Journal of Mathematical Modeling
Vol. 11, No. 1, 2023, pp. 19–34. Research Article

JMM

Global stability and Hopf bifurcation of delayed fractional-order complex-valued BAM neural network with an arbitrary number of neurons

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Abstract. In this paper, a general class of fractional-order complex-valued bidirectional associative memory neural network with time delay is considered. This neural network model contains an arbitrary number of neurons, i.e. one neuron in the X-layer and other neurons in the Y-layer. Hopf bifurcation analysis of this system will be discussed. Here, the number of neurons, i.e., n can be chosen arbitrarily. We study Hopf bifurcation by taking the time delay as the bifurcation parameter. The critical value of the time delay for the occurrence of Hopf bifurcation is determined. Moreover, we find two kinds of appropriate Lyapunov functions that under some sufficient conditions, global stability of the system is obtained. Finally, numerical examples are also presented.

Keywords: neural network, fractional ordinary differential equations, Hopf bifurcation, time delay, Lyapunov function, global stability.
AMS Subject Classification 2010: 92B20, 34A08, 34C23, 34Kxx, 93D30, 34D23.

1 Introduction

Kosko, for the first time, introduced the bidirectional associative memory (BAM) neural networks. The advantages of BAM neural networks are that they are able to store multiple patterns, but most of neural networks have only one storage pattern or memory pattern. In fact, BAM neural networks can be applied in storing paired patterns or memories. It should be point out that these kinds of networks are able to search the desired patterns through both forward and backward directions [12]. It is well-known that a great number of periodic solutions show multiple memory patterns and also, periodic solutions can

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Received: 21 May 2022/ Revised: 15 September 2022/ Accepted: 23 September 2022
DOI: 10.22124/JMM.2022.22299.1972

be obtained from Hopf bifurcation in delayed differential equations. That is why the study of Hopf bifurcation is very important in BAM neural networks. The dynamical behaviors of simplified forms of BAM neural networks have been studied (see e.g., [3, 5, 10, 11, 20, 23–25]). In [23] a simplified tri-neuron BAM neural network with two time delays and in [20, 24, 25] simplified delayed five-neuron and six-neuron BAM neural networks have been considered. It should be noted that these models consist of one neuron in the X-layer and other three, five or six neurons in the Y-layer. However, in [10, 11], the authors studied a five-neuron model with two neurons in the X-layer.

It should be noted that there are a lot of papers on real-valued neural networks but a few on complex-valued neural networks (CVNN). These kinds of networks have complex-valued state variables, connection weights and activation functions. In fact, complex states contain two different kinds of information. Therefore, CVNNs can solve some problems that cannot be solved by their real-valued ones [8]. For example, CVNNs have applications in filtering, image processing, speech synthesis and computer vision [1, 14, 16]. Because of the above mentioned applications, the dynamics of CVNNs have been also studied. For example, in [7], the authors proposed several sufficient conditions to obtain the existence, uniqueness and global asymptotic stability of delayed CVNNs with two classes of complex-valued activation functions.

Fractional calculus was first introduced more than three centuries ago. It has attracted the attention of a lot of researchers and has been extensively applied in physics and engineering [2, 6, 15]. It is worth noticing that using fractional-order differential equations for modeling neural networks has two main advantages. First, it makes possible the description of the memory and hereditary properties of various processes. The second one is that the fractional-order parameter is a powerful tool for the performance of a system to have one more degree of freedom. In fact, the models of neural networks with fractional derivatives have neurons with a fundamental and general computation ability. Then, it leads to efficient information processing [13].

The dynamics of fractional-order BAM neural networks are also interesting for researchers. Because of the importance of periodic solutions in fractional-order BAM neural networks, some researchers have studied Hopf bifurcation in these systems. See for example [19, 21, 22]. In [17], stability and Hopf bifurcation of a class of fractional-order complex-valued single neuron model with time delay was studied. The authors in [18], investigated the global asymptotic stability of impulsive fractional-order BAM neural networks with time delay where the activation functions are real-valued. In fact, [18] is devoted to presenting a sufficient criterion for asymptotic stability of fractional-order BAM neural networks. We would like to point out that there have been a lot of works on dynamic analysis of integer-order CVNNs but there are few works on the dynamical behavior of fractional-order CVNNs.

Here, in this paper, a general class of fractional-order complex-valued bidirectional associative memory neural network with time delay is studied. This model contains an arbitrary number of neurons, i.e. one neuron in the X-layer and other neurons in the Y-layer. Here, the number of neurons i.e. n can be chosen arbitrarily. To the best of our knowledge, this model has not been studied up to now. Hopf bifurcation analysis of this system will be discussed and the associated characteristic equation is studied. To discuss the model, we assume a set of new conditions. First, some preliminaries and model description are presented. Then, Hopf bifurcation is investigated and the critical value of the time delay for the occurrence of Hopf bifurcation is determined. Moreover, we find two kinds of appropriate Lyapunov functions that under some sufficient conditions, result in global asymptotic stability of the system. Finally, numerical simulations are also given.

2 Preliminaries

In this section, we state the definitions, methods and models that are needed for other sections. The following two definitions have been stated in [15].

Definition 1. *The Caputo fractional derivative with order q of a function $f(t)$ is defined as*

$$ {}_0^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau. \tag{1} $$

Definition 2. *The Laplace transform of the Caputo fractional derivative is*

$$ L\{{}_0^C D_t^q f(t); s\} = s^q F(s) - \sum_{k=0}^{n-1} s^{q-k-1} f^k(0), \quad n-1 < q \le n, \tag{2} $$

where $F(s)$ is the Laplace transform of $f(t)$, and $f^k(0)$, $k = 0,1,2,\ldots,n-1$, are the initial conditions. If $f^k(0) = 0$, $k = 0,1,2,\ldots,n-1$, then

$$ L\{{}_0^C D_t^q f(t); s\} = s^q F(s). $$

Consider the following linear fractional-order system with time delay:

$$ {}_0^C D_t^q x(t) = -Ax(t) + Kx(t-\tau), \tag{3} $$

where $A = (a_{ij})_{n\times n}$, $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $K = (k_{ij})_{n\times n}$ and

$$ x(t-\tau) = (x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_n(t-\tau_n))^T. $$

Taking Laplace transform on both sides of (3), we have

$$ \Delta(s) = \begin{bmatrix} s^q - k_{11}e^{-s\tau_1} + a_{11} & -k_{12}e^{-s\tau_2} + a_{12} & \ldots & -k_{1n}e^{-s\tau_n} + a_{1n} \\ -k_{21}e^{-s\tau_1} + a_{21} & s^q - k_{22}e^{-s\tau_2} + a_{22} & \ldots & -k_{2n}e^{-s\tau_n} + a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -k_{n1}e^{-s\tau_1} + a_{n1} & -k_{n2}e^{-s\tau_2} + a_{n2} & \ldots & s^q - k_{nn}e^{-s\tau_n} + a_{nn} \end{bmatrix}. \tag{4} $$

By the distribution of the eigenvalues from $det(\Delta(s)) = 0$, the stability of the system can be determined.

