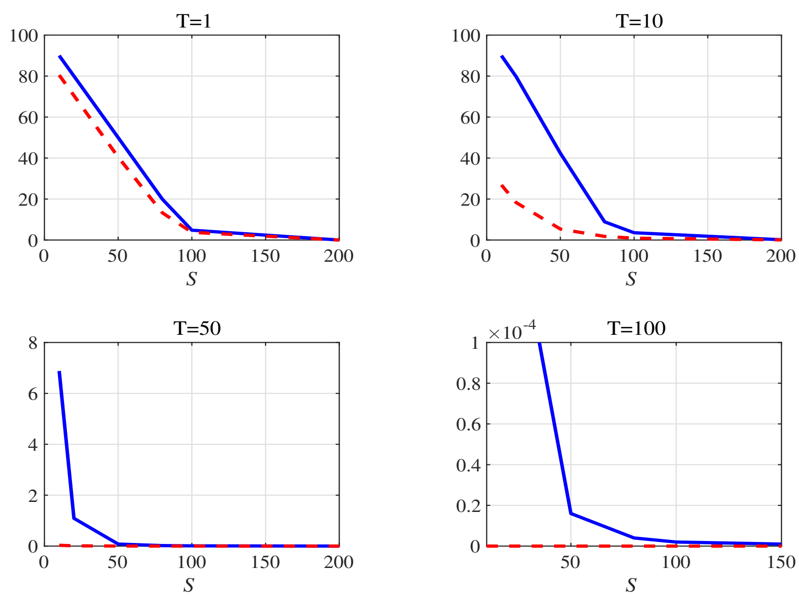

Figure 6: The asymptotic behavior of the American option when S tends to infinity for different time to maturity $\tau = T$ (Blue) and $\tau = 0$ (Red)

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