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Numerical solution of an influenza model with vaccination and antiviral treatment by the Newton-Chebyshev polynomial method

Bahman Babayar-Razlighi[*]

Department of Mathematics, Qom University of Technology, P.O.Box 1519-37195, Qom, Iran
Email(s): babayar@qut.ac.ir

Abstract. We consider a mathematical model of an influenza disease with vaccination and antiviral treatment. This model is expressed by a system of nonlinear ordinary differential equations. We linearize this system by the Newton's method and obtain a sequence of linear systems. The linear systems can be solved by the Chebyshev polynomial solutions, which is a convergence method for numerical solution of linear systems. We solve the problem on a union of many partial intervals. In each partial interval, we first obtain a crude approximation for starting the Newton's method, then solve the problem on current interval by using the lag intervals. An illustration of procedures, we give an algorithm for the initial guess and apply this algorithm for obtaining the total algorithm of the method. We investigate the convergence conditions of the Newton's method for the presented model. In the numerical examples section, we provide some numerical examples to illustrate of the accuracy of the method, and see that the main criterion of the convergence is true for such problems.

Keywords: The Newton's method, influenza model, Chebyshev polynomial solutions, long time, nonlinear nonaotonomous ODE.
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1 Introduction

In this paper, we consider the mathematical model of influenza disease. This model is described by the following system of ordinary differential equations

[*]Corresponding author.

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$$
\begin{cases}
\frac{dS}{dt} = \Lambda - (\mu+\nu)S - \lambda_S(t)S - \lambda_R(t)S + \omega R + \sigma V, \\
\frac{dV}{dt} = \nu S - (\sigma+\mu)V, \\
\frac{dI_{SU}}{dt} = (1-f)\lambda_S(t)S - (\mu+k_U)I_{SU}, \\
\frac{dI_{ST}}{dt} = f(1-c)\lambda_S(t)S - (\mu+k_T)I_{ST}, \\
\frac{dI_R}{dt} = \lambda_R(t)S + fc\lambda_S(t)S - (\mu+k_R)I_R, \\
\frac{dR}{dt} = k_T I_{ST} + k_U I_{SU} + k_R I_R - (\mu+\omega)R,
\end{cases} \tag{1}
$$

where $S, V, I_{SU}, I_{ST}, I_R, R$ are unknown functions of the model and denote the number of susceptible (S), vaccinated (V), infected with the sensitive strain and untreated (I_{SU}) or treated (I_{ST}), infected with the resistant strain (I_R), and recovered (R) subclasses of the total number of the population $\mathbf{N} = S+V+I_{SU}+I_{ST}+I_R+R$. The quantities $\Lambda, \mu, \nu, \omega, \sigma, f, c, k_U, k_T, k_R$ are known positive constants and

$$
\lambda_S(t) = \beta_S \frac{I_{SU} + \delta I_{ST}}{\mathbf{N}} \quad \lambda_R(t) = \beta_R \frac{I_R}{\mathbf{N}}, \tag{2}
$$

are known functions of their arguments, and β_R, β_S, δ are positive constants [9]. For more details, we refer the reader to [9], and for illustration of the medical sense of parameters, we rewrite Tables 1,3 from [9] as Table 1. The model presented in this paper has a similar structure as that in Lipsitch et al. [8].

Table 1: Definitions and Parameter values for system (1).

Parameter	Description	Estimated value
Λ	Recruitment rate of individuals	0.5
$\frac{1}{\mu}$	Average life span	20000 day
ν	Rate at which susceptible individuals are vaccinated	0.001 day^{-1}
$\frac{1}{\omega}$	Average time of losing immunity acquired by infection	$\frac{1000}{3}$ day
$\frac{1}{\sigma}$	Average time of losing vaccine-induced immunity	$\frac{1000}{3}$ day
β_S	Transmission coefficient of the untreated infected individuals	0.2835 day^{-1}
β_R	Transmission coefficient of the drug-resistant infected individuals	0.2551 day^{-1}
δ	Reduction factor in infectiousness due to the antiviral treatment	0.4
f	Fraction of the newly infected cases which are treated	0.9
c	Fraction of the treated infected cases which progress to the drug-resistant stage	0.02
$\frac{1}{k_U}$	Average infected length of the untreated cases	$\frac{10000}{1667}$ day
$\frac{1}{k_T}$	Average infected length of the treated cases	$\frac{10000}{1667}$ day
$\frac{1}{k_R}$	Average infected length of the drug-resistant cases	$\frac{10000}{1667}$ day

Transmission dynamics of the model with the stability of equilibria and asymptotic behavior analyses of the model is described in [9]. We add together all equations in (1) and obtain that the total population size $\mathbf{N}$ satisfies the following equation

$$
\mathbf{N}' = \Lambda - \mu \mathbf{N}. \tag{3}
$$

This equation has the exact solution

$$\mathbf{N}(t)=\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}, \tag{4}$$

where $N_0=\mathbf{N}(0)$. The function $\mathbf{N}(t)$ is a monotone increasing function with $\lim\limits_{t\to\infty}\mathbf{N}(t)=\Lambda/\mu$, and the biological feasible region

$$\Gamma=\left\{(S,V,I_{SU},I_{ST},I_R,R)^T:0\le S,V,I_{SU},I_{ST},I_R,R,\quad S+V+I_{SU}+I_{ST}+I_R+R\le\frac{\Lambda}{\mu}\right\}, \tag{5}$$

