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Ranking the Pareto frontiers of multi-objective optimization problems by a new quasi-Gaussian evaluation measure

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Abstract. The existence of different solution approaches that generate approximations to the optimal Pareto frontiers of a multi-objective optimization problem lead to different sets of non-dominated solutions. To evaluate the quality of these solution sets, one requires a comprehensive evaluation measure to consider the features of the solutions. Despite various evaluation measures, the deficiency caused by the lack of such a comprehensive measure is visible. For this reason, in this paper, by considering some evaluation measures, first we evaluate the quality of the approximations to the optimal Pareto front resulting from the decomposition-based multi-objective evolutionary algorithm equipped with four decomposition approaches and investigate the related drawbacks. In the second step, we use the concept of Gaussian degree of closeness to combine the evaluation measures, and hence, we propose a new evaluation measure called the quasi-Gaussian integration measure. The numerical results obtained from applying the proposed measure to the standard test functions confirm the effectiveness of this measure in examining the quality of the non-dominated solution set in a more accurate manner.

Keywords: Multi-objective optimization, evolutionary algorithm, evaluation measure, Pareto frontier, decomposition.
AMS Subject Classification 2010: 34A34, 65L05.

1 Introduction

Extensive studies have been conducted in multi-objective optimization problems (MOPs, for Multi-objective problem). Considering various constraints to adapt these problems to the real world indicates the importance of such issues in different sciences, especially in basic and engineering sciences. We refer the reader to [2, 8, 11, 14, 21] for more details.

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For instance, in 2022, Guo and Zhang proposed a simulation-based approach integrating Light Gradient Boosting Machine (LightGBM) and NSGA-III[1] to realize the automatic evacuation evaluation and adaptive optimization at metro stations in Singapore [8]. In 2022, Sharma and Chahar briefly explained multi-objective optimization algorithms and their variants together with their pros and cons [21]. In each category, they discussed in depth the representative algorithms, as well as applications of various multi-objective algorithms in various fields of engineering. In the same year, Manni and Nicolini conducted a synthetic review of the models of MOPs and their applications to the design of climate-responsive buildings and neighborhoods [14]. They identified the knowledge gaps and considered the future trends in automation in the design of buildings.

In recent years, due to the considerable development of various algorithms for the MOPs, various evaluation measures have been proposed to evaluate the generated approximate Pareto frontiers (PFs).

Generally, the quality of the approximations to the PF can be evaluated based on the following structural properties [25].

- The distance between the optimal PF and its approximations; how the points are distributed on the approximations?
- How do the approximations cover the whole PFs?

In some studies, including Knowles et al. [13] and Zitzler et al. [26], various approaches have been proposed to examine the evaluation measures. In these studies, the most appropriate measures are defined as assigning a vector of values, considering the types of quality characteristics for approximations to the PFs. In 2003, a survey was conducted [19] on several evaluation measures, in which the measures were categorized according to their structural characteristics.

Given that the evaluation measures can be considered as maps that assign scores to the approximations, Zeitler et al. studied three evaluation measures, namely, the quality measure, the ranking measure (using a standard statistical test), and the probability function measure. According to their findings, each measure provides different results, and hence, the rank assignments for approximations to PFs, which are derived from these measures, are different, although these three measures can be valid for the evaluation of the quality of solutions produced by the utilized solution approaches [27]. Collette and Siarry proposed some evaluation measures to evaluate and hence, compared the quality of two approximations to the PFs [4]. In [3], Cheng et al. first examined the PF evaluation measures of an MOP and then divided the evaluation measures into three categories: *approximate front-based, reference point-based*, and *Pareto optimal front-based*. Also, they analyzed the advantages and drawbacks of different evaluation measures. In 2014, Jiang et al. [12] studied the relationship between different evaluation measures and the processing time of concave and convex PFs.

In 2015, Riquelme et al. [20] examined the most common evaluation measures in various studies and ranked them according to their characteristics and applications. In this ranking, the top five measures were introduced as the *volume increase measure*, that is, the *Hyper Volume* (HV) *measure*, *Inverted Generation Distance* (IGD), *Diversity measure*, *Overall Non-dominated Vector Generation* (ONVG) *measure* and, the *c-* and *R-metric binary measures*. Although the HV and the IGD measures were the most well-known measures, the HV could not be an appropriate measure due to the exponential computational complexity. Recently, researchers have paid more attention to the binary measures (which examine the

[1] Non-dominated sorting genetic algorithm III (NSGA-III)

quality of two PFs) and the IGD measure (which evaluates the distance between the approximate and optimal frontiers) compared to other measures. In [1], Charles et al. reviewed fifty-seven measures to evaluate the performance of multi-objective evolutionary optimization methods up to 2020, of which HV and binary measures were the most common measures. According to their findings, as the number of objectives and reference points increases, the cost of computations increases dramatically.

In general, based on the studies conducted so far, each of the measures evaluates a PF by considering a specific feature. Accordingly, the rankings of different approximations to an optimal PF may be different, which leads to misleading results. Hence, in this paper, we propose a technique based on the concept of Gaussian closeness inspired by the concept of fuzzy dominance [18]. This approach provides some necessary conditions for a comprehensive measure to analyze the quality of different PFs, considering several characteristics simultaneously. To do so, this approach introduces a new integrated evaluation measure using the concept of Gaussian closeness and a weight vector corresponding to the utilized evaluation measures, as well as using a reference (ideal) point obtained from the data under consideration. In general, this scalar measure is a weighted combination of some essential factors that affect the quality of a set of non-dominated solutions.