Now, we state the delayed BAM neural network that is used for our model description. The delayed BAM neural network is described by the following system:

$$ \begin{cases} \dot{x}_i(t) = -\mu_i x_i(t) + \sum_{j=1}^{m} c_{ji} f_i(y_j(t-\tau_{ji})) + I_i, & i = 1,2,\ldots,n, \\ \dot{y}_j(t) = -\upsilon_j y_j(t) + \sum_{i=1}^{n} d_{ij} g_j(x_i(t-\sigma_{ij})) + J_j, & j = 1,2,\ldots,m, \end{cases} \tag{5} $$

where c_{ji} and d_{ij} are the connection weights through the neurons in two layers: the X-layer and the Y-layer. The stability of internal neuron processes on the X-layer and Y-layer are described by μ_i and υ_j, respectively. On the X-layer, the neurons whose states are denoted by $x_i(t)$ receive the input I_i and the inputs outputted by those neurons in the Y-layer via activation function f_i, while the similar process happens on the Y-layer. Also, τ_{ji} and σ_{ij} correspond to the finite time delays of neural processing and delivery of signals. For further details, see [12].

In [9], the following delayed BAM neural network model in general is considered:

$$\begin{cases} \dot{x_1}(t) = -\mu_1 x_1(t) + \sum_{j=1}^{n-1} c_{j1} f_1(y_j(t-\tau_2)) + I_1 \\ \dot{y_j}(t) = -\upsilon_j y_j(t) + d_{1j} g_j(x_1(t-\tau_1)) + J_j, \qquad j = 1,2,\dots,n-1, \end{cases} \tag{6}$$

where $\mu_1 > 0$, $\upsilon_j > 0$, for $j = 1,2,\dots,n-1$ and c_{j1}, d_{1j}, for $j = 1,2,\dots,n-1$ are real constants. The time delay from the X-layer to another Y-layer is τ_1, while the time delay from the Y-layer back to the X-layer is τ_2, and there are one neuron in the X-layer and other n-1 neurons in the Y-layer. It should be noted that system (6) has been studied without any special conditions on the number of neurons. In fact, the number of neurons is arbitrary.

Motivated by the model (6), in the next section, we propose a fractional complex-valued BAM neural network model with one time delay.

3 Model Description

Consider the following fractional complex-valued BAM neural network with time delay:

$$\begin{cases} {}_0^C D_t^q z_1(t) = -\mu_1 z_1(t) + \sum_{j=2}^{n} c_{j1} f_1(z_j(t-\tau)) + I_1, \\ {}_0^C D_t^q z_j(t) = -\upsilon_j z_j(t) + d_{1j} g_j(z_1(t)) + J_j, \qquad j = 2,\dots,n, \end{cases} \tag{7}$$

where the fractional order $q \in (0,1)$ and $z_j(t)$, $j = 1,2,\dots,n$ are the complex-valued states of the neurons. Also, $\mu_1 > 0$, $\upsilon_j > 0$, for $j = 2,\dots,n$ and c_{j1}, d_{1j}, $j = 2,\dots,n$ are real constants. The time delay from the Y-layer to the X-layer is τ, and in this model, there are one neuron in the X-layer and other n-1 neurons in the Y-layer. $f_1(.)$ and $g_j(.)$, $j = 2,\dots,n$ are the complex-valued activation functions.

To simplify the model (7), we rewrite $f_j = g_j, \mu_j = \upsilon_j$, $j = 2,\dots,n$ and $b_j = c_{j1}, a_j = d_{1j}$, $j = 2,\dots,n$. Now, let $I_1 = 0$ and $J_j = 0$, $j = 2,\dots,n$. Thus, we have the following system:

$$\begin{cases} {}_0^C D_t^q z_1(t) = -\mu_1 z_1(t) + b_2 f_1(z_2(t-\tau)) + b_3 f_1(z_3(t-\tau)) + \cdots + b_n f_1(z_n(t-\tau)), \\ {}_0^C D_t^q z_2(t) = -\mu_2 z_2(t) + a_2 f_2(z_1(t)), \\ {}_0^C D_t^q z_3(t) = -\mu_3 z_3(t) + a_3 f_3(z_1(t)), \\ \vdots \\ {}_0^C D_t^q z_n(t) = -\mu_n z_n(t) + a_n f_n(z_1(t)). \end{cases} \tag{8}$$

In order to facilitate the analysis of system (8), the following hypotheses are imposed:

(H1) $z_j = x_j + iy_j$ $j = 1,\dots,n$, where x_j and y_j are the real and imaginary parts of z_j, respectively, and $i = \sqrt{-1}$. The activation function f_j, $j = 1,\dots,n$ can be separated into real and imaginary parts as

$$f_j(z_k) = f_j^R(x_k, y_k) + i f_j^I(x_k, y_k), \quad j,k = 1,\dots,n.$$

(H2) $f_j^R(0,0) = 0$ and $f_j^I(0,0) = 0$ for $j = 1,\dots,n$.

(H3) f_j is differentiable at the equilibrium point $z_j^* = (x_j^*, y_j^*) = (0,0)$, $j = 1,\dots,n$.

4 Hopf Bifurcation Analysis

According to the hypotheses (H1) – (H3), separating the real and imaginary parts, system (8) can be written as

$$\begin{cases} {}_0^C D_t^q x_1(t) = -\mu_1 x_1(t) + \sum\limits_{j=2}^{n} b_j f_1^R(x_j(t-\tau), y_j(t-\tau)), \\ {}_0^C D_t^q y_1(t) = -\mu_1 y_1(t) + \sum\limits_{j=2}^{n} b_j f_1^I(x_j(t-\tau), y_j(t-\tau)), \\ {}_0^C D_t^q x_2(t) = -\mu_2 x_2(t) + a_2 f_2^R(x_1(t), y_1(t)), \\ {}_0^C D_t^q y_2(t) = -\mu_2 y_2(t) + a_2 f_2^I(x_1(t), y_1(t)), \\ \vdots \\ {}_0^C D_t^q x_n(t) = -\mu_n x_n(t) + a_n f_n^R(x_1(t), y_1(t)), \\ {}_0^C D_t^q y_n(t) = -\mu_n y_n(t) + a_n f_n^I(x_1(t), y_1(t)). \end{cases} \tag{9}$$