is positively invariant for the system (1). By substituting Eq. (4) in Eq. (2) and then Eq. (2) in Eq. (1), we obtain the following nonlinear nonautonomous system of differential equations

$$\begin{cases}\frac{dS}{dt}=\Lambda-(\mu+\nu)S-\beta_S\frac{(I_{SU}+\delta I_{ST})S}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}-\beta_R\frac{I_RS}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}+\omega R+\sigma V,\\ \frac{dV}{dt}=\nu S-(\sigma+\mu)V,\\ \frac{dI_{SU}}{dt}=(1-f)\beta_S\frac{(I_{SU}+\delta I_{ST})S}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}-(\mu+k_U)I_{SU},\\ \frac{dI_{ST}}{dt}=f(1-c)\beta_S\frac{(I_{SU}+\delta I_{ST})S}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}-(\mu+k_T)I_{ST},\\ \frac{dI_R}{dt}=\beta_R\frac{I_RS}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}+fc\beta_S\frac{(I_{SU}+\delta I_{ST})S}{\frac{\Lambda}{\mu}+\left(N_0-\frac{\Lambda}{\mu}\right)e^{-\mu t}}-(\mu+k_R)I_R,\\ \frac{dR}{dt}=k_TI_{ST}+k_UI_{SU}+k_RI_R-(\mu+\omega)R.\end{cases} \tag{6}$$

By [9], when the total population is assumed to be constant, $\mathbf{N}=\frac{\Lambda}{\mu}$, then the system (1) reduces to

$$\begin{cases}\frac{dS}{dt}=\Lambda-(\mu+\nu)S-\frac{\mu\beta_S}{\Lambda}(I_{SU}+\delta I_{ST})S-\frac{\mu\beta_R}{\Lambda}I_RS+\omega R+\sigma\left(\frac{\Lambda}{\mu}-S-I_{SU}-I_{ST}-I_R-R\right),\\ \frac{dI_{SU}}{dt}=(1-f)\frac{\mu\beta_S}{\Lambda}(I_{SU}+\delta I_{ST})S-(\mu+k_U)I_{SU},\\ \frac{dI_{ST}}{dt}=f(1-c)\frac{\mu\beta_S}{\Lambda}(I_{SU}+\delta I_{ST})S-(\mu+k_T)I_{ST},\\ \frac{dI_R}{dt}=\frac{\mu\beta_R}{\Lambda}I_RS+fc\frac{\mu\beta_S}{\Lambda}(I_{SU}+\delta I_{ST})S-(\mu+k_R)I_R,\\ \frac{dR}{dt}=k_TI_{ST}+k_UI_{SU}+k_RI_R-(\mu+\omega)R.\end{cases} \tag{7}$$

For investigation of the systems (6)-(7), we consider the following system of nonlinear ordinary differential equations

$$\begin{cases}U'(t)=g(t,U(t)),\ t\in[0,n_Il],\\ U(0)=U_0.\end{cases} \tag{8}$$

where $U(t)=(u_1(t),\dots,u_d(t))^T$ is unknown vector of functions,

$$g(t,U(t))=(g_1(t,U(t)),\dots,g_d(t,U(t)))^T,$$

is a known vector valude function of its arguments, and $U(0)=U_0=\left(u_1^{(0)},\dots,u_d^{(0)}\right)^T$ is the known initial condition. The positive integer d is dimension of the problem, and we are going to solve the problem on $t\in[0,n_Il]=\bigcup_{k=1}^{n_I}I_k$, where $l>0$ is the length of partial intervals, n_I is the number of intervals and $I_k = [(k-1)l, kl]$ is the kth partial interval. We call the proposed method the Newton-Chebyshev polynomial method (NC).

The organization of the paper is as follows: In Section 2, we describe the Newton's method and obtain a sequence of linear systems of ordinary differential equations. We solve the linear sequences of ordinary differential equations by the Chebyshev polynomial solutions technique, which is described in Section 3. For obtaining an initial guess of the Newton's method, we give an iterative procedure established on composite trapezoidal quadrature in Section 4. In Section 5, we provide total algorithm of the method and finally, in Section 6, we have solved the second example with another method and checked the advantages of the Newton's method. The main finding of both methods is that the second-order convergence of the Newton's process causes the error propagation to be negligible for a long period of time. We have provided these findings by two benchmark sample problems with some exact solutions, and analyzed method's accuracy.

2 The Newton's method

Suppose X, Y are two Banach spaces, and $F : X \to Y$ be a Frechet differentiable operator. With the initial guess $U^{(0)}$, the Newton's method for numerical solution of the following operator equation

$$F(U) = 0, \tag{9}$$

is as follows [1, 11]

$$F'(U^{(n)})U^{(n+1)} = F'(U^{(n)})U^{(n)} - F(U^{(n)}), \quad n = 0,1,2,\dots. \tag{10}$$

For the initial value problem (8), the associated operator is as follows

$$\begin{cases} F : (C^1[0,n_I l])^d \longrightarrow (C[0,n_I l])^d, \\ F(U)(t) := U'(t) - g(t,U(t)), \quad U \in (C^1[0,n_I l])^d, t \in [0,n_I l]. \end{cases} \tag{11}$$