This paper is organized as follows. In Section 2, we briefly describe the multi-objective evolutionary algorithm based on decomposition (MOEA/D) equipped with different decomposition approaches. Also, we recall some well-known evaluation measures from the literature. In Section 3, we introduce a new evaluation measure proposed for evaluating the performance of solution approaches used to solve the MOPs. In Section 4, considering three different evaluation measures, we analyze the experimental results in light of applying the new measure to the benchmark problems. Finally, we present a brief conclusion outlining our achievements and ideas for future works in Section 5.

2 Description of the problem

In order to show the necessity of suggesting a comprehensive evaluation measure, first of all, we need to briefly describe the general structure of the MOEA/D algorithm when it is equipped with four different decomposition approaches or solution approaches. Then, using the most common evaluation measures of the literature, we examine the quality of approximate PFs resulting from each solution approach.

Definition 1 ([24]). *In general, the mathematical model of an MOP is defined as*

$$Min \quad F(X) = (f_1(X), f_2(X), \ldots, f_m(X)) \quad s.t. \quad X \in \Omega. \tag{1}$$

Herein Ω is the solution space and $F : \Omega \to R^m$ is a vector function in R^m. If the objective functions are continuous and the solution space Ω is given by

$$\Omega = \{X \in R^n | h_j(X) \geq 0, j = 1, \ldots, p\},$$

in which the functions $h_j(X)$ are continuous, then the problem (1) is said to be a continuous MOP. Since the objectives of (1) are often conflicting, it is usually not possible to find a point in Ω that simultaneously minimizes all the objectives.

2.1 The variants of MOEA/D

In 1999, Miettinen studied the nonlinear MOP [17]. In 2001, some *scalarization* optimization approaches (to which we refer as *decomposition approaches* for short) were provided in [16]. Using these solution approaches, the MOEA/D generates an approximation to the optimal PF by solving some single-objective optimization sub-problems. We choose four arbitrary decomposition approaches which are used in the literature only to show the way our evaluation approach works. These are the *Tchebycheff* [17], *boundary intersection* (BI) [5], *penalty-based BI* (PBI) [15], and *Gaussian* (Ga) [23] decomposition approaches.

In order to express the main structure of our approach, without loss of generality, we begin with a discussion of the Ga approach.

In the following discussion, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ be a real-valued weight vector, and the vector $\boldsymbol{Z}^* = (z_1^*, z_2^*, \ldots, z_m^*)$ be the reference point (the ideal point) defined as follows.

$$z_i^* = \min\{f_i(X) | X \in \Omega\}, \quad i = 1, \ldots, m. \tag{2}$$

• The Ga approach [23]

In this decomposition approach, the single-objective optimization problem which is used in MOEA/D can be expressed as

$$\max \quad g^{Ga}(X \mid \boldsymbol{\lambda}, \boldsymbol{Z}^*) \quad s.t. \quad X \in \Omega, \tag{3}$$

where

$$g^{Ga}(X \mid \boldsymbol{\lambda}, \boldsymbol{Z}^*) = \prod_{i=1}^{m} q_i(X \mid \boldsymbol{\lambda}, \boldsymbol{Z}^*). \tag{4}$$

In the equation (4), the Gaussian values q_i, which are related to the functions f_i in (1), are calculated using the Gaussian function

$$q_i(X \mid \boldsymbol{\lambda}, \boldsymbol{Z}^*) = \exp\left(\frac{-1}{2}\left(\frac{\lambda_i f_i(X) - z_i^*}{\sigma}\right)^2\right). \tag{5}$$

Here, the vectors $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$ and $\boldsymbol{Z}^* = (z_1^*, z_2^*, \ldots, z_m^*)$ are defined as before, and the parameter σ is defined as a positive real number. A small σ indicates that the values q_i are more sensitive to the distance from the f_is to the reference point $\boldsymbol{Z}^*$, and a great value for σ can be interpreted similarly.

In (3), the optimal value of $g^{Ga}(X \mid \boldsymbol{\lambda}, \boldsymbol{Z}^*)$ belongs to $(0, 1]$, to which we refer to the degree of closeness of the vector $F(X)$ to the vector $\boldsymbol{Z}^*$. In other words, as the optimal value of (3) approaches to 1, the objective vector $F(X^*)$ tends to the reference point Z^*. It can be easily proved that the optimal solution of (3) is an optimal solution of (1) (see Theorem 1 of [23]).

2.2 The structure of MOEA/D

In the first step, MOEA/D decomposes the MOP (1) into N single-objective optimization sub-problems and then solves these sub-problems simultaneously. In each generation, this algorithm forms a population of the best solutions obtained for the sub-problems. The neighborhood relations among these sub-problems are defined based on the distances between their weight vectors. Since the optimal solutions

of the neighboring sub-problems are close to each other due to the continuity of the problem, each sub-problem in MOEA/D is optimized only using the information of the neighborhood of the corresponding sub-problem [24].