In order to get the linear part of system (9), expand the functions $f_1^R(x_j(t-\tau), y_j(t-\tau))$, $f_1^I(x_j(t-\tau), y_j(t-\tau))$ and $f_j^R(x_1(t), y_1(t))$, $f_j^I(x_1(t), y_1(t))$, where $j = 2, \ldots, n$, at the equilibrium point $z_j^* = (x_j^*, y_j^*) = (0,0)$, $j = 1, \ldots, n$, $Z^* = (z_1^*, \ldots, z_n^*)^T$:

$$\begin{aligned} f_1^R(x_j(t-\tau), y_j(t-\tau)) &= f_1^R(x_j^*, y_j^*) + \frac{\partial f_1^R(x_j^*, y_j^*)}{\partial x_j}(x_j(t-\tau) - x_j^*) + \frac{\partial f_1^R(x_j^*, y_j^*)}{\partial y_j}(y_j(t-\tau) - y_j^*) + h.o.t, \\ f_1^I(x_j(t-\tau), y_j(t-\tau)) &= f_1^I(x_j^*, y_j^*) + \frac{\partial f_1^I(x_j^*, y_j^*)}{\partial x_j}(x_j(t-\tau) - x_j^*) + \frac{\partial f_1^I(x_j^*, y_j^*)}{\partial y_j}(y_j(t-\tau) - y_j^*) + h.o.t, \\ f_j^R(x_1(t), y_1(t)) &= f_j^R(x_1^*, y_1^*) + \frac{\partial f_j^R(x_1^*, y_1^*)}{\partial x_1}(x_1(t) - x_1^*) + \frac{\partial f_j^R(x_1^*, y_1^*)}{\partial y_1}(y_1(t) - y_1^*) + h.o.t, \\ f_j^I(x_1(t), y_1(t)) &= f_j^I(x_1^*, y_1^*) + \frac{\partial f_j^I(x_1^*, y_1^*)}{\partial x_1}(x_1(t) - x_1^*) + \frac{\partial f_j^I(x_1^*, y_1^*)}{\partial y_1}(y_1(t) - y_1^*) + h.o.t. \end{aligned}$$

By using the above relations, the system (9) results in

$$\begin{cases} {}_0^C D_t^q x_1(t) = -\mu_1 x_1(t) + \sum\limits_{j=2}^{n} (q_{j1} x_j(t-\tau) + q_{j2} y_j(t-\tau)) \\ {}_0^C D_t^q y_1(t) = -\mu_1 y_1(t) + \sum\limits_{j=2}^{n} (w_{j1} x_j(t-\tau) + w_{j2} y_j(t-\tau)) \\ {}_0^C D_t^q x_2(t) = -\mu_2 x_2(t) + e_{21} x_1(t) + e_{22} y_1(t) \\ {}_0^C D_t^q y_2(t) = -\mu_2 y_2(t) + r_{21} x_1(t) + r_{22} y_1(t) \\ \vdots \\ {}_0^C D_t^q x_n(t) = -\mu_n x_n(t) + e_{n1} x_1(t) + e_{n2} y_1(t) \\ {}_0^C D_t^q y_n(t) = -\mu_n y_n(t) + r_{n1} x_1(t) + r_{n2} y_1(t) \end{cases} \tag{10}$$

where

$$q_{j1} = b_j \frac{\partial f_1^R(x_j^*, y_j^*)}{\partial x_j}, \quad q_{j2} = b_j \frac{\partial f_1^R(x_j^*, y_j^*)}{\partial y_j}, \quad w_{j1} = b_j \frac{\partial f_1^I(x_j^*, y_j^*)}{\partial x_j}, \quad w_{j2} = b_j \frac{\partial f_1^I(x_j^*, y_j^*)}{\partial y_j},$$

$$e_{j1}=a_j\frac{\partial f_j^R(x_1^*,y_1^*)}{\partial x_1},\quad e_{j2}=a_j\frac{\partial f_j^R(x_1^*,y_1^*)}{\partial y_1},\quad r_{j1}=a_j\frac{\partial f_j^I(x_1^*,y_1^*)}{\partial x_1},\quad r_{j2}=a_j\frac{\partial f_j^I(x_1^*,y_1^*)}{\partial y_1},$$

for $j=2,\dots,n$. Therefore, the characteristic equation of system (10), by using the hypothesis (H2) and Definition 2, can be obtained as

$$det\begin{bmatrix} s^q+\mu_1 & 0 & -q_{21}e^{-s\tau} & -q_{22}e^{-s\tau} & \dots & -q_{n1}e^{-s\tau} & -q_{n2}e^{-s\tau} \\ 0 & s^q+\mu_1 & -w_{21}e^{-s\tau} & -w_{22}e^{-s\tau} & \dots & -w_{n1}e^{-s\tau} & -w_{n2}e^{-s\tau} \\ -e_{21} & -e_{22} & s^q+\mu_2 & 0 & \dots & 0 & 0 \\ -r_{21} & -r_{22} & 0 & s^q+\mu_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -e_{n1} & -e_{n2} & 0 & 0 & \dots & s^q+\mu_n & 0 \\ -r_{n1} & -r_{n2} & 0 & 0 & \dots & 0 & s^q+\mu_n \end{bmatrix}=0. \tag{11}$$

For an arbitary n, the above equation can be expressed as

$$\begin{aligned} &s^{2nq}+A_1s^{(2n-1)q}+A_2s^{(2n-2)q}+A_3s^{(2n-3)q}+\cdots+A_{2n-1}s^q+A_{2n} \\ &+(B_ns^{nq}+B_{n-1}s^{(n-1)q}+\cdots+B_1s^q+B_0)e^{-s\tau}+Ce^{-2s\tau}=0. \end{aligned} \tag{12}$$

