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Second order spline method for fractional Bagley-Torvik equation with variable coefficients and Robin boundary conditions

S. Joe Christin Mary, Ayyadurai Tamilselvan*

Department of Mathematics, Bharathidasan University, Tiruchirappalli - 620 024, Tamilnadu, India
Email(s): joechristinmary@gmail.com, mathats@bdu.ac.in

Abstract. A fractional Bagley-Torvik equation of variable coefficients with Robin boundary conditions is considered in this paper. We prove the existence of the solution which is demonstrated by converting the boundary value problem into a Volterra integral equation of the second kind and also prove the uniqueness of the solution by using the minimum principle. We propose a numerical method that combines the second order spline approximation for the Caputo derivative and the central difference scheme for the second order derivative term. Meanwhile, the Robin boundary conditions is approximated by three-point endpoint formula. It is to be proved that this method is of second order convergent. Numerical examples are provided to demonstrate the accuracy and efficiency of the method.

Keywords: Fractional Bagley-Torvik equation, Caputo fractional derivative, Robin boundary conditions, spline method, convergence analysis.
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1 Introduction

Fractional order differential equations (FDEs) have been used to explain problems mostly in the fields of fluid mechanics [1, 5], science and engineering [9, 10, 12]. Fractional order boundary value problems appear in the modeling of several physical phenomena in stochastic transport, diffusion wave, control theory, and the oil industry [4, 8]. The Bagley-Torvik equation is used to simulate the motion of a rigid plate submerged in a Newtonian fluid.

Many people have investigated the existence and uniqueness of the solution to FDE. Wei et al. [6, 13] have discussed the uniqueness of solution for fractional Bagley-Torvik equations with variable coefficients.

*Corresponding author.
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The Bagley-Torvik equation has been solved using a number of methods that combine analytical and numerical approaches. Zahra et.al. [14] proposed cubic spline method for Bagley- Torvik equation. Zahra and Elkholy have used quadratic spline approximation to solve boundary value problem of fractional order [15]. Mustafa Gulsu et.al [5] applied Taylor matrix method for solving fractional Bagley-Torvik equation in fluid dynamics. Saha and Bera [11] developed an analytical method to solve the Bagley-Torvik equation using the Adomian decomposition method. Taylor series expansion method of Bagley-Torvik equation with variable coefficients is studied by Huang et al [6].

Many authours have solved Bagley-Torvik equation with constant coefficients using various numerical methods. In this work we have concentrated on solving Bagley-Torvik equation with variable coefficients and Robin boundary conditions. The main goal of this study is to solve Robin boundary value problems of the Begley-Torvik equation with variable coefficients using the second order linear spline method. A spline method is used to approximate the Caputo fractional derivative of order δ, where $0 < \delta < 1$, and the Robin boundary conditions is approximated by three-point endpoint formula. Accuracy of the method is demonstrated through numerical experiments. Convergence analysis and stability of the method are discussed.

The paper is organized as follows: Statement of the continuous problem is presented in Section 2. The existence and uniqueness of the solution to the problem are established and also the minimum principle is derived in Section 3. The discretization of the continuous problem is completely explained in Section 4. Convergence analysis is discussed in Section 5. The computational results are presented in Section 6 which demonstrate the second-order accuracy of the numerical method and in Section 7, conclusions are provided. **Notations:**

- Throughout the paper, C is a positive constant that is independent of N.
- The maximum norm is defined by $||u||_{\Omega} = \max\limits_{x\in\Omega} |u(x)|$.
- $C^n(\Omega)$ denotes the space of n times continuously differentiable functions defined on $\Omega = (0,1)$.

2 Continuous problem

Consider the following fractional Bagley-Torvik Robin boundary value problem of variable coefficients with the Caputo derivative as

$$\begin{cases} \mathscr{L}u(x) = u''(x) + b(x)^C D^{\delta} u(x) - c(x)u(x) = g(x), \;\; x \in \Omega = (0,1), \\ \mathscr{B}_0 u(0) = \alpha_0 u(0) - \beta_0 u'(0) = A_0, \\ \mathscr{B}_1 u(1) = \alpha_1 u(1) + \beta_1 u'(1) = A_1, \end{cases} \tag{1}$$

where $A_0,\ A_1$ are constants and $b(x),\ c(x)$ and the source term $g(x)$ are sufficiently smooth functions on $\overline{\Omega} = [0,1]$ with the assumptions

$$\begin{cases} b(x) < 0, \;\; c(x) > 0, \;\; \forall x \in \overline{\Omega}, \\ \forall x \in \overline{\Omega}, \;\; \alpha_0,\ \alpha_1 > 0,\ \alpha_0 - \beta_0 > 0, \;\; \beta_0,\ \beta_1 \geq 0. \end{cases} \tag{2}$$

The Caputo fractional derivative of order δ, is defined by

$${}^C D^{\delta} u(x) = \frac{1}{\Gamma(1-\delta)} \int_0^x (x-t)^{-\delta} u'(s)\, ds, \;\; 0 < \delta < 1.$$

Under the assumptions (2), the problem (1) will admit a unique solution in Ω.

3 Analytical results

In this section, we prove the existence of the solution of the fractional boundary value problem (1). Also, we present the minimum principle which will be used to prove the uniqueness of the solution.