By [4], one step of the Newton's method for the operator F in (11) is

$$\left(U^{(n+1)}(t)\right)' - g'(t,U^{(n)}(t))U^{(n+1)}(t) = g(t,U^{(n)}(t)) - g'(t,U^{(n)}(t))U^{(n)}(t), \quad n = 0,1,2,\dots, \tag{12}$$

where $g'(t,U^{(n)}(t)) = \left[\frac{\partial g_i}{\partial u_j}(t,U(t))|_{U(t)=U^{(n)}(t)}\right]_{d\times d}$ is the Jacobian or Frechet derivative matrix. From Section 2 of [4], the Newton's method is a convergent method for the initial problem (8) on each partial interval $I_k = [(k-1)l, kl]$, if

$$A(U(t)) = \left[\frac{\partial g_i}{\partial u_j}(t,U(t))\right]_{d\times d}, \tag{13}$$

is a Lipschitz continuous operator with a Lipschitz constant $L < \infty$, where

$$L = \max_{1\le i\le d} \sum_{j=1}^{d}\sum_{k=1}^{d} b_{ijk}, \quad b_{ijk} = \max_{t\in[0,n_I l], U\in\Gamma} \left|\frac{\partial^2 g_i}{\partial u_j \partial u_k}(t,U(t))\right|, \tag{14}$$

and Γ is the biologically feasible region defined by (5) [3, 5]. Whatever $L > 0$ is small, the accuracy of the method is better. We shall see that in our influenza model, this quantity is small.

3 Chebyshev polynomial solutions technique

Chebyshev polynomial solutions technique is one of the powerful tools in solving systems of linear differential equations [2, 10]. In this section, we deal with the numerical solution of the sequence of differential equations (12). We describe the method of solving these systems with the Chebyshev polynomial solutions technique. We consider equation (12) as follows

$$P_0(t)y(t)+P_1(t)y^{'}(t)=r(t), \tag{15}$$

where $P_0(t)=-g^{'}(t,U^{(n)}(t)),\ P_1(t)=I_d$, (the $d\times d$ identity matrix), $y(t)=U^{(n+1)}(t)$ and

$$r(t)=g(t,U^{(n)}(t))-g^{'}(t,U^{(n)}(t))U^{(n)}(t).$$

Akyüz and Sezer [2], are described the method on $\xi\in[-1,1]$, which is the domain of the Chebyshev polynomials $T_i(\xi)=\cos\left(i\cos^{-1}\xi\right)$, but we must apply the method on short time intervals $I_k=[(k-1)l,kl]$. For a given $k\in\{1,\dots,n_I\}$, suppose $a=(k-1)l,\ b=kl$, and we want to explain the method on $[a,b]$. For this purpose suppose the solution of (15) is expressed by a truncated Chebyshev series

$$y_i(t)=\sum_{j=0}^{N}a_{ij}T_j(\xi(t)), i=1,\dots,d,\ a\le t\le b, \tag{16}$$

where $\xi(t)=(2t-b-a)/(b-a)$ maps $[a,b]$ on to $[-1,1]$, N is a positive integer and a_{ij} are unknown Chebyshev coefficients. We can write the equation (16) in the following matrix form

$$y_i(t)=T(\xi(t))A_i, \quad i=1,\dots,d, \tag{17}$$

where $T(\xi)=[T_0(\xi),T_1(\xi),\dots,T_N(\xi)]$, $A_i=[a_{i0},a_{i1},\dots,a_{iN}]^T$. Now, let

$$y_i^{'}(t)=\sum_{j=0}^{N}a_{ij}^{(1)}T_j(\xi(t)), i=1,\dots,d,\ a\le t\le b, \tag{18}$$

then in accordance with [7], Appendix B.2.2, and for $i=1,\dots,d$, we have

$$\begin{aligned}
&a_{iN}^{(1)}=0,\\
&a_{i,N-1}^{(1)}=2sNa_{iN},\\
&a_{i,k-1}^{(1)}=a_{i,k+1}^{(1)}+2ska_{ik},\ k=N-1,N-2,\dots,2,\\
&a_{i0}^{(1)}=a_{i2}^{(1)}/2+sa_{i1},
\end{aligned} \tag{19}$$

where $s=\frac{2}{b-a}$. By an induction, we obtain

$$\begin{aligned}
a_{i0}^{(1)}&=2s\left(\sum_{j=2,4,6,\dots\le N}\frac{j}{2}a_{ij}+\frac{1}{2}a_{i1}\right),\\
a_{ik}^{(1)}&=2s\left(\sum_{j=k+1,k+3,k+5,\dots\le N}ja_{ij}\right),\ k=1,2,\dots,N-1\\
a_{iN}^{(1)}&=0.
\end{aligned} \tag{20}$$

Hence we can write

$$y_i'(t) = T(\xi(t))A_i^{(1)}, \quad i = 1,\dots,d, \tag{21}$$

where $A_i^{(1)} = [a_{i0}^{(1)}, a_{i1}^{(1)}, \dots, a_{iN}^{(1)}]^T = 2sMA_i$, $M = [m_{ij}]$ is a semi-sparse $(N+1)\times(N+1)$ matrix with the following nonzero components

$$\begin{aligned} m_{12} &= \frac{1}{2}, \\ m_{1j} &= \frac{j}{2}, \quad j = 3,5,7,\dots \le N+1, \\ m_{ij} &= j, \quad i = 2,3,4,\dots,N,\ j = i+1, i+3, i+5, \dots \le N+1. \end{aligned} \tag{22}$$