Now, for the convenience of further analysis, denote

$$F=s^{2nq}+A_1s^{(2n-1)q}+A_2s^{(2n-2)q}+A_3s^{(2n-3)q}+\cdots+A_{2n-1}s^q+A_{2n},$$

$$E=B_ns^{nq}+B_{n-1}s^{(n-1)q}+\cdots+B_1s^q+B_0.$$

Thus, Eq. (12) can be written as

$$F+Ee^{-s\tau}+Ce^{-2s\tau}=0. \tag{13}$$

Multiplying $e^{s\tau}$ on both sides of Eq. (13) leads to

$$Fe^{s\tau}+E+Ce^{-s\tau}=0. \tag{14}$$

Let $s=\omega i=\omega(\cos(\frac{\pi}{2})+i\sin(\frac{\pi}{2}))$, $\omega>0$, and F_1,F_2, E_1,E_2 be the real and imaginary parts of F,E, respectively. Then, Eq. (14) results in

$$(F_1+iF_2)(\cos(\omega\tau)+i\sin(\omega\tau))+(E_1+iE_2)+C(\cos(\omega\tau)-i\sin(\omega\tau))=0 \tag{15}$$

where

$$\begin{aligned} F_1 &= \omega^{2nq}\cos(q\pi n)+A_1\omega^{(2n-1)q}\cos((2n-1)q\frac{\pi}{2})+\cdots+A_{2n-1}\omega^q\cos(q\frac{\pi}{2})+A_{2n}, \\ F_2 &= \omega^{2nq}\sin(q\pi n)+A_1\omega^{(2n-1)q}\sin((2n-1)q\frac{\pi}{2})+\cdots+A_{2n-1}\omega^q\sin(q\frac{\pi}{2}), \\ E_1 &= B_n\omega^{nq}\cos(nq\frac{\pi}{2})+B_{n-1}\omega^{(n-1)q}\cos((n-1)q\frac{\pi}{2})+\cdots+B_1\omega^q\cos(q\frac{\pi}{2})+B_0, \\ E_2 &= B_n\omega^{nq}\sin(nq\frac{\pi}{2})+B_{n-1}\omega^{(n-1)q}\sin((n-1)q\frac{\pi}{2})+\cdots+B_1\omega^q\sin(q\frac{\pi}{2}). \end{aligned}$$

Separating the real and imaginary parts of Eq. (15), we have

$$\begin{cases} (F_1+C)\cos(\omega\tau)-F_2\sin(\omega\tau)+E_1=0, \\ F_2\cos(\omega\tau)+(F_1-C)\sin(\omega\tau)+E_2=0. \end{cases} \tag{16}$$

We solve (16) as follows

$$\begin{cases} \cos(\omega\tau) = \dfrac{\det\begin{pmatrix} -E_1 & -F_2 \\ -E_2 & F_1-C \end{pmatrix}}{\det\begin{pmatrix} F_1+C & -F_2 \\ F_2 & F_1-C \end{pmatrix}} = \frac{-E_1F_1+E_1C-F_2E_2}{F_1^2-C^2+F_2^2}, \\ \sin(\omega\tau) = \dfrac{\det\begin{pmatrix} F_1+C & -E_1 \\ F_2 & -E_2 \end{pmatrix}}{\det\begin{pmatrix} F_1+C & -F_2 \\ F_2 & F_1-C \end{pmatrix}} = \frac{-F_1E_2-E_2C+E_1F_2}{F_1^2-C^2+F_2^2}. \end{cases} \tag{17}$$

Now, by considering the fact $\sin^2\omega\tau+\cos^2\omega\tau=1$, one can easily compute the value of ω, as long as all the parameters are provided. Moreover, by using (17), one can get

$$\tau_1^k = \frac{1}{\omega}\left(\arccos(\frac{-E_1F_1+E_1C-F_2E_2}{F_1^2-C^2+F_2^2})+2k\pi\right), \quad k=0,1,2,\ldots,$$
$$\tau_2^k = \frac{1}{\omega}\left(\arcsin(\frac{-F_1E_2-E_2C+E_1F_2}{F_1^2-C^2+F_2^2})+2k\pi\right), \quad k=0,1,2,\ldots.$$

Let

$$\tau_0 = \min\{\tau_1^k,\tau_2^k\}, \quad k=0,1,2,\ldots,$$

as the bifurcation point.

When $\tau=0$, let $\lambda=s^q$. Then, the characteristic equation (12) can be written as

$$\lambda^{2n}+A_1\lambda^{2n-1}+\cdots+A_{2n-1}\lambda+A_{2n}+B_n\lambda^n+B_{n-1}\lambda^{n-1}+\cdots+B_1\lambda+B_0+C=0. \tag{18}$$

The above equation gives

$$\lambda^{2n}+p_1\lambda^{2n-1}+p_2\lambda^{2n-2}+\cdots+p_n\lambda^n+\cdots+p_{2n-1}\lambda+p_{2n}=0, \tag{19}$$

where $p_1=A_1$, $p_2=A_2$, …, $p_n=A_n+B_n, \ldots$, $p_{2n-1}=A_{2n-1}+B_1$ and $p_{2n}=A_{2n}+B_0+C$.

Now, by the well-known Routh-Hurwitz criteria, we can find the following set of conditions

(H4) $p_1>0, \quad \det\begin{pmatrix} p_1 & 1 \\ p_3 & p_2 \end{pmatrix}>0, \quad \det\begin{pmatrix} p_1 & 1 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{pmatrix}>0, \ldots, p_{2n}>0.$

The hypotheses (H4) and the well-known Routh-Hurwitz criteria ensure that all the roots of Eq. (12), when $\tau=0$, have negative real parts.

To obtain the main result, the following assumption is also needed

(H5) $Re(\frac{ds}{d\tau})|_{\tau=\tau_0}\neq 0$

Taking the derivative of Eq. (12) with respect to τ, we have

$$\begin{aligned} &2nqs^{2nq-1}\frac{ds}{d\tau}+A_1(2n-1)qs^{(2n-1)q-1}\frac{ds}{d\tau}+\cdots+qA_{2n-1}s^{q-1}\frac{ds}{d\tau} \\ &\quad+e^{-s\tau}(\frac{ds}{d\tau})(nqB_ns^{nq-1}+(n-1)qB_{n-1}s^{(n-1)q-1}+\cdots+qB_1s^{q-1}) \\ &\quad+(B_ns^{nq}+B_{n-1}s^{(n-1)q}+\cdots+B_1s^q+B_0)(-se^{-s\tau}-\tau\frac{ds}{d\tau}e^{-s\tau}) \\ &\quad+C(-2se^{-2s\tau}-2\tau\frac{ds}{d\tau}e^{-2s\tau})=0. \end{aligned} \tag{20}$$

From (20), one can acquire

$$\frac{ds}{d\tau} = \frac{X(s)}{Y(s)}, \tag{21}$$

where

$$X(s) = s(B_n s^{nq} + B_{n-1} s^{(n-1)q} + \cdots + B_1 s^q + B_0)e^{-s\tau} + 2sCe^{-2s\tau}, \tag{22}$$

and

$$\begin{aligned} Y(s) = {} & 2nqs^{2nq-1} + A_1(2n-1)qs^{(2n-1)q-1} + \cdots + qA_{2n-1}s^{q-1} \\ & + e^{-s\tau}(nqB_n s^{nq-1} + (n-1)qB_{n-1}s^{(n-1)q-1} + \cdots + qB_1 s^{q-1} \\ & - \tau(B_n s^{nq} + B_{n-1}s^{(n-1)q} + \cdots + B_1 s^q + B_0)) - 2C\tau e^{-2s\tau}. \end{aligned} \tag{23}$$

Also, we have

$$Re(\frac{X(s)}{Y(s)}) = Re(\frac{X_1(s) + iX_2(s)}{Y_1(s) + iY_2(s)} \times \frac{Y_1(s) - iY_2(s)}{Y_1(s) - iY_2(s)}) = \frac{X_1 Y_1 + X_2 Y_2}{Y_1^2 + Y_2^2}, \tag{24}$$

where $X_i, Y_i (i = 1,2)$ are the real and imaginary parts of $X(s), Y(s)$, respectively.