More precisely, we assume that the decomposition approach used in MOEA/D is the Gaussian decomposition approach. We also assume that $\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \ldots, \boldsymbol{\lambda}^N$ is a set of weighted vectors with uniformly distribution, and the vector Z^* is the reference point obtained from (2). In other words, the approximate PF of problem (1) can be resulted from solving the N single-objective optimization sub-problems concerning each of the decomposition approaches. Here, we use the Ga approach (3) for instance, where the objective function j of the sub-problem related to $\boldsymbol{\lambda}^j$ is defined as

$$g^{Ga}(X \mid \boldsymbol{\lambda}^j, \boldsymbol{Z}^*) = \exp\left(\frac{-1}{2}\left(\frac{\boldsymbol{\lambda}^j f_j(X) - Z^*}{\sigma}\right)^2\right). \tag{6}$$

It is worth noting that, since the function $g^{Ga}(\cdot)$ is continuous withe respect to the vector $\boldsymbol{\lambda}$, the optimal solution $g^{Ga}(X \mid \boldsymbol{\lambda}^j, \boldsymbol{Z}^*)$ approaches the optimal solution $g^{Ga}(X \mid \boldsymbol{\lambda}^k, \boldsymbol{Z}^*)$ as the weight vector $\boldsymbol{\lambda}^j$ approaches the weight vector $\boldsymbol{\lambda}^k$. Hence, any information about g^{Ga} obtained from the weight vectors close to $\boldsymbol{\lambda}^j$ is helpful for the optimization of problem (1). In MOEA/D, we define a neighborhood of the weight vector $\boldsymbol{\lambda}^j$ as a set of weight vectors with respect to the Euclidean distance from $\boldsymbol{\lambda}^j$, and accordingly, the neighborhood of the j^{th} sub-problem is considered as all the sub-problems corresponding to the neighborhood of weight vectors $\boldsymbol{\lambda}^j$. The final PF contains the best solution found among all the sub-problems. To optimize a sub-problem in MOEA/D, only the current solutions of the neighboring sub-problems are examined. To understand how the MOEA/D works, we refer the reader to [24].

The process of embedding the Ga approach (3) in MOEA/D is described in Algorithm 1. In Step, (2-4) of the MOEA/D described in [24], we use the Ga approach to compare the two Pareto solutions.

It is worth noting that for a more careful investigation, we choose the values of the parameters N and T the same as those in [24]. Moreover, we consider the operators used in Steps (2-1) and (2-2) as a single-point crossover and a traditional mutation with a probability of 0.01, respectively. In Step (2-4), according to the concept of Gaussian closeness, newly generated solutions are added to the set of Pareto solutions (namely, the APF) if the Gaussian degree of closeness of the corresponding vector of objective functions to the reference point Z^* is higher than all neighboring solutions in the APF.

As mentioned before, many evaluation measures exist in the literature that have been used to evaluate and hence, compare the performance of MOEAs. These measures examine and rank different PFs concerning some characteristics of the structure of the PFs. In what follows, it is necessary to briefly introduce three well-known evaluation measures used by many researchers (see [7, 23], and [24] for example).

- **The measure of distance from the reference point (The measure M_1) [10]**
  In this measure, a reference point of an MOP is considered, and the (Euclidean) distances from the points on the PF to the reference point are calculated. The lower the average distance of the PF points from the reference point, the higher the quality of the PF.

- **The IGD measure (The measure M_2) [24]**
  Suppose that A is an approximation to the PF, and P^* is a set of points along the optimal PF which are uniformly distributed. The average distance of A from P^* is defined by

$$D(A, P^*) = \frac{\sum_{v \in P^*} d(v, A)}{|P^*|},$$

Algorithm 1 MOEA/D based on the Ga approach (MOEA/GD)

Input: The number of sub-problems (N), the cardinality of each neighborhood set (T), the weight vectors $\{\lambda^1, \lambda^2, \ldots, \lambda^N\}$ with uniform distribution, and the stopping criterion

Output: An approximate PF (APF).

Step 1. Initialization phase

(1-1) APF $:= \{\ \}$.

(1-2) Find the T weight vectors which are closest to $\boldsymbol{\lambda}^i$ with respect to the Euclidean distance for each weight vector $\boldsymbol{\lambda}^i$. Let $B(i) = \{i_1, i_2, \ldots, i_T\}$ be the set of indices of vectors $\boldsymbol{\lambda}^{i_1}, \boldsymbol{\lambda}^{i_2}, \ldots, \boldsymbol{\lambda}^{i_T}$, the T closest weight vectors to $\boldsymbol{\lambda}^i$.

(1-3) Generate an initial population $X^1, X^2, \ldots, X^N$ of solutions randomly, and calculate the objective values $FV^i = F\left(X^i\right) \quad i = 1, \ldots, N$ of (1).

(1-4) Calculate the initial reference vector $\boldsymbol{Z}^* = (z_1^*, z_2^*, \ldots, z_m^*)^T$ by considering (2).

Step 2. Updating phase

(2-1) **Reproduction**: Choose two random solutions X^k and X^l with $k, l \in B(i)$, and use a crossover operator to generate a new solution Y.