Hence the Eq. (21) reduces to

$$y_i'(t) = 2sT(\xi(t))MA_i, \quad i = 1,\dots,d. \tag{23}$$

We can represent (17) and (23) by the following matrix form

$$y^{(i)}(t) = \begin{bmatrix} y_1^{(i)}(t) \\ y_2^{(i)}(t) \\ \vdots \\ y_d^{(i)}(t) \end{bmatrix} = 2^i s^i T^*(\xi(t))M_i^* A, \quad i = 0,1, \tag{24}$$

where

$$T^*(\xi) = \begin{bmatrix} T(\xi) & 0 & 0 & 0 \\ 0 & T(\xi) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & T(\xi) \end{bmatrix},\ M_i^* = \begin{bmatrix} M^i & 0 & 0 & 0 \\ 0 & M^i & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & M^i \end{bmatrix},\ A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_d \end{bmatrix}.$$

Here $T^*(\xi), M_i^*$ are $d\times d$ blocks matrices, and A is a $d\times 1$ blocks vector. The Chebyshev collocation points on $[-1,1]$ are

$$\xi_j = \cos\frac{j\pi}{N}, \quad j = 0,1,\dots,N, \tag{25}$$

and hence the Chebyshev collocation points on $[a,b]$ are

$$t_j = \frac{1}{2}\left(b + a + (b-a)\cos\frac{j\pi}{N}\right), \quad j = 0,1,\dots,N. \tag{26}$$

By setting $t = t_j,\ j = 0,1,\dots,N$ in (15), we obtain

$$P_0 Y^{(0)} + P_1 Y^{(1)} = R, \tag{27}$$

where

$$P_i = \begin{bmatrix} P_i(t_0) & 0 & 0 & 0 \\ 0 & P_i(t_1) & 0 & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & P_i(t_N) \end{bmatrix},\ Y^{(i)} = \begin{bmatrix} y^{(i)}(t_0) \\ y^{(i)}(t_1) \\ \vdots \\ y^{(i)}(t_N) \end{bmatrix},\ R = \begin{bmatrix} r(t_0) \\ r(t_1) \\ \vdots \\ r(t_N) \end{bmatrix}.$$

By using the Eq. (24) and the associated Chebyshev collocation points from (25)-(26), we get

$$y^{(i)}(t_j) = 2^i s^i T^*(\xi_j) M_i^* A, \quad j = 0,1,\ldots,N,\ i = 0,1,$$

and hence

$$Y^{(i)} = 2^i s^i \mathbf{T} M_i^* A, \quad i = 0,1, \tag{28}$$

where $\mathbf{T} = \begin{bmatrix} T^*(\xi_0) & T^*(\xi_1) & \ldots & T^*(\xi_N) \end{bmatrix}$. Substituting Eq. (28) in Eq. (27) implies that

$$P_0 \mathbf{T} A + 2sP_1 \mathbf{T} M_1^* A = R. \tag{29}$$

By defining $W = [w_{ij}]_{(N+1)d\times(N+1)d} := P_0\mathbf{T} + 2sP_1\mathbf{T}M_1^*$, we must solve the following system of linear algebric equations

$$WA = R. \tag{30}$$

Eq. (30) gives the general solution for $y(t)$. For obtaning a particular solution that passes the initial condition $y(a) = \widetilde{\lambda} = (\lambda_1,\ldots,\lambda_d)^T$, from (24) for $i = 0$, we have

$$\widetilde{V}A = \widetilde{\lambda}, \tag{31}$$

where $\widetilde{V} = T^*(\xi(a)) = T^*(-1) = [v_{ij}]_{d\times(N+1)d}$. Eq. (31) is the fundamental matrix form of initial condition. By replacing the rows of the matrices $\widetilde{V}$ and $\widetilde{\lambda}$, by the last rows of the matrices W and R, respectively, we get $\widetilde{W}A = \widetilde{R}$, where

$$\widetilde{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1,(N+1)d} \\ w_{21} & w_{22} & \cdots & w_{2,(N+1)d} \\ \vdots & \vdots & \ddots & \vdots \\ w_{Nd,1} & w_{Nd,2} & \cdots & w_{Nd,(N+1)d} \\ v_{11} & v_{12} & \cdots & v_{1,(N+1)d} \\ v_{21} & v_{22} & \cdots & v_{2,(N+1)d} \\ \vdots & \vdots & \ddots & \vdots \\ v_{d,1} & v_{d,2} & \cdots & w_{d,(N+1)d} \end{bmatrix}, \quad \widetilde{R} = \begin{bmatrix} r(t_0) \\ r(t_1) \\ \vdots \\ r(t_{N-1}) \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_d \end{bmatrix}.$$

4 An initial guess to the Newton's method

In the partial interval $I_k = [(k-1)l, kl]$, we put $a = (k-1)l,\ b = kl$. Now we are going to obtain an initial guess for

$$\begin{cases} U'(t) = g(t,U(t)),\ t \in [a,b], \\ U(a) = U_a. \end{cases} \tag{32}$$

Integration of (32) on $a \le \tau \le t$ implies that

$$U(t) = U_a + \int_a^t g(\tau, U(\tau))d\tau. \tag{33}$$

Suppose

$$t_i^* = t_{N-i} \quad i = 0,1,\ldots,N, \tag{34}$$

where $t_n,\ n=0,1,\ldots,N$ are defined by (26). Set $t=t_i^*,\ i=1,\ldots,N$ in (33) and get

$$U(t_i^*)=U_a+\int_a^{t_i^*} g(\tau,U(\tau))d\tau=U_a+\sum_{j=1}^{i}\int_{t_{j-1}^*}^{t_j^*} g(\tau,U(\tau))d\tau. \tag{35}$$