Now, by the above discussion, we can state the following theorem:

Theorem 1. *Suppose (H1) – (H5) hold. As τ increases from zero, there exists a value τ_0 such that the zero equilibrium point is locally asymptotically stable when $\tau \in [0, \tau_0)$ but unstable when $\tau > \tau_0$. In fact, system (8) undergoes a Hopf bifurcation at the origin when τ passes through τ_0.*

5 Global stability analysis

In this section, we introduce a suitable Lyapunov function for system (9) that is model (8) under the assumptions. In fact, we discuss on the global stability of the equilibrium point of system (9).

In order to get the main result, the following hypothesis is imposed

(H6) Suppose the functions f_i^R and f_i^I, $i = 1,2,\ldots,n$ are Lipschitz continuous. That is, there exist positive constants L_i^R and M_i^I, $i = 1,2,\ldots,n$ such that

$$|f_i^R(u_1,y) - f_i^R(u_2,y)| \leq L_i^R |u_1 - u_2|,$$

$$|f_i^I(x,v_1) - f_i^I(x,v_2)| \leq M_i^I |v_1 - v_2|.$$

Now, we define a Lyapunov function as follows:

$$V(t) = {}_0^C D_t^{-(1-q)}(\sum_{i=1}^{n}(|x_i(t)| + |y_i(t)|)) + \sum_{j=2}^{n}|b_j||L_1^R|\int_{t-\tau}^{t}|x_j(s)|ds + \sum_{j=2}^{n}|b_j||M_1^I|\int_{t-\tau}^{t}|y_j(s)|ds. \tag{25}$$

Calculating the derivative of V along the solution of the system (9) gives

$$\dot{V}(t) = \frac{d}{dt}({}_0^C D_t^{-(1-q)}(\sum_{i=1}^{n}(|x_i(t)| + |y_i(t)|))) + \frac{d}{dt}(\sum_{j=2}^{n}|b_j|(|L_1^R|\int_{t-\tau}^{t}|x_j(s)|ds + |M_1^I|\int_{t-\tau}^{t}|y_j(s)|ds)). \tag{26}$$

So,

$$\dot{V}(t) = {}_0^C D_t^q \Big(\sum_{i=1}^{n}(|x_i(t)|+|y_i(t)|)\Big) + \sum_{j=2}^{n}|b_j|(|L_1^R|(|x_j(t)|-|x_j(t-\tau)|) + |M_1^I|(|y_j(t)|-|y_j(t-\tau)|)). \quad (27)$$

Then,

$$\begin{aligned}\dot{V}(t) \le &\sum_{i=1}^{n}(sgn(x_i(t)){}_0^C D_t^q x_i(t) + sgn(y_i(t)){}_0^C D_t^q y_i(t)) + \sum_{j=2}^{n}|b_j|(|L_1^R|(|x_j(t)|-|x_j(t-\tau)|)\\ &+|M_1^I|(|y_j(t)|-|y_j(t-\tau)|)).\end{aligned} \quad (28)$$

Using system (9), we have

$$\begin{aligned}\dot{V}(t) \le\ & sgn(x_1(t))(-\mu_1 x_1(t) + \sum_{j=2}^{n} b_j f_1^R(x_j(t-\tau), y_j(t-\tau)))\\ &+ sgn(y_1(t))(-\mu_1 y_1(t) + \sum_{j=2}^{n} b_j f_1^I(x_j(t-\tau), y_j(t-\tau)))\\ &+ \sum_{j=2}^{n}((sgn(x_j(t)))(-\mu_j x_j(t) + a_j f_j^R(x_1(t),y_1(t))) + (sgn(y_j(t)))(-\mu_j y_j(t) + a_j f_j^I(x_1(t),y_1(t))))\\ &+ \sum_{j=2}^{n}|b_j|(|L_1^R|(|x_j(t)|-|x_j(t-\tau)|) + |M_1^I|(|y_j(t)|-|y_j(t-\tau)|)).\end{aligned} \quad (29)$$

By (H6) and (H2), one can obtain

$$\begin{aligned}\dot{V}(t) \le &-\sum_{i=1}^{n}\mu_i(|x_i(t)|+|y_i(t)|) + \sum_{j=2}^{n}|b_j|(|L_1^R||x_j(t-\tau)| + |M_1^I||y_j(t-\tau)|)\\ &+\sum_{j=2}^{n}|a_j|(|L_j^R||x_1(t)| + |M_j^I||y_1(t)|)\\ &+\sum_{j=2}^{n}|b_j|(|L_1^R|(|x_j(t)|-|x_j(t-\tau)|) + |M_1^I|(|y_j(t)|-|y_j(t-\tau)|)).\end{aligned} \quad (30)$$

Thus,

$$\begin{aligned}\dot{V}(t) \le &-\sum_{i=1}^{n}\mu_i(|x_i(t)|+|y_i(t)|) + \sum_{j=2}^{n}|a_j|(|L_j^R||x_1(t)| + |M_j^I||y_1(t)|)\\ &+\sum_{j=2}^{n}|b_j|(|L_1^R||x_j(t)| + |M_1^I||y_j(t)|).\end{aligned} \quad (31)$$

Now, we rewrite (31) as follows:

$$\begin{aligned}\dot{V}(t) \le &\sum_{j=2}^{n}|L_j^R|\Big(\frac{-\mu_1}{(n-1)|L_j^R|} + |a_j|\Big)|x_1(t)| + \sum_{j=2}^{n}|M_j^I|\Big(\frac{-\mu_1}{(n-1)|M_j^I|} + |a_j|\Big)|y_1(t)|\\ &+\sum_{j=2}^{n}|L_1^R|\Big(\frac{-\mu_j}{|L_1^R|} + |b_j|\Big)|x_j(t)| + \sum_{j=2}^{n}|M_1^I|\Big(\frac{-\mu_j}{|M_1^I|} + |b_j|\Big)|y_j(t)|.\end{aligned} \quad (32)$$