Theorem 1 (Existence of solution)**.** *Consider problem (1) with $b(x),\ c(x)$ and $g(x) \in C(\overline{\Omega})$. A function $u(x) \in C^2(\overline{\Omega})$ is a solution, if and only if, $v(x)$ satisfies the equation*

$$v(x) = Hv(x) + p(x), \tag{3}$$

where

$$Hv(x) = -b(x)\left[I^{(2-\delta)}v(x) - \frac{\alpha_0 M_2 M_3}{M_1}\right] + c(x)\left[I^2 v(x) - \frac{M_2}{M_1}(\beta_0 + \alpha_0 x)\right],$$
$$p(x) = g(x) - b(x)\frac{M_3}{M_1}(\alpha_0 A_1 - \alpha_1 A_0) + \frac{c(x)}{M_1}\left[A_0(\alpha_1 + \beta_1 - \alpha_1 x) + A_1(\beta_0 + \alpha_0 x)\right],$$

and I^δ is the Riemann-Liouville fractional integral operator of order $\delta \in \mathbb{R}_+$, defined by

$$I^\delta f(x) := \frac{1}{\Gamma(\delta)} \int_0^x (x-t)^{\delta-1} f(t)\, dt.$$

Proof. Let $u(x)$ be a solution of (1). Consider the new function

$$v(x) := u''(x). \tag{4}$$

Then by integrating both sides of (4), we obtain

$$u(x) = I^2 v(x) + k_1 + k_2 x, \tag{5}$$

where $k_1,\ k_2$ are constants. By applying the boundary conditions of the problem (1), we find that

$$\alpha_0 k_1 - \beta_0 k_2 = A_0, \tag{6}$$
$$\alpha_1 k_1 + (\alpha_1 + \beta_1) k_2 = A_1 - \alpha_1 I^2 v(1) - \beta_1 I v(1). \tag{7}$$

By solving these two equations (6) and (7), we have

$$\begin{aligned} k_1 &= \frac{\beta_0(A_1 - M_2) + A_0(\alpha_1 + \beta_1)}{M_1}, \\ k_2 &= \frac{\alpha_0(A_1 - M_2) - \alpha_1 A_0}{M_1}, \end{aligned} \tag{8}$$

where

$$M_1 = \alpha_1 \beta_0 + \alpha_0(\alpha_1 + \beta_1), \quad M_2 = \alpha_1 I^2 v(1) + \beta_1 I v(1), \quad M_3 = \frac{x^{(1-\delta)}}{\Gamma(2-\delta)}.$$

Substitution of k_1 and k_2 back into (5) gives

$$u(x) = I^2 v(x) + \frac{(A_1 - M_2)}{M_1}(\beta_0 + \alpha_0 x) + \frac{A_0}{M_1}(\alpha_1 + \beta_1 - \alpha_1 x). \tag{9}$$

Hence, by inserting (9) into (1), we finally write the form

$$v(x) = Hv(x) + p(x). \tag{10}$$

Conversely, let $v(x)$ be a solution of (3). Define a function $u(x)$ in the form (9) which satisfies the fractional Bagley-Torvik equation (1). □

Theorem 2 (Minimum principle)**.** *Consider $\mathscr{L}$ to be the differential operator defined in (1) and φ be any function satisfying $\mathscr{B}_0\varphi(0)$, $\mathscr{B}_1\varphi(1) \geq 0$ and $\mathscr{L}\varphi(x) \leq 0$ for $x \in \Omega$. Then $\varphi(x) \geq 0$, $\forall x \in \overline{\Omega}$.*

Proof. The proof is by contradiction. Let x_* be such that $\varphi(x_*) = \min\limits_{x\in\overline{\Omega}} \varphi(x)$ with $\varphi(x_*) < 0$.
Case(i): $x_* = 0$, we have $\mathscr{B}_0\varphi(x_*) = \alpha_0\varphi(x_*) - \beta_0\varphi(x_*) < 0$, a contradiction.
Case(ii): $x_* = 1$, we have $\mathscr{B}_1\varphi(x_*) = \alpha_1\varphi(x_*) - \beta_1\varphi(x_*) < 0$, a contradiction.
Case(iii): $x_* \in \Omega$. Then, by Theorem 1 of [7], we have

$$\mathscr{L}\varphi(x_*) = \varphi''(x_*) + b(x_*)^C D^\delta \varphi(x_*) - c(x_*)\varphi(x_*) > 0,$$

a contradiction. Hence the theorem is proved. □

Theorem 3 (Uniqueness of solution)**.** *Suppose the hypothesis of Theorem 1 is satisfied. Then, problem (1) has at most a single solution.*

Proof. Let $u_1(x)$ and $u_2(x)$ be two solutions of (1). Then, $z := u_1(x) - u_2(x)$ satisfies the homogeneous equation

$$\mathscr{L}z(x) = 0, \;\; x \in \Omega, \;\; \mathscr{B}_0 z(0) = 0, \;\; \mathscr{B}_1 z(1) = 0.$$

By Theorem 2, we get $z(x) = 0$. This shows that $u_1(x) = u_2(x)$ and problem (1) has at most single solution. □

4 Numerical scheme

In this section, we derive the numerical scheme for problem (1) by using linear spline to approximate the Caputo fractional derivative and central difference scheme for second order derivative term and the three-point formula for the Robin boundary conditions.

4.1 Discretization of the Continuous problem

Let N be a positive integer. The uniform mesh is discretized by dividing the domain $\overline{\Omega}$ into N subintervals, defined by the mesh points $x_i = ih$, $i = 0(1)N$ and the mesh width is $h = 1/N$.