Using the composite trapezoidal rule implies that

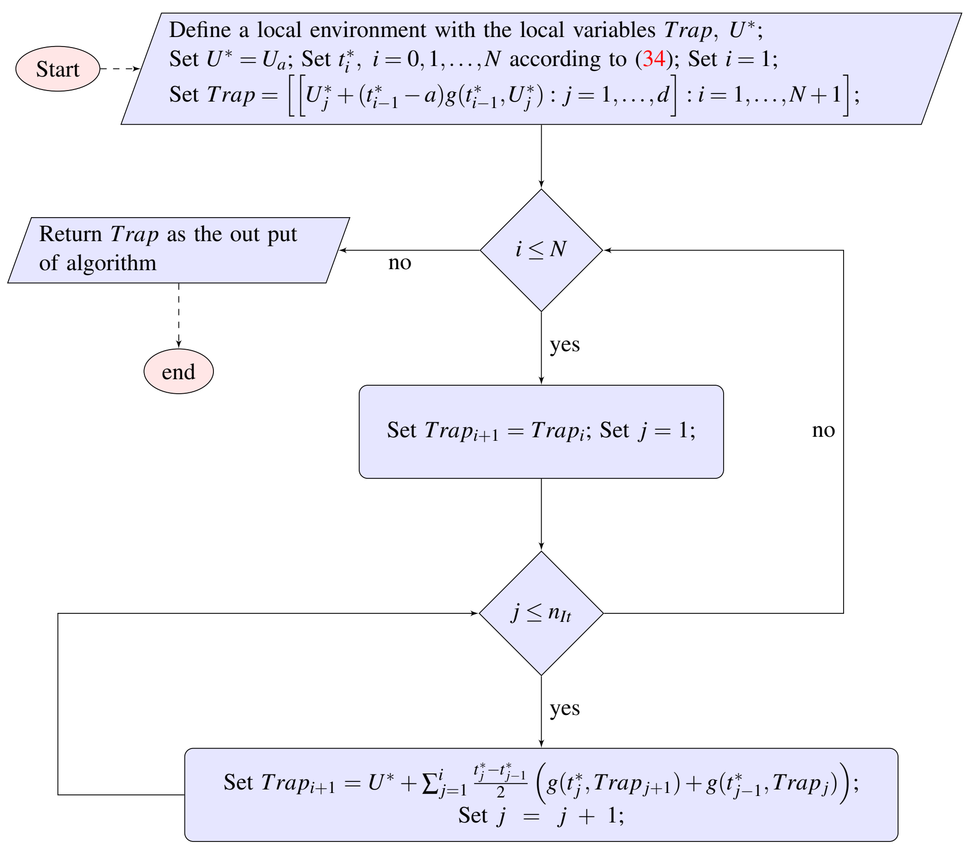
$$U_i=U_a+\sum_{j=1}^{i}\frac{t_j^*-t_{j-1}^*}{2}\left(g(t_j^*,U_j)+g(t_{j-1}^*,U_{j-1})\right). \tag{36}$$

where $U_j=U(t_j^*),\ j=0,1,\ldots,N$, are the initial guess for (32) on mesh points (34). Equations (36) are a nonlinear diagonal system of algebraic equations in $U_i,\ i=1,\ldots,N$, and we solve this system by the following iterative algorithm.

Algorithm 1 (Composite Trapezoidal iteration process)
We give a notation for denoting column vectors, and for simplicity of the mathematical programming we present the algorithm by a flowchart.

Definition 1. *Suppose V be a $d-$column vector of objects $v_i,\ i=1,\ldots,d$. Every v_i can be a scaler, vector, matrix, or other things. We denote V by $V=[v_i:i=1,\ldots,d]$.*

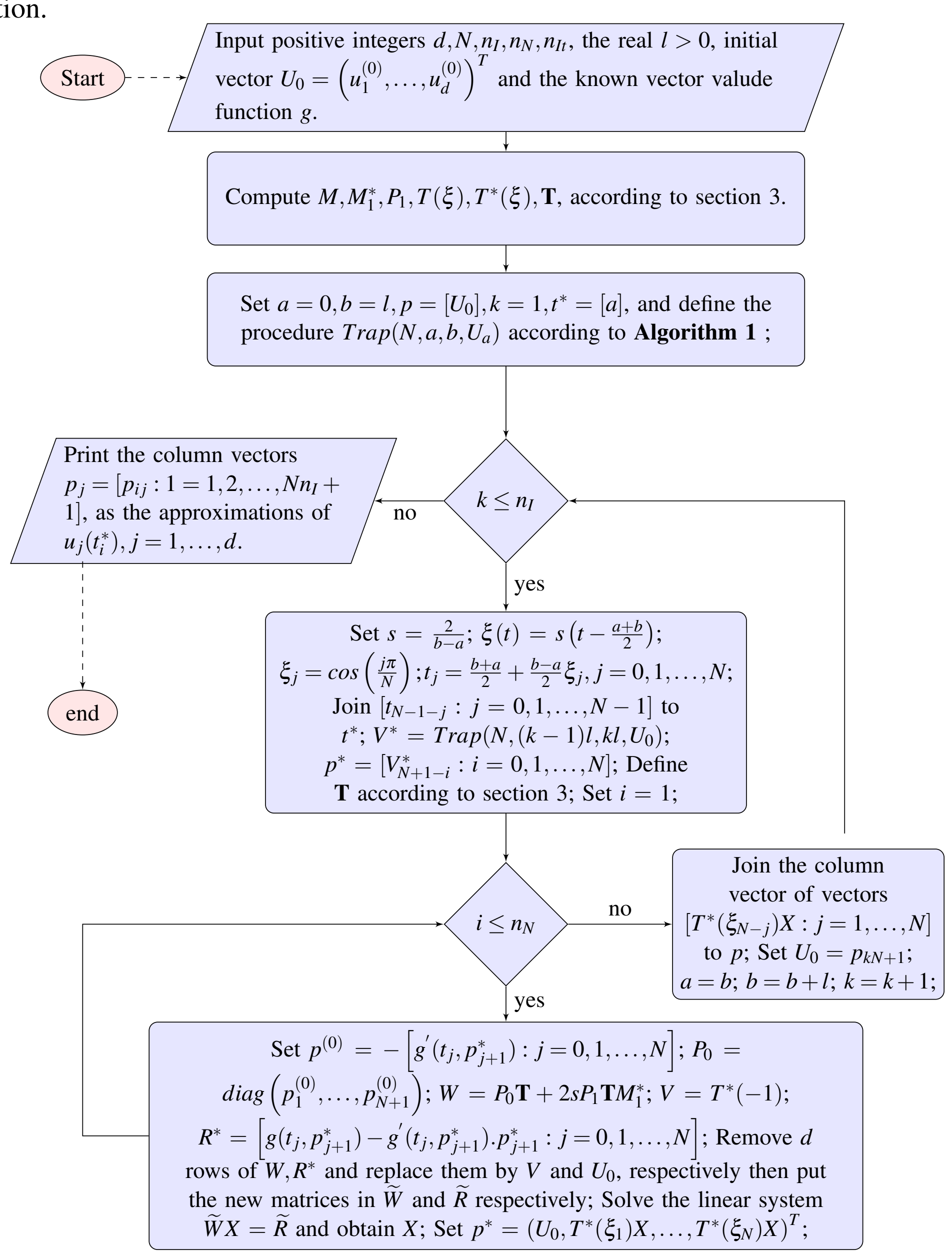
The following flowchart gives the above initial guess by a column vector of vectors.