(2-2) **Improvement**: Apply a local search method to Y and generate an improved solution Y'.

(2-3) **Update the vector** Z^*: For any j, if $z_j^* > f_j(Y')$, then set $z_j^* := f_j(Y')$, $j = 1, \ldots, m$.

(2-4) **Update the Neighboring solutions**: For each $j \in B(i)$, if $g^{Ga}(Y'\lambda^j, Z^*) \geq g^{Ga}(X^j\lambda^j, Z^*)$, set $X^j := Y'$ and $FV^j := F(Y')$.

(2-5) **Update the** APF: If no vectors in APF dominate $F(Y')$, then APF $:=$ APF$\cup F(Y')$. Remove all the vectors which are dominated by $F(Y')$ from the APF.

Step 3. Stopping criteria If the stopping criteria are met, consider the APF as an approximation to the PF and then stop. Otherwise, return to **Step 2**.

where $d(v, A)$ is the minimum Euclidean distance v from the points in A.
It is noteworthy that one can evaluate both the convergence and diversity of A, if the cardinality of P^* is large enough to define the PF [24]. To obtain a small value for $D(A, P^*)$, the set A should be close to the PF and also, no part of the PF should be missed.

- **The GEM measure (The measure M_3) [23]**

  Considering the structural nature of the measure M_1, and based on the Gaussian degree of closeness, Zahiri et al. introduced the *Gaussian evaluation measure* (*GEM*) to maintain the benefits of M_1 and calculate the degree of dominance of points on the PFs. In this measure, after examining the Gaussian closeness of each A_j to the vector Z_j^*, which is denoted by g_j^{Ga} $(j = 1, \ldots, K)$, the frontiers are sorted in the descending order of g_j^{Ga} $(j = 1, \ldots, K)$ (the arrangement $A_1, A_2, \ldots, A_K$

results from the implementation of K different decomposition approaches in the MOEA/D), and the PF with the maximum value of g^{Ga} is selected as the best approach.

2.3 Some notes on the evaluation measures

Although different evaluation measures can be found in the literature to evaluate the performance of various approaches available for solving the MOP, taking the structures of these measures into account, it may not be possible to use a specific measure to evaluate the PFs comprehensively. More precisely, the measure M_1 only considers the distance or closeness of the non-dominated solutions to a reference point, and is less sensitive to the structure of the PF [10]. The measure M_2 does not work well for the PFs which are not uniformly distributed, or those for which the diversity of solutions to the PF is low [22]. Also, when we consider the evaluation measure M_3, no attention is paid to the distribution or diversification of the solutions as factors that affect the quality of the PF, even though a scalar value (including the concepts of Pareto dominance and degree of closeness of the PF solutions to a reference point) is assigned to each PF [23].

In general, measuring the quality and ranking the PFs resulting from different solution approaches depend on the structure of the problems. Hence, it makes sense to design an evaluation measure considering the structural characteristics. To do that, in the next section, we present a technique to integrate the evaluation measures while preserving the advantages of each utilized measure. Doing so, we are going to provide some necessary conditions to rank the solution approaches, irrespective of the number of utilized evaluation measures.

3 The integrated evaluation measure

As mentioned before, each evaluation measure focuses on some special characteristics of the structure of a PF. For example, the measure M_2 implicitly addresses the dispersion and diversity of a frontier. Also, some measures in the literature, such as the quality measure (see [7, 24] for more details), only consider the number of non-dominated solutions on a frontier. Based on the nature of M_3, even though this measure focuses on some factors affecting the quality of a PF, one observes that the measure cannot to consider all the structural factors needed for evaluating the quality of non-dominated solutions set. Therefore, to carefully study the performance of different solution approaches generating different PFs, it is necessary to use different evaluation measures. Nevertheless, based on the experimental results, making such an effort might produce misleading results. We describe it in more detail in Section 4.

Accordingly, in this section, using the concept of degree of closeness to the best, we integrate numerical results obtained from the different evaluation measures. This approach can be introduced as a new evaluation measure that on the one hand evaluates the quality of different approximations to the PFs concerning different characteristics and on the other hand, overcomes the challenge of using several measures simultaneously to evaluate the quality of PFs. In what follows, we explain the details of this proposed approach as a new evaluation measure.

3.1 The quasi-Gaussian integrated measure

Assume that S evaluation measures are used to evaluate approximations to the PF resulting from implementing of K solution approaches for an MOP. Moreover, assume that the measure vector $V^k =$

$(v_1^k,\ldots,v_S^k)$ represents the scalar values of the S evaluation measures produced by the k^{th} solution approach $(k=1,\ldots,K)$. To evaluate, and hence to rank, the vectors $V^1,V^2,\ldots,V^K$ corresponding to the K solution approaches, the following three steps are considered.

In the first step, we calculate the components of the reference point $Z^*=(z_1^*,\ldots,z_S^*)$ by calculating the best scalar value among the K solution approaches. In other words, by considering K solution approaches, we calculate the z_i^* corresponding to the measure i as

$$z_i^* := \text{opt}_{k=1}^{K}\left(v_i^k\right), \quad i=1,\ldots,S.$$

Here, opt indicates maximum or minimum, depending on the type of the evaluation measure.

