Journal of Mathematical Modeling
Vol. 10, No. 4, 2022, pp. 535-553. Research Article

JMM

On the solution of parameterized Sylvester matrix equations

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Abstract. In this paper, parametric Sylvester matrix equations whose elements are linear functions of interval parameters are considered. In contrast to deterministic problems, when a system of equations is derived from a stochastic model, its coefficients may depend on some parameters and so the parameterized system of equations appears. This work considers the parameterized Sylvester matrix equations and tries to propose some methods containing a direct method and two iterative methods to obtain outer estimations of the solution set.

Keywords: Interval computation, matrix equation, Sylvester matrix equation, parameterized linear systems.
AMS Subject Classification 2010: 65G30, 15A24.

1 Introduction

In the problem of interval system of equations which is of particular importance in interval computing, often it is assumed that the elements of the system perturb independently within the given intervals. But practically this assumption is often not established and the involved components may not operate independently and leads to the parameterized interval system of equations. In this paper, we investigate the parametric Sylvester matrix equation

$$A(p)X + XB(p) = C(p), \tag{1a}$$

where p is an s-dimensional parameter vector and $A(p)$, $B(p)$ and $C(p)$ are, respectively, m-by-m, n-by-n and m-by-n matrices whose their elements are defined as

$$A_{ij}(p) = \alpha_{ij} + \sum_{k=1}^{s} \alpha_{ijk} p_k, \tag{1b}$$

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Received: 17 April 2022/ Revised: 21 July 2022/ Accepted: 15 August 2022
DOI: 10.22124/JMM.2022.22153.1950

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$$B_{ij}(p) = \beta_{ij} + \sum_{k=1}^{s} \beta_{ijk} p_k, \tag{1c}$$

$$C_{ij}(p) = \gamma_{ij} + \sum_{k=1}^{s} \gamma_{ijk} p_k, \tag{1d}$$

and $p_k \in \mathbf{p}_k$, $k = 1, \dots, s$.

The Sylvester matrix equations appear frequently in a variety of subjects such as control theory [1,6], image restoration [4], vibration theory [5,7,41], implementation of implicit numerical methods for ordinary differential equations, model reduction and so on , see [12, 15–22]. The parametric Sylvester matrix equation (1) generalizes both real and interval cases of the Sylvester matrix equations. About the interval Sylvester matrix equations, we refer the interested reader to [8, 10, 34–36]. The solution set of (1) is defined as

$$\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big) = \Big\{X \in \mathbb{R}^{m\times n} : A(p)X + XB(p) = C(p),\ \ p \in \mathbf{p}\Big\}.$$

The first paper on parametric interval systems was appeared by Jansson [23]. Rump [33] for the first time investigated the general problem of parametric linear systems. There are a collection of papers on the topic of parametric systems, see [9, 14, 23, 26, 27, 30, 31, 33, 37–40].

Notations: In this note, bold face letters denote interval quantities and ordinary letters stand for the real quantities. $\mathbb{IR}$ stands for the set of real intervals and the set of m-by-n real interval matrices is denoted by $\mathbb{IR}^{m\times n}$. Kaucher [24] extended the set of proper intervals $\mathbb{IR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \le \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$ by the set $\overline{\mathbb{IR}} := \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \ge \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$ of improper intervals, obtaining the set of generalized intervals $\mathbb{KR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$. The "dual" is an important monadic operator that reverses the endpoints of the generalized intervals. For interval $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ the midpoint of $\mathbf{x}$ is defined as $\text{mid}(\mathbf{x}) = \frac{\overline{\mathbf{x}}+\underline{\mathbf{x}}}{2}$, its radius is $\text{rad}(\mathbf{x}) = \frac{\overline{\mathbf{x}}-\underline{\mathbf{x}}}{2}$ and also we have dual($\mathbf{x}$)=$[\overline{\mathbf{x}}, \underline{\mathbf{x}}]$. The hull of a bounded set $\Omega \subseteq \mathbb{R}^{m\times n}$ is the interval matrix $\square\Omega = [\inf\Omega, \sup\Omega]$.

If $\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big)$ is a bounded set, then its interval hull exists and is called the interval hull solution to (1) and is denoted by $\mathbf{x}^*$. Any other interval matrix $\mathbf{x}$ containing $\mathbf{x}^*$, i.e., $\mathbf{x}^* \subseteq \mathbf{x}$ is referred to as an outer estimation of the solution set. Computing the exact solution of an interval linear system and even its interval hull solution is NP-hard [32], so providing some approximations for the solution set is considered by most researchers.

2 Theoretical results and algorithms

In this section, we introduce and analyze some methods involving direct and iterative methods for enclosing the solution set of the parameterized Sylvester matrix equation (1).