Now, we derive the numerical approximation to the fractional derivative of the problem (1). Consider the Caputo derivative

$$^C D^\delta u(x_i) = \frac{1}{\Gamma(1-\delta)} \int_0^{x_i} (x_i - \zeta)^{-\delta} u'(\zeta)\, d\zeta, \;\; i = 1(1)N-1. \tag{11}$$

The linear spline $\Pi_i(\zeta)$,whose nodes and knots are named at x_k, $k = 0(1)i$, is of the form

$$\Pi_i(\zeta) = \sum_{k=0}^{i} \frac{du}{d\zeta}(x_k)\Pi_{i,k}(\zeta), \tag{12}$$

where $\Pi_{i,k}(\zeta)$, in each interval $x_{k-1} \leq \zeta \leq x_{k+1}$, for $k = 1(1)i-1$, given by

$$\Pi_{i,k}(\zeta) = \begin{cases} \dfrac{\zeta - x_{k-1}}{x_k - x_{k-1}}, & \text{for } x_{k-1} \leq \zeta \leq x_k, \\ \dfrac{x_{k+1} - \zeta}{x_{k+1} - x_k}, & \text{for } x_k \leq \zeta \leq x_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

For $k = 0$ and $k = i$, we have

$$\Pi_{i,0}(\zeta) = \begin{cases} \dfrac{x_1 - \zeta}{x_1 - x_0}, & \text{for } x_0 \leq \zeta \leq x_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Pi_{i,i}(\zeta) = \begin{cases} \dfrac{\zeta - x_{i-1}}{x_i - x_{i-1}}, & \text{for } x_{i-1} \leq \zeta \leq x_i, \\ 0, & \text{otherwise.} \end{cases}$$

From (11) and after some calculations, we have

$$\begin{aligned} \frac{1}{\Gamma(1-\delta)} \int_0^{x_i} \Pi_i(\zeta)(x_i - \zeta)^{-\delta} d\zeta &= \frac{1}{\Gamma(1-\delta)} \sum_{k=0}^{i} \frac{du}{d\zeta}(x_k) \int_0^{x_i} (x_i - \zeta)^{-\delta} \Pi_{i,k}(\zeta) d\zeta \\ &= \frac{h^{1-\delta}}{\Gamma(3-\delta)} \sum_{k=0}^{i} \frac{du}{d\zeta}(x_k) w_{i,k}, \end{aligned} \tag{13}$$

where

$$w_{i,k} = \begin{cases} (i-1)^{2-\delta} - i^{1-\delta}(i-2+\delta), & k = 0, \\ (i-k+1)^{2-\delta} - 2(i-k)^{2-\delta} + (i-k-1)^{2-\delta}, & 1 \leq k \leq i-1, \\ 1, & k = i. \end{cases} \tag{14}$$

Remark 1. *The coefficients $w_{i,k}$ implies that $\sum_{k=0}^{i} w_{i,k} = (2-\delta)i^{1-\delta}$.*

The first order derivative of (13) can be approximated by left side finite difference as

$$\frac{du(x_0)}{dx} = \frac{-3U_0 + 4U_1 - U_2}{2h}, \quad k = 0.$$

For $k = 1(1)N-1$, the central difference approximation of the first order derivative of (13) is

$$\frac{du(x_k)}{dx} = \frac{U_{k+1} - U_{k-1}}{2h}.$$

Finally, the approximation of ${}^{C}D^{\delta}(x_i)$ can be rewritten as

$$
{}^{C}D_S^{\delta} := \frac{h^{-\delta}}{2\Gamma(3-\delta)}\left\{w_{i,1}(-3U_0+4U_1-U_2)+\sum_{k=1}^{i} w_{i,k}(U_{k+1}-U_{k-1})\right\}. \tag{15}
$$

The first derivatives in the Robin boundary conditions are approximated by the three-point endpoint formula [2] as follows:

$$
\begin{aligned}
\frac{du(0)}{dx} &= \frac{1}{2h}(-3U_0+4U_1-U_2),\\
\frac{du(1)}{dx} &= \frac{1}{2h}(3U_N-4U_{N-1}+U_{N-2}).
\end{aligned}
$$

Hence, the discretization of (1) is: Find $\{U_i\}_{i=0}^{N}$ such that

$$
\begin{cases}
\mathscr{L}^N U_i = \delta^2 U_i + b_i^C D_S^{\delta} U_i - c_i U_i = g_i, & i = 1(1)N-1,\\
\mathscr{B}_0 U_0 = \alpha_0 U_0 - \dfrac{\beta_0}{2h}(-3U_0+4U_1-U_2) = A_0,\\
\mathscr{B}_1 U_N = \alpha_1 U_N + \dfrac{\beta_1}{2h}(3U_N-4U_{N-1}-U_{N-2}) = A_1,
\end{cases} \tag{16}
$$

where $b_i := b(x_i)$ and similarly for c_i and f_i, and $\delta^2 U_i = (U_{i+1}-2U_i+u_{i-1})/h^2$.