In the second step, we consider the weight vector $\overrightarrow{\lambda}=(\lambda_1,\ldots,\lambda_S)\in(0,1]^S$. A scalar value Q_k obtained from integrating the S evaluation measures considering the solution approach k can be calculated as

$$Q_k = Q(V^k \mid \boldsymbol{\lambda}, Z^*) = \prod_{i=1}^{S} \exp\left(\frac{-1}{2}\left(\frac{\lambda_i v_i^k - z_i^*}{\sigma_k}\right)^2\right). \tag{7}$$

Here, the value of the parameter σ_k is set to 0.5. In general, this parameter depends on the values of all the measures used to evaluate the solution approach k.

Since we are going to introduce an evaluation technique for approximations to the PF, sensitivity analysis of this parameter is not considered in this paper.

It is clear that the value of Q_k corresponding to the solution approach k is a scalar in $(0,1]$. Therefore, as Q_k gets closer to 1, the quality of solutions generated by the solution approach k increases.

It is worth noting that in general, the components of the vector $\boldsymbol{\lambda}$ can be defined as a convex linear combination that depends on the importance of the evaluation measures. In this paper, the importance of all evaluation measures is identical and equals 1 (that is, $\overrightarrow{\lambda} =: \overrightarrow{1} = (1,\ldots,1)\in\mathbb{R}^S$).

In the third and so, final step, the maximum values Q_k are considered as an evaluation measure for ranking the solution approaches with respect to their performance to generate the non-dominated solutions set. Intuitively speaking, according to Table 1, we can consider the solution approach p as the best approach among the K solution approaches if $Q_p = \max\limits_{k=1,\ldots,K} Q_k$.

In what follows, for a more convenient description, we call the proposed integrated measure as the *quasi-Gaussian integration measure* and denote it by *GIM*.

Table 1: Integration of several evaluation measures to determine the best solution approach.

Measures / **Solution approach**	measure 1	measure 2	$\cdots$	measure K	Q_k
Approach 1	v_1^1	v_2^1	$\cdots$	v_S^1	Q_1
$\vdots$					$\vdots$
Approach p	v_1^p	v_2^p	$\cdots$	v_S^p	Q_P
$\vdots$					$\vdots$
Approach K	v_1^K	v_2^K	$\cdots$	v_S^K	Q_K
The solution approach p is selected as the best one.					$Q_p = \max\limits_{k=1,\ldots,K} Q_k$

3.2 Theoretical discussion

In this section, we discuss some theoretical issues related to the proposed GIM. First of all, we show that if all utilized evaluation measures select a solution approach p as the best solution approach, then *GIM* also introduces the same solution approach as the superior one. Without loss of generality, assume that all evaluation measures are of the minimum type (that is, the minimum value is chosen as the best). To prove this claim, suppose that all S utilized evaluation measures find the solution approach p as the best solution approach. In other words, according to the notations used in Table 1, the measure vector of the solution approach p, namely, $V^p = \left(v_1^p, v_2^p, \ldots, v_S^p\right)$, is obtained as follows.

$$v_i^p = \min_{k=1}^{K} v_i^k, \quad i = 1,2,\ldots,S.$$

Thus,

$$\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^p - z_i^*\right)^2 > \frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^k - z_i^*\right)^2, \quad p \neq k, \quad k = 1,2,\ldots,K.$$

Since the exponential function $\exp(x)$ is strictly increasing, we obtain

$$\begin{aligned}
&\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^p - z_i^*\right)^2\right) > \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^k - z_i^*\right)^2\right) \\
\Rightarrow &\prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(v_i^p - z_i^*\right)^2\right) > \prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(v_i^k - z_i^*\right)^2\right), \quad p \neq k, \quad k = 1,2,\ldots,K.
\end{aligned}$$

Now, considering equation (7), we obtain Q_p as follows

$$Q_p = Q\left(V^p|\overrightarrow{1}, Z^*\right) > Q_k = Q\left(V^k|\overrightarrow{1}, Z^*\right), \quad p \neq k, \quad k = 1,2,\ldots,K.$$

Therefore, based on GIM, we observe that the solution approach p outperforms the other solution approaches.

Lemma 1. *Let the vectors $V^1 = (v_1^1, \ldots, v_S^1)$ and $V^2 = (v_1^2, \ldots, v_S^2)$ be two non-dominated measure vectors corresponding to the solution approaches 1 and 2, respectively. Also, let us assume that all the evaluation measures are of the minimum type. Then, the solution approach 1 outperforms the solution approach 2 if*

$$\sum_{i=1}^{S}\left(v_i^1\right)^2 < \sum_{i=1}^{S}\left(v_i^2\right)^2.$$

Proof. Since $z_i^* = \min_{k=1}^{K} v_i^k \;\; (i = 1,2,\ldots,S)$, for all $i = 1,2,\ldots,S$ we obtain $\left(v_i^k - z_i^*\right) \geq 0$. Now, by considering the assumption, it is straightforward to show that

$$\sum_{i=1}^{S}\left(v_i^1 - z_i^*\right)^2 < \sum_{i=1}^{S}\left(v_i^2 - z_i^*\right)^2.$$