Let

$$\zeta_j^{(1)} = |L_j^R|(\frac{\mu_1}{(n-1)|L_j^R|} - |a_j|), \quad \zeta_j^{(2)} = |M_j^I|(\frac{\mu_1}{(n-1)|M_j^I|} - |a_j|),$$
$$\alpha_j^{(1)} = |L_1^R|(\frac{\mu_j}{|L_1^R|} - |b_j|), \quad \alpha_j^{(2)} = |M_1^I|(\frac{\mu_j}{|M_1^I|} - |b_j|).$$

Therefore,

$$\dot{V}(t) \leq -\sum_{j=2}^{n} \zeta_j^{(1)}|x_1(t)| - \sum_{j=2}^{n} \zeta_j^{(2)}|y_1(t)| - \sum_{j=2}^{n} \alpha_j^{(1)}|x_j(t)| - \sum_{j=2}^{n} \alpha_j^{(2)}|y_j(t)|. \tag{33}$$

Now, suppose

$$\hat{\zeta}^{(1)} = \min_j\{\zeta_j^{(1)}\} > 0, \quad \hat{\zeta}^{(2)} = \min_j\{\zeta_j^{(2)}\} > 0,$$
$$\hat{\alpha}^{(1)} = \min_j\{\alpha_j^{(1)}\} > 0, \quad \hat{\alpha}^{(2)} = \min_j\{\alpha_j^{(2)}\} > 0.$$

So,

$$\dot{V}(t) \leq -(n-1)\hat{\zeta}^{(1)}|x_1(t)| - (n-1)\hat{\zeta}^{(2)}|y_1(t)| - \sum_{j=2}^{n} \hat{\alpha}^{(1)}|x_j(t)| - \sum_{j=2}^{n} \hat{\alpha}^{(2)}|y_j(t)|. \tag{34}$$

Hence, $\dot{V} < 0$ and we can state the following theorem:

Theorem 2. *Suppose that (H2) and (H6) hold. Then the equilibrium point of system (9) is globally asymptotically stable if $\hat{\zeta}^{(1)} > 0$, $\hat{\zeta}^{(2)} > 0$, $\hat{\alpha}^{(1)} > 0$ and $\hat{\alpha}^{(2)} > 0$, where*

$$\hat{\zeta}^{(1)} = \min_j\{\zeta_j^{(1)}\}, \quad \hat{\zeta}^{(2)} = \min_j\{\zeta_j^{(2)}\}, \quad \hat{\alpha}^{(1)} = \min_j\{\alpha_j^{(1)}\}, \quad \hat{\alpha}^{(2)} = \min_j\{\alpha_j^{(2)}\}$$

and

$$\zeta_j^{(1)} = |L_j^R|(\frac{\mu_1}{(n-1)|L_j^R|} - |a_j|), \quad \zeta_j^{(2)} = |M_j^I|(\frac{\mu_1}{(n-1)|M_j^I|} - |a_j|),$$
$$\alpha_j^{(1)} = |L_1^R|(\frac{\mu_j}{|L_1^R|} - |b_j|), \quad \alpha_j^{(2)} = |M_1^I|(\frac{\mu_j}{|M_1^I|} - |b_j|).$$

To find the other kind of Lyapunov function and so global asymptotic stability, we need the following lemma.

Lemma 1. *Suppose that $\omega_1, \omega_2 : R \to R$ are continuous nondecreasing functions, $\omega_1(s), \omega_2(s)$ are positive for $s > 0$, and $\omega_1(0) = \omega_2(0) = 0$. If there exists a continuously differentiable function $V : R \to R$ such that $\omega_1(\|x(t)\|) \leq V(t, x(t)) \leq \omega_2(\|x(t)\|)$ holds, and there exist two constants $r > p > 0$ such that for any given $t_0 \in R$ the fractional derivative of V along the solution $x(t)$ of the fractional system*

$$D^{\alpha}x(t) = f(t, x(t), x(t-\tau)),$$

satisfies

$$D^{\alpha}V(t, x(t)) \leq -rV(t, x(t)) + p \sup_{-\tau \leq \theta \leq 0} V(t+\theta, x(t+\theta)),$$

for $t \geq t_0$, then the fractional system is globally asymptotically stable.

Proof. For the proof, see [4]. □

For system (9), consider the following Lyapunov function

$$V(t) = \sum_{i=1}^{n} (|x_i(t)| + |y_i(t)|).$$

By calculating the derivative of $V(t)$ along the solutions of system (9), we get

$$\begin{aligned}
D^q V(t) &= \sum_{i=1}^{n} (D^q |x_i(t)| + D^q |y_i(t)|) \\
&= \sum_{i=1}^{n} (sgn(x_i(t)) D^q x_i(t) + sgn(y_i(t)) D^q y_i(t)) \\
&= sgn(x_1(t))(-\mu_1 x_1(t) + \sum_{j=2}^{n} b_j f_1^R(x_j(t-\tau), y_j(t-\tau))) + sgn(y_1(t))(-\mu_1 y_1(t) \\
&\quad + \sum_{j=2}^{n} b_j f_1^I(x_j(t-\tau), y_j(t-\tau))) + \sum_{i=2}^{n} (sgn(x_i(t))(-\mu_i x_i(t) + a_i f_i^R(x_1(t), y_1(t))) \\
&\quad + sgn(y_i(t))(-\mu_i y_i(t) + a_i f_i^I(x_1(t), y_1(t)))).
\end{aligned} \tag{35}$$

Now, by using (H2) and (H6), we get

$$\begin{aligned}
D^q V(t) \leq &-\mu_1 |x_1(t)| - \mu_1 |y_1(t)| + \sum_{j=2}^{n} (|b_j| L_1^R |x_j(t-\tau)| + |b_j| M_1^I |y_j(t-\tau)|) \\
&- \sum_{i=2}^{n} \mu_i (|x_i(t)| + |y_i(t)|) + \sum_{i=2}^{n} (|a_i| L_i^R |x_1(t)| + |a_i| M_i^I |y_1(t)|).
\end{aligned} \tag{36}$$

So,

$$\begin{aligned}
D^q V(t) \leq &- \sum_{i=1}^{n} \mu_i (|x_i(t)| + |y_i(t)|) + \sum_{j=2}^{n} |b_j| (L_1^R |x_j(t-\tau)| + M_1^I |y_j(t-\tau)|) \\
&+ \sum_{j=2}^{n} |a_j| (L_j^R |x_1(t)| + M_j^I |y_1(t)|).
\end{aligned} \tag{37}$$

Now, let

$$r = \min_i \{\mu_i\}, \quad \hat{b} = \max_j \{|b_j|\}, \quad \hat{a} = \max_j \{|a_j| L_j^R\}, \quad \tilde{a} = \max_j \{|a_j| M_j^I\},$$
$$\hat{c} = \max\{\hat{b} L_1^R, \hat{a}\}, \quad \hat{d} = \max\{\hat{b} M_1^I, \tilde{a}\}, \quad p = \max\{\hat{c}, \hat{d}\}.$$