We denote the output of the above algorithm by $Trap(N,a,b,U_a)$. The nonnegative integer n_{It} is the number of the iterative process. The above algorithm itself creates a convergence method that makes the initial guess close enough to the exact answer [6, Theorem 4.5]. Therefore, this algorithm is a suitable starter for Newton's method.

5 Total Algorithm

In this section, we give the total algorithm for the numerical solution of (1) with an appropriate initial condition.



6 Numerical Examples

Example 1. In the problem (6), with the data given in Table 1 and $U_0 = (9000, 906, 1, 1, 1, 1)^T$, we use the proposed method. In Table 2, columns 2,3 show absolute and relative errors of $\widetilde{\mathbf{N}}$ at $t_i = 80i, i = 1, \ldots, 10$. $\mathbf{N}$ is exact solution of total population size and $\widetilde{\mathbf{N}}$ is evaluated by the proposed method with $N = 4, n_I = 1600, l = 0.5, n_{it} = 5, n_N = 5$ and hence we obtain the solution on $[0, 800]$. As Table 2 shows, the error propagation is negligible in this long time interval and this is the main advantage of the method. Figure 1 shows variation of the total number of the population size is a function of t. Figure 2 shows the variation of all solutions in the model (6) as functions of t. In this example, the quantity L in Eq. (14) is $L = 1.32 \times 10^{-4}$, which is excellent for numerical computations.

Table 2: Absolute and Relative errors of $\widetilde{\mathbf{N}}$ at $t_i = 80i, i = 1, \ldots, 10$, for Example 1.

i	Absolute errors of $\widetilde{\mathbf{N}}$	Relative errors of $\widetilde{\mathbf{N}}$
1	2.24×10^{-6}	2.26×10^{-10}
2	4.46×10^{-6}	4.50×10^{-10}
3	6.67×10^{-6}	6.73×10^{-10}
4	8.86×10^{-6}	8.94×10^{-10}
5	1.10×10^{-5}	1.11×10^{-9}
6	1.32×10^{-5}	1.32×10^{-9}
7	1.53×10^{-5}	1.55×10^{-9}
8	1.74×10^{-5}	1.76×10^{-9}
9	1.95×10^{-5}	1.97×10^{-9}
10	2.16×10^{-5}	2.18×10^{-9}

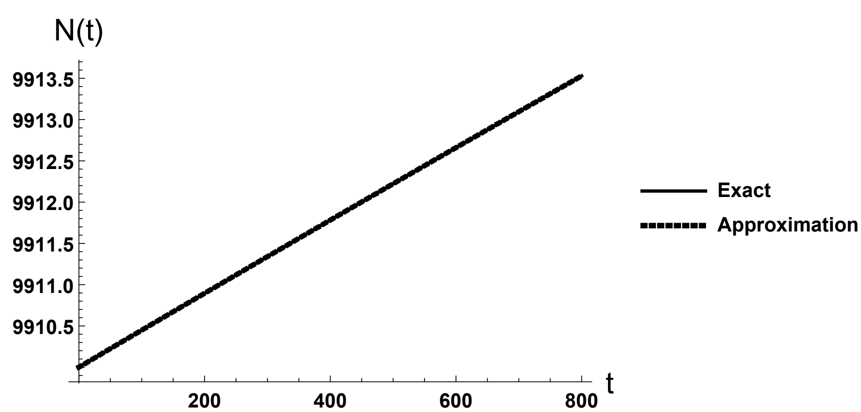


Figure 1: Variations of $\mathbf{N}$ and $\widetilde{\mathbf{N}}$ as functions of t, for Example 1.