Therefore, having in mind that $\exp(x)$ is strictly increasing, we have

$$\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^1-z_i^*\right)^2\right) > \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(v_i^2-z_i^*\right)^2\right)$$
$$\Rightarrow \prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(v_i^1-z_i^*\right)^2\right) > \prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(v_i^2-z_i^*\right)^2\right).$$

From equation (7) we conclude that

$$Q_1 = Q\left(V^1|\overrightarrow{1},Z^*\right) > Q_2 = Q\left(V^2|\overrightarrow{1},Z^*\right).$$

Therefore, the solution approach 1 outperforms the solution approach 2. □

Remark 1. *From Lemma 1 we easily conclude that by considering the proposed GIM, the superiority of different solution approaches depends on the number of non-domination points and also the amount of Gaussian closeness of non-dominated points to the reference point.*

The following lemma generalizes Lemma 1 to the case where different preferences are allowed for the evaluation measures.

Lemma 2. *Suppose that the vectors $V^1=(v_1^1,\ldots,v_S^1)$ and $V^2=(v_1^2,\ldots,v_S^2)$ are two non-negative and non-dominated measure vectors corresponding to the two different solution approaches 1 and 2, respectively. Furthermore, assume that all evaluation measures are of the minimum type. Then, the solution approach 1 outperforms the solution approach 2 if the following conditions are satisfied:*

i) $\sum_{i=1}^{S}\left(v_i^1\right)^2 < \sum_{i=1}^{S}\left(v_i^2\right)^2$,

ii) $\lambda_i \geq \max_{k=1}^{S}\left\{\frac{z_k^*}{v_k^1}\right\}, \quad i=1,\ldots,S.$

Herein, the weight vector $\overrightarrow{\lambda}=(\lambda_1,\ldots,\lambda_S)\in(0,1]^S$ corresponding to the S evaluation measures and the reference points z_i^* $(i=1,2,\ldots,S)$ is defined as in Lemma 1.

Proof. According to the assumption,

$$\lambda_i \geq \max_{k=1}^{S}\left\{\frac{z_k^*}{v_k^1}\right\}, \quad i=1,\ldots,S.$$

Then, for all $i=1,\ldots,S$, we get

$$\lambda_i \geq \left\{\frac{z_i^*}{v_i^1}\right\} \Rightarrow \left(\lambda_i v_i^1 - z_i^*\right) \geq 0.$$

As a result, by assuming $0<\lambda_i\leq 1$, from $\left(v_i^1\right)^2 < \left(v_i^2\right)^2$ we obtain $(\lambda_i v_i^1 - z_i^*)^2 < (\lambda_i v_i^2 - z_i^*)^2$, for $i=1,\ldots,S$, and therefore,

$$\sum_{i=1}^{S}(\lambda_i v_i^1 - z_i^*)^2 < \sum_{i=1}^{S}(\lambda_i v_i^2 - z_i^*)^2.$$

Consequently,

$$\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(\lambda_i v_i^1 - z_i^*\right)^2\right) > \exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{S}\left(\lambda_i v_i^2 - z_i^*\right)^2\right)$$
$$\Rightarrow \prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(\lambda_i v_i^1 - z_i^*\right)^2\right) > \prod_{i=1}^{S}\exp\left(\frac{-1}{2\sigma^2}\left(\lambda_i v_i^2 - z_i^*\right)^2\right).$$

By considering (7), we obtain Q_1 as follows

$$Q_1 = Q\left(V^1|\overrightarrow{\lambda},Z^*\right) > Q_2 = Q\left(V^2|\overrightarrow{\lambda},Z^*\right).$$

Therefore, the solution approach 1 outperforms the solution approach 2. □

According to Lemma 2, making changes to the values of $\boldsymbol{\lambda}_i$'s makes changes to the closeness of the v_i's to the reference point. This conclusion reveals the role played by the vector $\boldsymbol{\lambda}$ for the ranking of the PFs.

4 The experimental results

According to what we discussed in Section 3, we numerically examine the performance and accuracy of the proposed evaluation measure, namely, GIM. We use a standard set of library problems. The benchmarks enable us to investigate the performance of MOEA intuitively. In 2002, Deb et al. [6] introduced one of the most common test problems conceived for the MOPs, called DTLZ, which took its name from the researchers who proposed it, namely, Deb, Thiele, Laumanns and Zitzler [6]. These problems are of minimum type, each of which includes special features such as convex or concave PFs, and spherical fronts with uniform or non-uniform distribution, as well as various local optimal PFs.

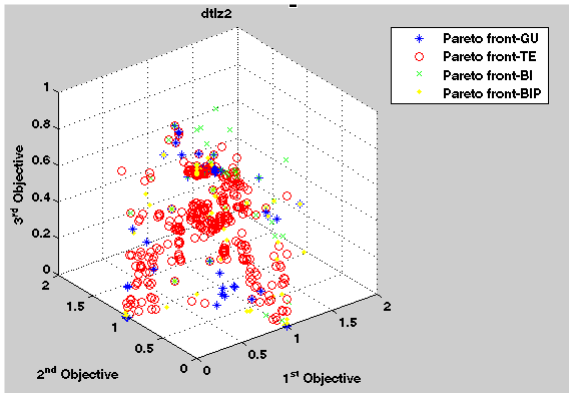
Here, as mentioned before, we embed four decomposition approaches, namely, the Gaussian, Tchebycheff, BI, and BIP approaches, in MOEA/D (see Section 2), and denote them by D_{Ge}, D_{Te}, D_{BI} and D_{BIP} respectively, to generate the four approximations to the PFs.