4.2 Matrix representation of the numerical scheme

We can write the discrete problem (16) in the matrix form as follows

$$
[\theta_1 K + \mu B(M+Q) + P + \theta_2 V]U = G, \tag{17}
$$

where $\theta_1 = 1/h^2$, $\theta_2 = 1/2h$, $\mu = h^{-\delta}/\Gamma(3-\delta)$, $G = (A_0, g_1, \ldots, g_{N-1}, A_1)^T$, $U = (U_0, U_1, \ldots, U_N)$ and the matrices K, B, M, Q, P and V are given below

$$
K = \begin{pmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\
1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0\\
0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0\\
 & & \ddots & \ddots & \ddots & & & \\
0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0\\
0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1\\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 & & & & \\
 & b_1 & & & \\
 & & b_2 & & \\
 & & & \ddots & \\
 & & & & b_{N-1} & \\
 & & & & & 0
\end{pmatrix},
$$

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ -3w_{1,0} & 4w_{1,0} & -w_{1,0} & 0 & \dots & 0 \\ -3w_{2,0} & 4w_{2,0} & -w_{2,0} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -3w_{N-2,0} & 4w_{N-2,0} & -w_{N-2,0} & 0 & \dots & 0 \\ -3w_{N-1,0} & 4w_{N-1,0} & -w_{N-1,0} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ q_{10} & q_{11} & q_{12} & \cdots & q_{1,N-2} & q_{1,N-1} & q_{1,N} \\ q_{20} & q_{21} & q_{22} & \cdots & q_{2,N-2} & q_{2,N-1} & q_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ q_{N-2,0} & q_{N-2,1} & q_{N-2,2} & \cdots & q_{N-2,N-2} & q_{N-2,N-1} & q_{N-2,N} \\ q_{N-1,0} & q_{N-1,1} & q_{N-1,2} & \cdots & q_{N-1,N-2} & q_{N-1,N-1} & q_{N-1,N} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} &q_{i,0} = -w_{i,1},\ \text{for } i = 1(1)N-1\ ; \quad q_{1,1} = 0\ ; \quad q_{i,1} = -w_{i,2},\ \text{for } i = 2(1)N-1\ ; \\ &q_{i,i} = w_{i,i-1},\ \text{for } i = 2(1)N-1\ ; \quad q_{i,i+1} = w_{i,i},\ \text{for } i = 1(1)N-1\ ; \\ &q_{i,k} = w_{i,k+1} - w_{i,k-1},\ \text{for } i = 3(1)N-1,\ k = 2(1)i-1, \\ &q_{i,k} = 0,\ i = 1(1)N-1,\ k = i+2, i+3, \dots, N, \end{aligned}$$

and

$$P = \begin{pmatrix} \alpha_0 & & & & & \\ & c_1 & & & & \\ & & c_2 & & & \\ & & & \ddots & & \\ & & & & c_{N-1} & \\ & & & & & \alpha_1 \end{pmatrix}, \quad V = \begin{pmatrix} 3\beta_0 & -4\beta_0 & \beta_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \beta_1 & -4\beta_1 & 3\beta_1 \end{pmatrix}.$$

5 Convergence analysis

In this section, we investigate the truncation error bound and the convergence of the proposed method.

Theorem 4. *Suppose $u(x) \in C^3(\overline{\Omega})$ and $0 < \delta < 1$, and the discrete operator Λ_δ is defined by*

$$\Lambda_\delta u(x_i) = \frac{1}{\Gamma(3-\delta)} \left(w_{i,0}\Lambda_0 u(x_0) + \sum_{k=1}^{i} w_{i,k}\Lambda^2 u(x_k) \right), \tag{18}$$

where $\Lambda_0 u(x_0) = -3u(x_0) + 4u(x_1) - u(x_2)$ and $\Lambda^2 u(x_k) = u(x_{k+1}) - u(x_{k-1})$. Then

$$\frac{1}{h^\delta}\Lambda_\delta u(x_i) = {}^C D^\delta u(x_i) + \eta_1(x_i) + \eta_2(x_i),$$

with

$$\max_{0\le x_i\le 1}|\eta_r(x_i)| \le \frac{x_i^{1-\delta}}{\Gamma(2-\delta)}O(h^2), \quad r=1,2.$$

Proof. From (18),

$$\frac{1}{h^\delta}\Lambda_\delta u(x_i) = \frac{h^{-\delta}}{\Gamma(3-\delta)}\left(w_{i,0}\Lambda_0 u(x_0) + \sum_{k=1}^{i} w_{i,k}\Lambda^2 u(x_k)\right).$$

Using the Taylor expansion [2] we have

$$\frac{1}{2h}\Lambda_0 u(x_0) = \frac{du}{dx}(x_0) + O(h^2); \quad \frac{1}{2h}\Lambda^2 u(x_k) = \frac{du}{dx}(x_k) + O(h^2).$$

Hence

$$\begin{aligned}\frac{1}{h^\delta}\Lambda_\delta u(x_i) &= \frac{h^{1-\delta}}{\Gamma(3-\delta)}\sum_{k=0}^{i} w_{i,k}\left(\frac{du}{dx}(x_k) + O(h^2)\right)\\ &= \frac{h^{1-\delta}}{\Gamma(3-\delta)}\sum_{k=0}^{i} w_{i,k}\frac{du}{dx}(x_k) + \eta_1(x_i),\end{aligned}$$

with

$$\eta_1(x_i) = \frac{h^{1-\delta}}{\Gamma(3-\delta)}\sum_{k=0}^{i} w_{i,k}O(h^2) = \frac{x_i^{1-\delta}}{\Gamma(2-\delta)}O(h^2),$$

from Remark 1. Now,

$$\begin{aligned}\frac{1}{h^\delta}\Lambda_\delta u(x_i) &= \frac{1}{\Gamma(1-\delta)}\int_0^{x_i}(x_i-\zeta)^{-\delta}\Pi_i(\zeta)d\zeta + \eta_1(x_i)\\ &= \frac{1}{\Gamma(1-\delta)}\int_0^{x_i}(x_i-\zeta)^{-\delta}\frac{du}{dx}(\zeta)d\zeta + \eta_2(x_i) + \eta_1(x_i),\end{aligned}$$

and hence

$$\begin{aligned}\eta_2(x_i) &= \frac{1}{\Gamma(1-\delta)}\left|\int_0^{x_i}\left(\Pi_i(\zeta) - \frac{du}{dx}(\zeta)\right)(x_i-\zeta)^{-\delta}d\zeta\right|\\ &\le \frac{1}{\Gamma(1-\delta)}\max_{\zeta\in[0,1]}\left|\frac{du}{dx}(\zeta) - \Pi_i(\zeta)\right|\int_0^{x_i}(x_i-\zeta)^{-\delta}d\zeta.\end{aligned}$$