Thus,

$$D^q V(t) \leq -rV(t) + p \sup_{t-\tau \leq s \leq t} V(s).$$

Lemma 1 and above calculations result in global asymptotic stability of the equilibrium point of system (9), and we can state the following theorem:

Theorem 3. *Suppose (H2) and (H6) hold. Then the equilibrium point of system (9) is globally asymptotically stable if $r > p > 0$, where $r = \min_i\{\mu_i\}$, $p = \max\{\hat{c}, \hat{d}\}$, $\hat{c} = \max\{\hat{b} L_1^R, \hat{a}\}$, $\hat{d} = \max\{\hat{b} M_1^I, \tilde{a}\}$, $\hat{b} = \max_j\{|b_j|\}$, $\hat{a} = \max_j\{|a_j| L_j^R\}$ and $\tilde{a} = \max_j\{|a_j| M_j^I\}$.*

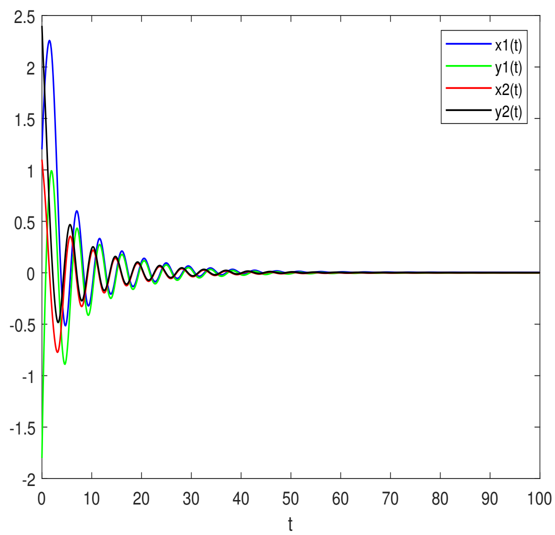

Figure 1: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.4 < \tau_0 = 0.55$.

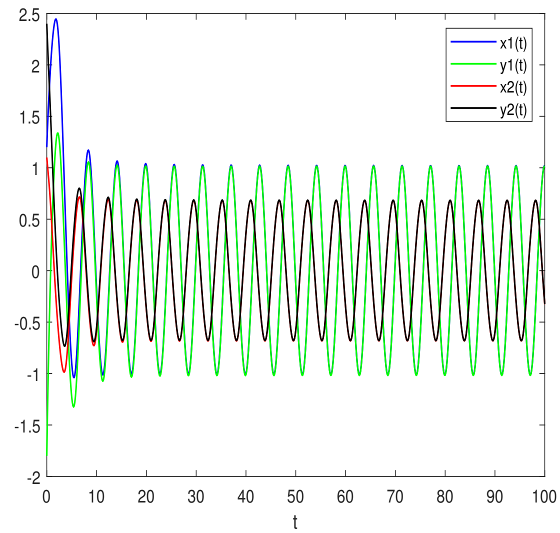

Figure 2: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.8 > \tau_0$. A family of periodic solutions bifurcate from the origin and Hopf bifurcation occurs.

6 Numerical simulations

In this section, to illustrate our theoretical results, three numerical examples are given. The numerical simulations are based on the Adams-Bashforth-Moulton predictor-corrector algorithm.

Example 1. Select $n = 2$, the activation functions $f_j(z) = \tanh(x) + \tanh(y) + i(\tanh(x) + \tanh(y))$ for $j = 1,2$, $q = 0.95$, $\mu_1 = \mu_2 = 0.5$, $b_2 = 1$, $a_2 = -0.5$ and the initial condition $(1.2, -1.8, 1.1, 2.4)$ in system (9). By some calculations, we get the critical bifurcation value $\tau_0 = 0.55$. Then, the zero equlibrium point is asymptotically stable when $\tau = 0.4 < \tau_0$, as shown in Figure 1. When $\tau = 0.8 > \tau_0$, Hopf bifurcation occurs and a family of periodic solutions appeares, as depicted in Figure 2. To illustrate more, three dimensional phase portraits are given in Figures 3, 4, 5 and 6.

Example 2. Consider system (9) with $n = 3$ and activation functions $f_1(z) = f_3(z) = \tanh(x) + i\tanh(y)$, $f_2(z) = -\tanh(x) - i\tanh(y)$. Set $\mu_1 = 1.5$, $\mu_2 = \mu_3 = 0.5$, $b_2 = -0.05$, $b_3 = 0.1$, $a_2 = -0.1$ and $a_3 = 0.02$. By some simple computations, one has $L_j^R = M_j^I = 1; j = 1,2,3$, $\zeta_2^{(1)} = \zeta_2^{(2)} = 0.65$, $\zeta_3^{(1)} = \zeta_3^{(2)} = 0.73$, $\alpha_2^{(1)} = \alpha_2^{(2)} = 0.45$, $\alpha_3^{(1)} = \alpha_3^{(2)} = 0.4$, and so $\hat{\zeta}^{(1)} = \hat{\zeta}^{(2)} = 0.65 > 0$, $\hat{\alpha}^{(1)} = \hat{\alpha}^{(2)} = 0.4 > 0$. Thus, the conditions in Theorem 2 are satisfied. So, the equilibrium point of system (9) is globally asymptotically stable. Figure 7 shows the trajectories of the system in this example.

Example 3. In this case, consider system (9) with $n = 3$ and activation functions $f_1(z) = -\tanh(x) + i\tanh(y)$, $f_2(z) = \tanh(x) + i\tanh(y)$, $f_3(z) = \tanh(x) - i\tanh(y)$. Set $\mu_1 = 1.5$, $\mu_2 = 1.2$, $\mu_3 = 0.6$, $b_2 = -0.1$, $b_3 = 0.3$, $a_2 = 0.45$ and $a_3 = -0.1$. By some simple computations, it is easy to see that $L_j^R = M_j^I = 1; j = 1,2,3$, $r = 0.6$, $\hat{b} = 0.3$, $\hat{a} = \tilde{a} = 0.45$, $\hat{c} = \hat{d} = 0.45$ and so $p = 0.45$. Thus, $r =$

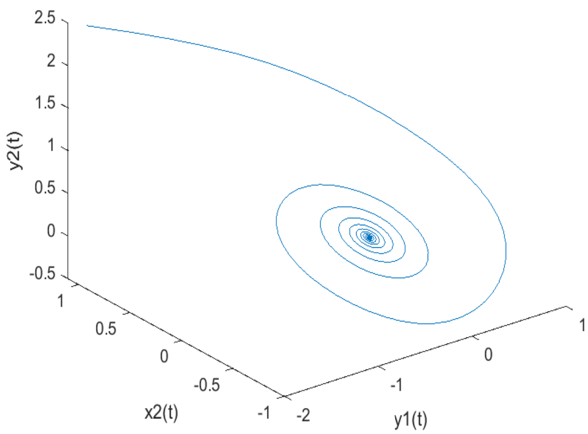

Figure 3: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.4 < \tau_0 = 0.55$.