To discuss the performance of the four decomposition approaches generating approximate PFs, similar to [9] and [22], the value of σ is set to 0.5. Moreover, the related parameters including the problem dimensions, the number of iterations, and the cardinality of the initial population are set to 3, 100 and 50, respectively. Moreover, all the weight vectors and also, the initial populations utilized for the four decomposition approaches, are the same.

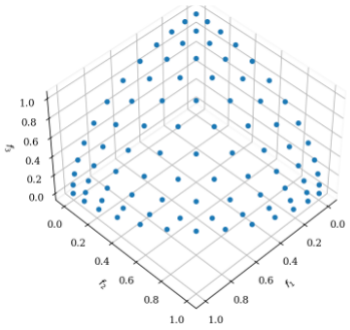
We analyzed the performance of the four solution approaches applied to the DTLZ instances with respect to the three evaluation measures (i.e., M_1, M_2, M_3)[1], and then re-evaluated them by using the proposed GIM. Accordingly, we ranked the approximate PFs and hence, identified the best solution approach. The numerical results are reported in Table 2[1]. Note that the measures M_1 and M_2 are of minimum type, and M_3 is of maximum type.

[1] The value P^* (the optimal PF) of DTLZs is adapted from https://www.cs.cinvestav.mx/~emoobook/apendix-e/apendix-e.html.

[1]Computations are done in a system with CPUE5_2650v3@2.3 *GHz intel(R) Xeon(R)RAM32GB*, and MOEA/D was coded in *MATLAB* software.



(a) Approximations of the PFs of DTLZ2 using Algorithm 1 with the four solution approaches



(b) Pareto optimal PF of DTLZ2

Figure 1: The PF of DTLZ2.

By considering DTLZ1, Table 2 allows us to compare the quality of approximations to the PF that are generated by the four solution approaches. Based on these results, considering the measure M_3, D_{Ga} performs better than the three others, D_{BIP} is in the second rank and, D_{Te} and D_{BI} are in the third position, that is, their performances are the same. Also, D_{Ga} is found as the superior solution approach considering the other two evaluation measures, namely, M_1 and M_2. However, considering the measure M_2, D_{BIP}, D_{Te} and D_{BI} take the second to the fourth ranks, respectively. Note that by considering the measure M_1, the solution approaches D_{Te}, D_{BIP} and D_{BI} are in the second to the fourth ranks, respectively. As shown in this table, all the three measures M_1,M_2, M_3, and hence the proposed GIM, find D_{Ge} as the superior solution approach. This is in line with Lemma 1.

The related results of DTLZ2 can be described as follows. As the second test function indicates, the ranks of the decomposition approaches depend on the utilized evaluation measures. In other words, various evaluation measures lead to different rankings (conflicting ranks). A closer look at Table 2 reveals the fact that considering the measures M_1 and M_3 for DTLZ2, non-dominated solutions generated by D_{Ga}, D_{Te} and D_{BIP} have the same quality. In this case, D_{BI} is the weakest solution approach. Nevertheless, if we choose the measure M_2, the PF related to D_{BI} has the highest quality and hence, D_{BI} outperforms three others.

The results of GIM are in line with the above discussion. Since the results of the three measures are nearly the same, GIM indicates that all solution approaches have the same performance and are slightly superior to D_{Te}.

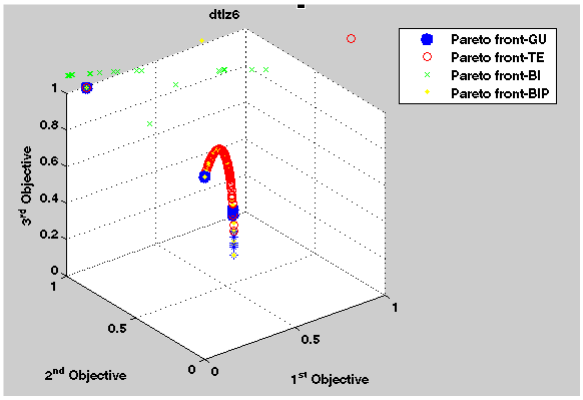
Now, let us consider the approximations to the PF which are generated by the four solution approaches. Although we observe that these approximations are not particularly superior to each other (see Figure 1), Remark 2 below may be derived.

Remark 2. *Referring to Figure 1 and Table 2, we observe that by integrating the four approximations of PFs, we obtain the definition of the final PF. In other words, for some cases, considering both criteria of diversity and quality, the frontiers which are generated by several solution approaches (with different or even the same ranks) seem to be complement to each other to form a good approximation to the whole PF.*

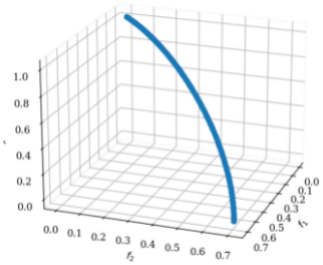
Table 2: Ranking of the solution approaches by using the proposed GIM.