Example 2. In this example, we consider model (7). Suppose $K = S + I_{SU} + I_{ST} + I_R + R$, then adding together all equations in (7) imply that

$$\frac{dK}{dt} = \sigma\left(\frac{\Lambda}{\mu} - K\right) + \Lambda - (\mu + \nu)S - \mu\left(I_{SU} + I_{ST} + I_R + R\right).$$

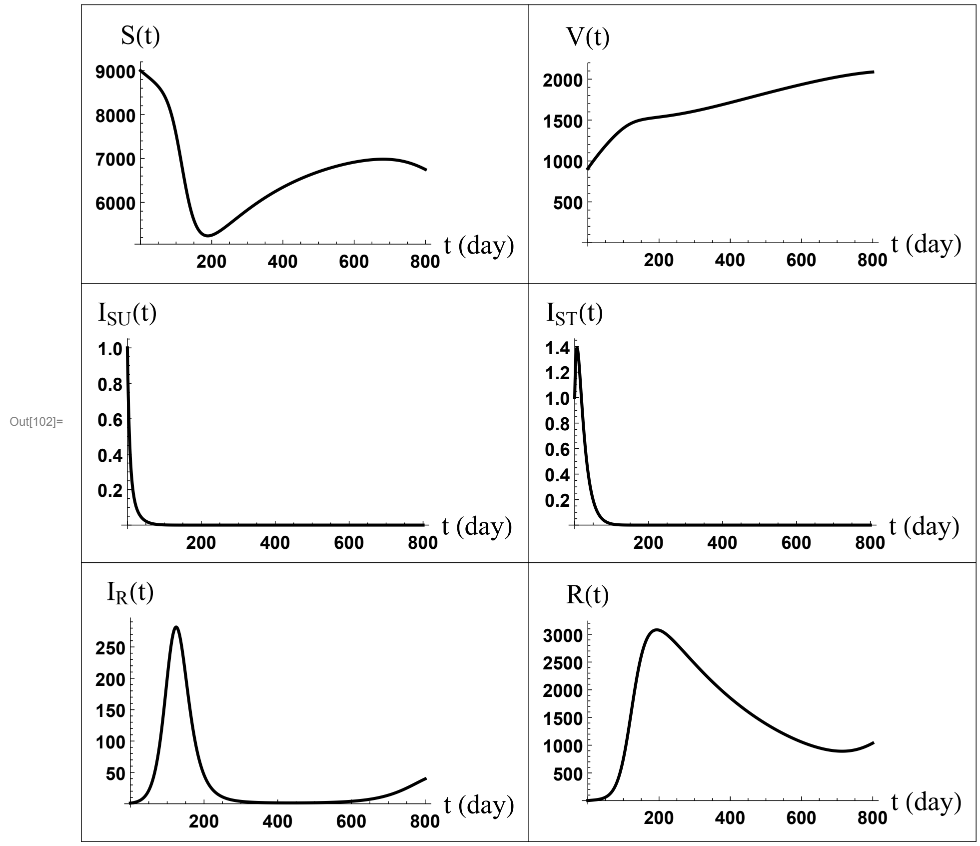


Figure 2: Variations of all classes as functions of t, for Example 1.

If we let $\nu = 0$, then we obtain

$$\frac{dK}{dt} = \frac{\sigma\Lambda}{\mu} + \Lambda - (\sigma+\mu)K, \tag{37}$$

which has the exact solution $K(t) = \frac{\Lambda}{\mu} + \left(K(0) - \frac{\Lambda}{\mu}\right) e^{-(\sigma+\mu)t}$. Other parameters are as the data given in Table 1 and $U_0 = (9000,1,1,1,1)^T$. To improve the accuracy, an initial guess obtained by the extrapolation process can be used [5]. This action increases the complexity of the algorithm, and therefore more time is required to run the program. This technique is described in reference [5] under the title of Newton-Taylor polynomial and extrapolation method (NTE). In Table 3, columns $2,3,4,5$ show absolute and relative errors of $\widetilde{K}$ at $t_i = 80i, i = 1,\ldots,10$, where $\widetilde{K}$ is the approximated of K evaluated by two methods: the proposed and reference [5] methods with $N = 4, n_I = 1600, l = 0.5, n_{it} = 5, n_N = 5$. Hence we obtain the solution on $[0,800]$. As table 3 shows, the error propagation is negligible in this long period of time and this is the main advantage of Newton's methods. Figure 3 shows the comparison between K and $\widetilde{K}$ in a period of 800 days. Figures 4 and 5 show the comparison between the solutions obtained by two methods: the Newton Chebyshev polynomial solutions method (NC) and reference [5] method (NTE). In this example, the quantity L in Eq. (14) is $L = 1.32 \times 10^{-4}$, which is excellent for numerical computations.

Table 3: Comparison of numerical results between NC and NTE methods, at $t_i = 80i, i = 1,\ldots,10$, for Example 2.