Then under the continuity of $\frac{d^3u}{dx^3}(\zeta)$, we have from [2]

$$\frac{du}{dx}(\zeta) = \Pi_i(\zeta) + O(h^2),$$

and

$$\eta_2(x_i) \le \frac{x_i^{1-\delta}}{\Gamma(2-\delta)}O(h^2),$$

which completes the proof. $\square$

5.1 Truncation error bound

Theorem 5. *The truncation error of the numerical scheme (16) is of $O(h^2)$.*

Proof. Let $u(x_i)$ be the solution of (1) and $U(x_i)$ be the solution of the discrete problem (16) on the mesh points x_i, $i = 0(1)N$. The truncation error is defined as $\Upsilon := (\Upsilon_0, \Upsilon_1, \dots, \Upsilon_N)$, where $\Upsilon = u(x_i) - U(x_i)$ and the Υ_i's can be written as

$$\begin{aligned}
\Upsilon_0 &:= \beta_0\left(u'(x_0) - \frac{-3U_0 + 4U_1 - U_2}{2h}\right),\\
\Upsilon_i &:= \left(u'(x_i) - \delta^2 U(x_i)\right) + b(x_i)\left({}^{C}D^{\delta}u(x_i) - {}^{C}D_S^{\delta}U(x_i)\right), \quad i = 1(1)N-1,\\
\Upsilon_N &:= \beta_1\left(u'(x_N) - \frac{3U_{N-2} - 4U_{N-1} + U_N}{2h}\right).
\end{aligned}$$

Now, we prove the bounds for the error function Υ.

- Case $i = 0$: For the left Robin boundary conditions, we have [2]

  $$\frac{du}{dx}(x_0) = \frac{-3U_0 + 4U_1 - U_2}{2h} + O(h^2).$$

  Therefore, $|\Upsilon_0| \le Ch^2$.

- Case $i = N$: Similarly as in Case $i = 0$, we have $\Upsilon_N \le Ch^2$.

- Case $i = 1(1)N-1$: Using Theorem 4 and the truncation error the second order derivative can be approximated by the central difference scheme and it can be written as $\Upsilon_i \le Ch^2$, $i = 1(1)N-1$.

□

5.2 Error estimate

In order to derive an error bound of the scheme, we need the following lemmas.

Lemma 1. *[15] If A is a square matrix of order N and $||A||_\infty < 1$, then $(I + A)^{-1}$ exists and*

$$||(I + A)^{-1}||_\infty < \frac{1}{1 - ||A||_\infty}.$$

Note: Using binomial expansion, we can prove

$$(1 - x)^y \le 1 - xy, \quad \text{for } x, y \in [0, 1]. \tag{19}$$

Lemma 2. *The entries $w_{i,0}$, $i = 1(1)N-1$ of the matrix M are decreasing.*

Proof. Let
$$Y(x) = (x-1)^{2-\delta} - x^{1-\delta}(x-2+\delta), \quad \forall x \geq 1. \tag{20}$$
Next, we show that $Y'(x) < 0$. Differentiating the function $Y(x)$ with respect to x, we have
$$Y'(x) = (2-\delta)(1-\delta)x^{-\delta}\left(1 - \frac{x^{1-\delta} - (x-1)^{1-\delta}}{(1-\delta)x^{-\delta}}\right).$$
Take $f(x) = \dfrac{x^{1-\delta} - (x-1)^{1-\delta}}{(1-\delta)x^{-\delta}}$. Then,
$$f(x) = \frac{x}{1-\delta}\left(1 - \left(1 - \frac{1}{x}\right)^{1-\delta}\right) > 1,$$
where by using (19) for $\dfrac{1}{x} \in (0,1]$ and $0 < 1-\delta < 1$. Hence, $Y'(x) < 0$. Therefore $Y(x)$ is a decreasing function. $\square$

Our main purpose now is to calculate a bound for $||E||$. Let $E_i = u(x_i) - U(x_i),\ i = 0(1)N$. We can write the error term in the following form
$$E = \left(I + \theta_1 P^{-1}K + \mu P^{-1}B(M+Q) + \theta_2 P^{-1}V\right)^{-1} P^{-1}\Upsilon, \tag{21}$$
where $E = (E_0, E_1, \ldots, E_N)^T$.