Test problems	Evaluation measure / Solution approach	M_1	M_2	M_3	GIM
DTLZ1	D_{Ga}	0.4666	0.0102	0.8454	1
	D_{Te}	2.6349	1.4533	0.0024	3.1×10^{-6}
	D_{BI}	79.5844	13.3375	0.0024	0
	D_{BIP}	2.8918	1.2708	0.0044	1.02×10^{-6}
	The best solution approach				D_{Ga}
DTLZ2	D_{Ga}	1.000	0.4014	0.1353	0.9021
	D_{Te}	1.002	0.1859	0.1353	0.9997
	D_{BI}	1.0298	0.1744	0.1201	0.9978
	D_{BIP}	1.000	0.4014	f0.1353	0.9021
	The best solution approach				D_{Te}
DTLZ3	D_{Ga}	1.9995	1.2824	0.1353	1.000
	D_{Te}	2.5067	1.7531	0.0024	0.3838
	D_{BI}	138.4056	f7.7071	f0.1348	0
	D_{BIP}	2.0006	1.3925	0.0044	0.9761
	The best solution approach				D_{Ga}
DTLZ4	D_{Ga}	1.000	0.4222	0.1353	1.000
	D_{Te}	1.000	0.4206	0.1353	1.000
	D_{BI}	1.2800	0.5127	0.1353	0.8405
	D_{BIP}	1.000	0.5521	0.1353	0.9660
	The best solution approach				D_{Ga} and D_{Te}
DTLZ5	D_{Ga}	1.000	0.4997	0.1353	1.000
	D_{Te}	1.0006	0.3178	0.1353	0.9496
	D_{BI}	1.0488	0.5461	0.1353	1.000
	D_{BIP}	1.000	0.4997	0.1353	1.000
	The best solution approach				D_{Ga}, D_{BI},D_{BIP}
DTLZ6	D_{Ga}	1.000	0.0471	0.1353	1.000
	D_{Te}	1.000	0.1935	0.1353	0.9581
	D_{BI}	2.7600	1.9451	0	0
	D_{BIP}	1.000	0.0471	0.1353	1.000
	The best solution approach				D_{BIP} and D_{Ga}
DTLZ7	D_{Ga}	2.8817	0.3512	0.6137	0.9960
	D_{Te}	3.9942	0.3063	0.6056	0.0841
	D_{BI}	16.6344	2.5208	0.6056	0
	D_{BIP}	2.8817	0.9527	0.6139	0.4335
	The best solution approach				D_{Ga}

For the case DTLZ3, similar to DTLZ2, the approximations to the PF resulting from the four decomposition approaches are compared. By considering the three measures M_1, M_2 and M_3, we are unable to rank the decomposition approaches, appropriately. In particular, when the measure M_1 is considered, we



(a) Approximations of the PFs of DTLZ6 using Algorithm 1 with the four decomposition approaches



(b) The optimal PF of DTLZ6

Figure 2: Approximations to the PFs for DTLZ6.

observe that D_{Ga} and D_{BIP} perform identically, and they outperform both D_{Te} and D_{BI}. In this case, D_{BI} is the worst one. However, if the measure M_2 is considered, D_{BI} and D_{Ga} outperform the others.

Remark 3. *The rank of a PF depends on how the approximations of the PFs are evaluated, and it seems that GIM overcomes this drawback.*

Similar arguments apply to the instances of DTLZ3, DTLZ4, DTLZ5, DTLZ6 and DTLZ7. The approximations of the PFs resulting from applying the four decomposition approaches to DTLZ5 and DTLZ6 show that the solution approaches are not particularly superior to each other. Hence, the idea of merging the different fronts to make a better approximation of the PF becomes more evident. For the sake of brevity, we only depicted the approximations for DTLZ6 in Figure 2.

For DTLZ6, the numerical results presented in Table 2 indicate the superiority of the performance of the three approaches D_{Ga}, D_{Te} and D_{BIP} over D_{BI}, irrespective of the utilized measure. So, GIM obtains the same result.

Remark 4. *The obtained experimental results are in line with the lemmas stated in Section 3.1.*

Remark 5. *Based on the obtained results, considering several evaluation measures, no particular heuristic algorithm exists to perform well for all MOPs. Therefore, the issue of integrating the PFs resulting from all the utilized solution approaches can be the basis of introducing a new technique that produces a non-dominant set of solutions, considering the diversity and quality criteria simultaneously. This idea underlies the future studies of the researchers.*

5 Conclusion

According to the experimental results, finding different evaluation measures to evaluate the quality of Pareto frontiers produced by multi-objective optimization methods is a challenging task. In other words, as the number of evaluation measures increases, the ranking of the solution approaches is made more complex, and in some cases leads to misleading results. To overcome this drawback, in this paper, first

we investigated the quality of Pareto frontiers generated by several solution approaches by considering various evaluation measures. In the next phase, a new evaluation measure, namely, the quasi-Gaussian integration measure was proposed to evaluate the structure of Pareto frontiers from different perspectives. Based on the theoretical and experimental results, we observed that the rankings of different frontiers by calling the proposed measure aligned with the obtained results. The idea of integrating all the approximations of the PF generated by several multi-objective optimization methods is an issue that underlies the future studies of the researchers.

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