	Absolute errors of $\widetilde{K}$		Relative errors of $\widetilde{K}$	
i	NC solution	NTE solution	NC solution	NTE solution
1	7.26×10^{-2}	5.46×10^{-12}	7.88×10^{-6}	5.92×10^{-16}
2	1.14×10^{-1}	1.09×10^{-11}	1.21×10^{-5}	1.16×10^{-15}
3	1.34×10^{-1}	1.46×10^{-11}	1.40×10^{-5}	1.53×10^{-15}
4	1.40×10^{-1}	7.28×10^{-12}	1.45×10^{-5}	7.56×10^{-16}
5	1.37×10^{-1}	5.46×10^{-12}	1.41×10^{-5}	5.62×10^{-16}
6	1.29×10^{-1}	9.09×10^{-12}	1.32×10^{-5}	9.31×10^{-16}
7	1.18×10^{-1}	3.64×10^{-12}	1.20×10^{-5}	3.70×10^{-16}
8	1.05×10^{-1}	1.82×10^{-12}	1.07×10^{-5}	1.85×10^{-16}
9	9.28×10^{-2}	5.46×10^{-12}	9.38×10^{-6}	5.52×10^{-16}
10	8.08×10^{-2}	5.46×10^{-12}	8.15×10^{-6}	5.50×10^{-16}

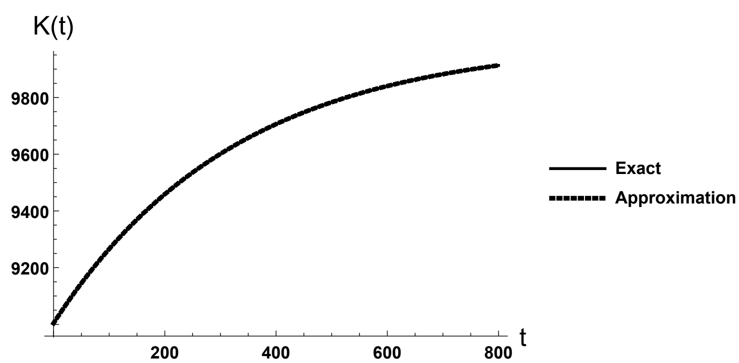


Figure 3: Variations of K and $\widetilde{K}$ as functions of t, for Example 2.

7 Conclusion

Vaccination and antiviral treatment are two important actions in preventing the spread of influenza. For this purpose, sufficient knowledge of the mathematical models of the disease is effective in determining the dose of drugs and preventive measures. Different strains of the disease can be mentioned in these models and it is also possible to prevent the occurrence of other strains that may arise later [9]. In this paper, we have presented the applicability of the proposed method for the mathematical model of such influenza diseases. The proposed method is a combination of several ways that together form a powerful technique to solve the problem. Error analysis of each method, is referred to relevant sources. So to better understand the technique, we have brought two flowcharts. The first flowchart is related to the initial guess of Newton's method and the second one is related to the total algorithm of the method. The most crucial criterion for the applicability and accuracy of the process are that the quantity in relation (14) is bounded. The smaller this positive quantity, the more ideal it is. As we can see in the sample problems, these quantities have a small positive values for influenza under study, which means that the proposed method has been chosen quite correctly.

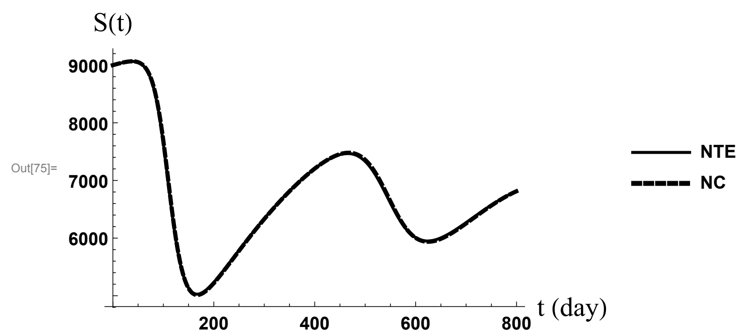


Figure 4: Comparison of numerical results between NC and NTE methods, for Example 2.

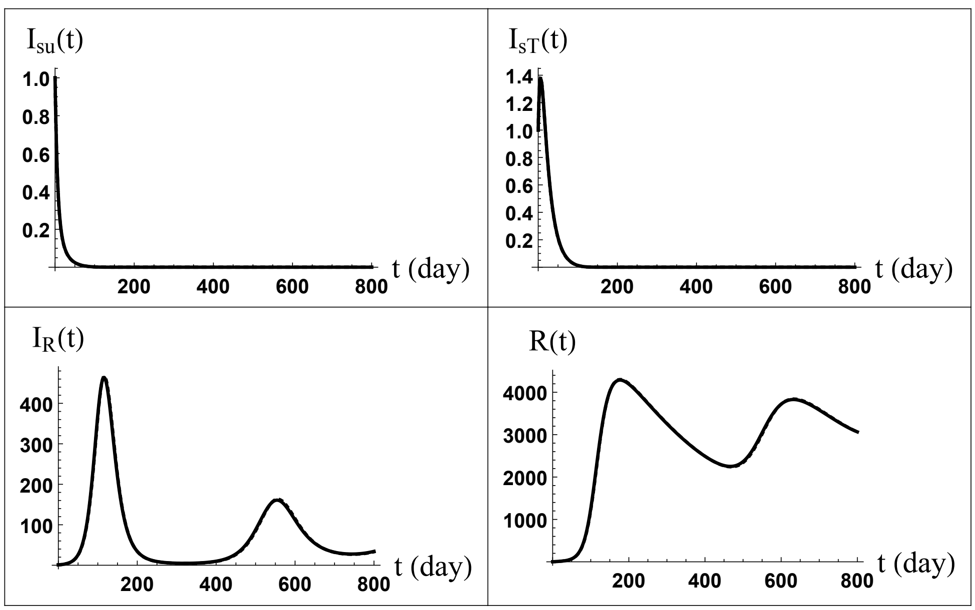


Figure 5: Comparison of numerical results between NC (dashed) and NTE (solid) methods, for Example 2.

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