Lemma 3. *The matrix* $[\theta_1 K + \mu B(M+Q) + P + \theta_2 V]$ *is regular, provided that*
$$\max\left\{\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, ||c^{-1}||\right\}\left(4\theta_1 + \mu||b||_\infty(8(1-\delta) + ||Q||_\infty) + \theta_2||V||_\infty\right) < 1. \tag{22}$$

Proof. Let
$$H = \theta_1 P^{-1}K + \mu P^{-1}B(M+Q) + \theta_2 P^{-1}V. \tag{23}$$
From the matrix P, we have
$$P \text{ is regular and } ||P^{-1}||_\infty \leq \max\left\{\frac{1}{\alpha_0}, \frac{1}{\alpha_1}, ||c^{-1}||\right\}.$$
Now,
$$||K||_\infty \leq 4, \quad ||B||_\infty \leq ||b||, \quad ||V||_\infty \leq 8\max\{\beta_0, \beta_1\}. \tag{24}$$
Then using Lemma 2, we get
$$||M||_\infty \leq 8||w|| = 8(1-\delta), \tag{25}$$
where $||w|| = \max\{w_{i,0},\ i = 1(1)N-1\}$. We have,
$$||Q||_\infty = \max_{1\leq i\leq N-1} \sum_{k=0}^{i+1} q_{i,k}. \tag{26}$$

Substituting (24-26) into (23) and using our assumption (22), we obtain

$$||H|| \leq 1.$$

Then using Lemma 1, $(I+\theta_1 P^{-1}K+\mu P^{-1}B(M+Q)+\theta_2 P^{-1}V)$ exists

$$\begin{aligned}
&||(I+\theta_1 P^{-1}K+\mu P^{-1}B(M+Q)+\theta_2 P^{-1}V)^{-1}|| \\
&\leq \frac{1}{1-\max\left\{\frac{1}{\alpha_0},\frac{1}{\alpha_1},||c^{-1}||\right\}\left(4\theta_1+\mu||b||_\infty(8(1-\delta)+||Q||_\infty)+\theta_2||V||_\infty\right)}.
\end{aligned}$$

□

Theorem 6. *If $u(x)$ is the solution of the continuous problem (1) and $U(x_i)$ is the numerical solution of the problem (16), then we have*

$$||E|| \cong O(h^2).$$

Proof. From equation (21), we can write

$$\begin{aligned}
||E|| &= ||\left(I+\theta_1 P^{-1}K+\mu P^{-1}B(M+Q)+\theta_2 P^{-1}V\right)^{-1}||_\infty \cdot ||P^{-1}||_\infty \cdot ||\Upsilon|| \\
&\leq \frac{||P^{-1}||_\infty ||\Upsilon||}{1-||P^{-1}||_\infty(\theta_1||K||_\infty+\mu||B||_\infty(||M||_\infty+||Q||_\infty)+\theta_2||V||_\infty)} \\
&\leq \frac{\max\left\{\frac{1}{\alpha_0},\frac{1}{\alpha_1},||c^{-1}||\right\}||\Upsilon||}{1-\max\left\{\frac{1}{\alpha_0},\frac{1}{\alpha_1},||c^{-1}||\right\}\left(4\theta_1+\mu||b||_\infty(8(1-\delta)+||Q||_\infty)+\theta_2||V||_\infty\right)},
\end{aligned}$$

by using Lemma 3. Applying Theorem 5, we have $||E|| \cong O(h^2)$. which is the desired result. □

6 Computational results

In this section, we solved two fractional Bagley-Torvik boundary value problems to illustrate our numerical method which demonstrate its order of convergence.

Example 1. Consider the fractional Bagley-Torvik boundary value problem:

$$\begin{cases}
u''(x)-(1-x^2)\,{}^{C}D^{\delta}u(x)-(1+\sin 2\pi x)u(x)=g(x), \quad x\in\Omega, \\
2u(0)-u'(0)=A_0, \\
u(1)+u'(1)=A_1,
\end{cases}$$

where the function $g(x)$ and the constants A_0, A_1 are chosen such that the exact solution $u_e(x)$ is

$$u_e(x)=x^{\delta+3}-x^{\delta+4}+2.$$

The maximum absolute error and the corresponding rate of convergence are given as below

$$D_\delta^N=\max_{0\leq i\leq N}|u_e(x_i)-U(x_i)| \quad \text{and} \quad p_\delta^N=\log_2\left(\frac{D_\delta^N}{D_\delta^{2N}}\right).$$

Also, the uniform errors and the corresponding rate of convergence for various values of δ are calculated by

$$D^N = \max_{\delta} D^N_\delta \quad \text{and} \quad p^N = \log_2\left(\frac{D^N}{D^{2N}}\right).$$

The exact and approximate solutions are exhibited in Figure 1 for the case of $n = 2^7$ and $\delta = 0.3$. The maximum error and order of convergence for various value of $\delta \in \{0.1, 0.2, \dots, 0.9\}$ are represented in Table 1. From the error plot shown in Figure 2, the maximum errors reduces as N increases. The loglog plot of the maximum error of the numerical solution U_i is given in Figure 3.

Table 1: Computed maximum errors and uniform errors D^N_δ, D^N and orders of convergence p^N_δ, p^N of Example 1 for the values of δ and N.