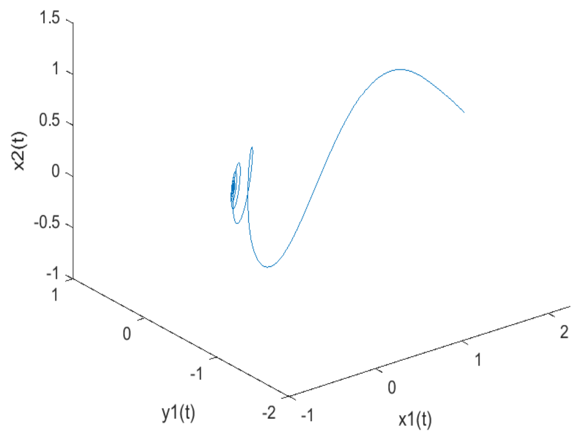

Figure 4: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.4 < \tau_0 = 0.55$.

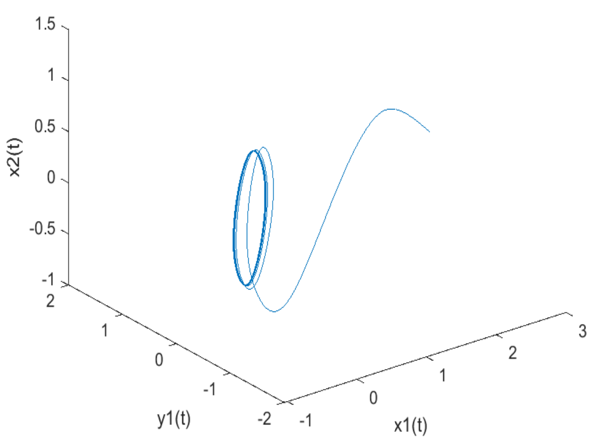

Figure 5: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.8 > \tau_0$. A family of periodic solutions bifurcate from the origin and Hopf bifurcation occurs.

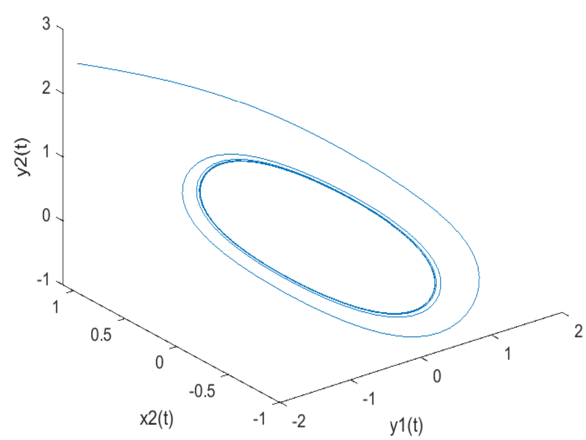

Figure 6: Curves of the real and imaginary parts of system (9) with $q = 0.95$ and $\tau = 0.8 > \tau_0$. A family of periodic solutions bifurcate from the origin and Hopf bifurcation occurs.

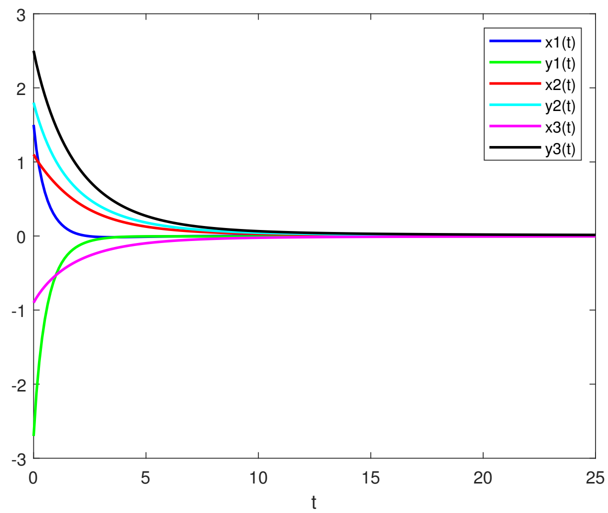

Figure 7: Time response of state variables of system (9) in Example 2.

$0.6 > p = 0.45 > 0$, the conditions in Theorem 3 are satisfied. So, the equilibrium point of system (9) is globally asymptotically stable. The trajectories of this system are shown in Figure 8.

7 Conclusions

In this paper, a general class of fractional-order complex-valued bidirectional associative memory neural network with time delay is first proposed. The model contains an arbitrary number of neurons, i.e. one neuron in the X-layer and other neurons in the Y-layer. We investigated Hopf bifurcation and global asymptotic stability of the system. Taking the time delay as the bifurcation parameter, the occurrence of Hopf bifurcation and the critical value of the time delay for Hopf bifurcation have been determined. Furthermore, two sets of sufficient conditions were obtained to ensure the system to be globally asymptotically stable. In fact, we constructed two kinds of appropriate Lyapunov functions to get the results. The new results were easy to test in the practical fields. Finally, in the three representative numerical examples, the correctness and effectiveness of the theoretical results of the theorems have been verified.

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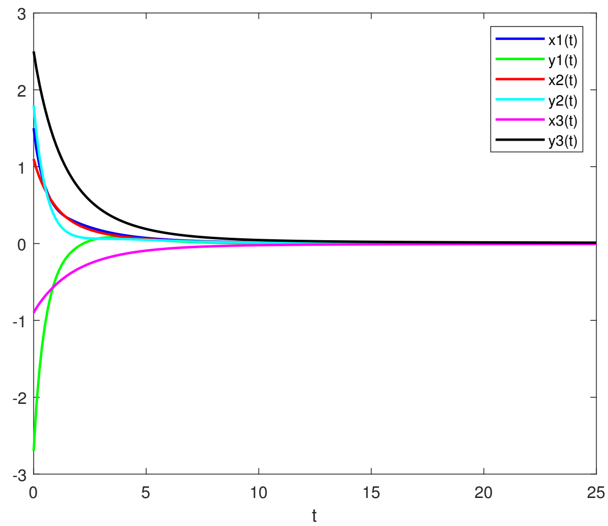

Figure 8: Time response of state variables of system (9) in Example 3.

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