δ/N	32	64	128	256	512	1024	2048	4096
0.1	3.740e-03 1.982	9.463e-04 1.994	2.375e-04 2.000	5.937e-05 2.002	1.481e-05 2.003	3.693e-06 2.004	9.205e-07 2.004	2.295e-07 -
0.2	3.945e-03 1.984	9.967e-04 1.996	2.498e-04 2.001	6.238e-05 2.003	1.555e-05 2.004	3.876e-06 2.004	9.660e-07 2.004	2.407e-07 -
0.3	4.162e-03 1.984	1.051e-03 1.995	2.637e-04 2.000	6.589e-05 2.002	1.644e-05 2.003	4.100e-06 2.003	1.022e-06 2.003	2.551e-07 -
0.4	4.391e-03 1.982	1.111e-03 1.994	2.788e-04 1.999	6.975e-05 2.001	1.742e-05 2.001	4.349e-06 2.001	1.086e-06 2.001	2.712e-07 -
0.5	4.628e-03 1.980	1.173e-03 1.992	2.949e-04 1.997	7.384e-05 1.999	1.846e-05 2.000	4.614e-06 2.000	1.152e-06 2.000	2.880e-07 -
0.6	4.872e-03 1.977	1.237e-03 1.989	3.115e-04 1.996	7.811e-05 1.998	1.954e-05 1.999	4.887e-06 2.000	1.221e-06 2.000	3.054e-07 -
0.7	5.123e-03 1.973	1.304e-03 1.988	3.287e-04 1.994	8.249e-05 1.997	2.065e-05 1.999	5.1672e-06 1.999	1.2920e-06 1.999	3.2304e-07 -
0.8	5.379e-03 1.971	1.372e-03 1.986	3.463e-04 1.993	8.698e-05 1.997	2.179e-05 1.998	5.452e-06 1.999	1.363e-06 1.999	3.408e-07 -
0.9	5.643e-03 1.967	1.442e-03 1.984	3.645e-04 1.992	9.160e-05 1.996	2.295e-05 1.998	5.745e-06 1.999	1.436e-06 1.999	3.593e-07 -
D^N p^N	5.643e-03 1.967	1.442e-03 1.984	3.645e-04 1.992	9.160e-05 1.996	2.295e-05 1.998	5.745e-06 1.999	1.436e-06 1.999	3.593e-07 -

Example 2. Consider the fractional Bagley-Torvik boundary value problem:

$$\begin{cases} u''(x) - (1+x)\,{}^C D^\delta u(x) - (16x^{0.7}+1) = x, \quad x \in \Omega, \\ 2u(0) - u'(0) = 0, \\ 3u(1) + u'(1) = 1. \end{cases}$$

The exact solution of this problem is not known. Therefore, we use the double- mesh principle [3] to determine the maximum errors and the order of convergence. Let U^N, U^{2N} be the numerical solutions with N and $2N$ mesh points. Then the two mesh differences is defined as

$$E^N_\delta = \max_{0\le i\le N} |U^N_i - U^{2N}_i|, \quad \text{and} \quad E^N = \max_{\delta} E^N_\delta.$$

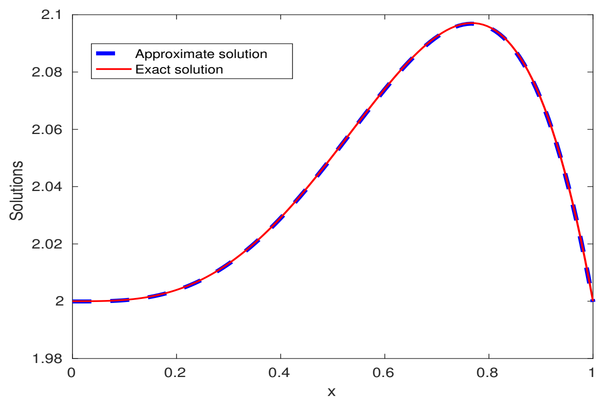

Figure 1: Exact and numerical solutions of Example 1 for $N = 2^7$ and $\delta = 0.3$.

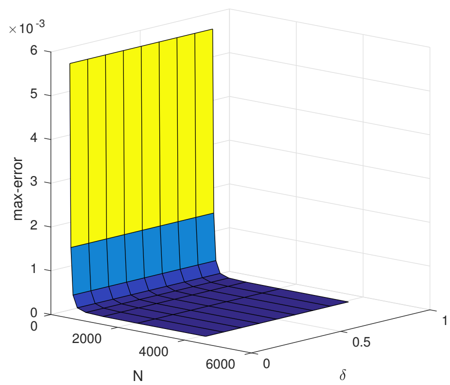

Figure 2: Error plot of Example 1.

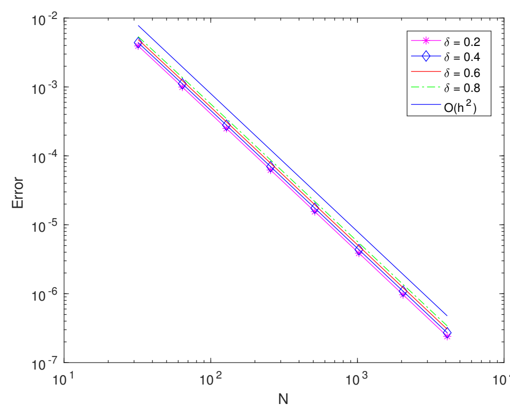

Figure 3: Loglog plot of Example 1.

Further, we determine the order of convergence as

$$q_\delta^N = \log_2\left(\frac{E_\delta^N}{E_\delta^{2N}}\right) \quad \text{and} \quad q^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

The graph of the numerical solution is shown in Figure 4 for different values of $\delta \in \{0.1, 0.5, 0.9\}$ and $n = 2^7$. The maximum errors and rate of convergence for the values $\delta \in \{0.1, 0.2, \ldots, 0.9\}$ are demonstrated in Table 2, which prove this proposed method is of second order convergence, as well as from the error plot and loglog plot displayed in Figures 5 and 6, respectively.

7 Conclusions

Approximate solution for a class of fractional Bagley-Torvik equations of variable coefficients with Robin boundary conditions are found through our proposed numerical method comprised of linear spline and three-point endpoint formula. It is proved that our numerical method is of second order convergent.

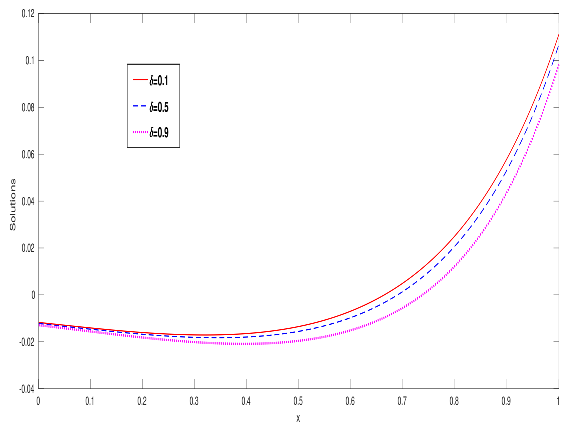

Figure 4: Numerical solutions of Example 2 for $N = 2^7$ and $\delta \in \{0.1, 0.5, 0.9\}$.

Table 2: Computed maximum errors and uniform errors E^N_δ, E^N and orders of convergence q^N_δ, q^N of Example 2 for the values of δ and N.

δ/N	32	64	128	256	512	1024	2048	4096
0.1	5.044e-04 1.912	1.339e-04 1.953	3.457e-05 1.975	8.795e-06 1.985	2.220e-06 1.990	5.588e-07 1.992	1.404e-07 1.993	3.528e-08 -
0.2	5.096e-04 1.911	1.354e-04 1.953	3.497e-05 1.974	8.899e-06 1.985	2.247e-06 1.990	5.656e-07 1.992	1.4218e-07 1.993	3.572e-08 -
0.3	5.159e-04 1.910	1.372e-04 1.952	3.545e-05 1.974	9.023e-06 1.985	2.279e-06 1.989	5.739e-07 1.991	1.443e-07 1.992	3.628e-08 -
0.4	5.235e-04 1.909	1.394e-04 1.951	3.603e-05 1.973	9.174e-06 1.984	2.318e-06 1.989	5.840e-07 1.990	1.469e-07 1.991	3.697e-08 -
0.5	5.329e-04 1.907	1.420e-04 1.950	3.673e-05 1.972	9.358e-06 1.983	2.366e-06 1.988	5.965e-07 1.989	1.502e-07 1.989	3.789e-08 -
0.6	5.445e-04 1.906	1.452e-04 1.949	3.760e-05 1.972	9.583e-06 1.982	2.425e-06 1.986	6.123e-07 1.985	1.545e-07 1.986	3.912e-08 -
0.7	5.593e-04 1.904	1.493e-04 1.949	3.868e-05 1.971	9.866e-06 1.980	2.499e-06 1.983	6.321e-07 1.986	1.601e-07 1.986	4.078e-08 -
0.8	5.787e-04 1.903	1.547e-04 1.948	4.010e-05 1.970	1.023e-05 1.979	2.593e-06 1.980	6.570e-07 1.975	1.671e-07 1.980	4.289e-08 -
0.9	6.053e-04 1.900	1.621e-04 1.947	4.204e-05 1.970	1.072e-05 1.980	2.717e-06 1.981	6.882e-07 1.984	1.751e-07 1.986	4.506e-08 -
E^N q^N	6.053e-04 1.900	1.621e-04 1.947	4.204e-05 1.970	1.072e-05 1.980	2.717e-06 1.981	6.882e-07 1.984	1.751e-07 1.986	4.506e-08 -

Error estimates and convergence analysis are given. Computational results of two examples were presented by means of graphs and tables which validates our theoretical results.

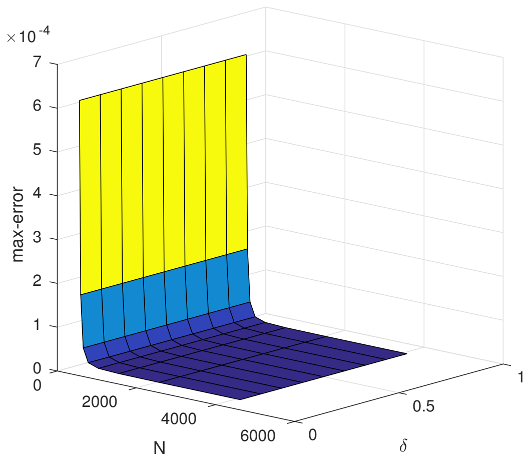

Figure 5: Error plot of Example 2

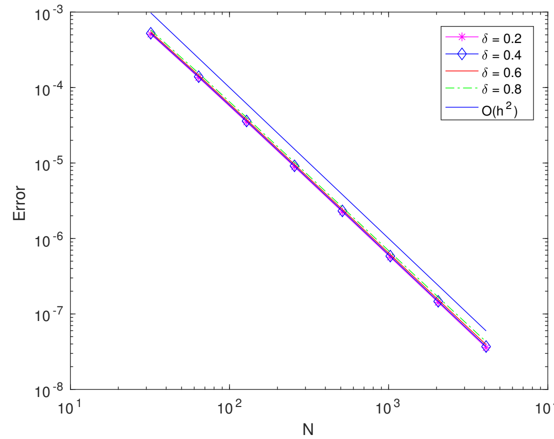

Figure 6: Loglog plot of Example 2

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