For any $p \in \mathbf{p}$, the equation $A(p)X + XB(p) = C(p)$ can be written in the equivalent form

$$G(p)x = f(p), \tag{2}$$

where

$$G(p) = (A(p) \otimes I_n) + (I_m \otimes B^T(p)), \qquad x = \text{vec}(X) \ \text{ and } \ f(p) = \text{vec}(C(p)).$$

Herein, $\otimes$ denotes the Kronecker product. The Kronecker product $A \otimes B$ of two matrices A and B is the block matrix whose its (i,j)-th block is $A_{ij}B$. For $C = (C_{ij}) \in \mathbb{R}^{m\times n}$, the vector $\text{vec}(C) \in \mathbb{R}^{mn}$ is obtained by stacking the rows of C, i.e, $\text{vec}(C) = (C_{11},\ldots,C_{1n},C_{21},\ldots,C_{2n},\ldots,C_{m1},\ldots,C_{mn})^T$.

Since for any $p \in \mathbf{p}$, the equation $A(p)X + XB(p) = C(p)$ is equivalent to (2), the parameterized Sylvester matrix equation (1) can be transformed to the following parameterized linear system

$$G(p)x = f(p), \quad p \in \mathbf{p}. \tag{3}$$

Thus the algorithms for solving the parameterized linear systems can be used to enclose the solution set $\Xi(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p}))$. Note that Eq. (3) is a large parameterized linear system, specially when m or n is large and so in computational point of view, is not efficient. For instance, some methods for handling system (3) need to compute (approximate) inverse of $A^0 \otimes I_n + I_m \otimes B^{0^T}$ where $A^0 = A(p^0)$, $B^0 = B(p^0)$, $C^0 = C(p^0)$, and $p^0 =\text{mid}(\mathbf{p})$. It is obvious that, when m or n is large, computing such approximate inverse is too costly since $A^0 \otimes I_n + I_m \otimes B^{0^T}$ is an mn-by-mn matrix. This motivates us to propose some methods which deal with the main problem, not its corresponding system (3).

Let p^0 and r denote the center and radius of $\mathbf{p}$, respectively, also let $A^0 = A(p^0)$, $B^0 = B(p^0)$, and $C^0 = C(p^0)$. We can redefine p, $A(p)$, $B(p)$, and $C(p)$ in the following form

$$p = p^0 + u, \qquad u \in \mathbf{u} = [-r,r], \tag{4a}$$

$$A(p) = A^0 + \Delta(u), \qquad \text{where } \Delta_{ij}(u) = \sum_{k=1}^{s} \alpha_{ijk}u_k, \tag{4b}$$

$$B(p) = B^0 + \nabla(u), \qquad \text{where } \nabla_{ij}(u) = \sum_{k=1}^{s} \beta_{ijk}u_k, \tag{4c}$$

$$C(p) = C^0 + \delta(u), \qquad \text{where } \delta_{ij}(u) = \sum_{k=1}^{s} \gamma_{ijk}u_k. \tag{4d}$$

Let $X \in \Xi\big(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p})\big)$ be written in the form

$$X = X^0 + V, \tag{5}$$

where X^0 is the solution of the real Sylvester matrix equation $A^0X + XB^0 = C^0$ (we assume that the solution X^0 exists and is unique, also for solving such systems, one can use the proposed methods in [6, 11] and references therein). Using (4) and this fact that $A^0X^0 + X^0B^0 = C^0$, system (1) can be written in the following equivalent form

$$A^0V + \Delta(u)X^0 + \Delta(u)V + VB^0 + X^0\nabla(u) + V\nabla(u) - \delta(u) = 0. \tag{6}$$

Now, if we introduce the m-by-n matrix E_k as

$$E_{ijk} \equiv E_k(i,j) = \sum_{t=1}^{m} \alpha_{itk}X^0_{tj} + \sum_{t=1}^{n} X^0_{it}\beta_{tjk} - \gamma_{ijk},$$

for $k = 1,\ldots,s$, then we have

$$\sum_{k=1}^{s} u_kE_k = \Delta(u)X^0 + X^0\nabla(u) - \delta(u),$$

and so (6) is equivalent to

$$A^0V + VB^0 + \Delta(u)V + V\nabla(u) + \sum_{k=1}^{s} u_k E_k = 0, \qquad u \in \mathbf{u}. \tag{7}$$

Suppose $G = -(A^0)^{-1}$ and $F_k = GE_k$, for $k = 1, \dots, s$, then (7) is equivalent to

$$V = G\Delta(u)V + GV(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \qquad u \in \mathbf{u}. \tag{8}$$

Let Ξ denote the solution set of (8), i.e.

$$\Xi = \Big\{V \in \mathbb{R}^{m\times n} : V = G\Delta(u)V + GV(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \quad u \in \mathbf{u}\Big\}. \tag{9}$$

It is obvious that if $\mathbf{V}$ is an outer estimation for the solution set Ξ, then $\mathbf{X} = X^0 + \mathbf{V}$ is an outer estimation for $\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big)$. So the problem of enclosing the solution set of (1) is equivalent to enclosing the solution set of (8).

In the next subsections we propose some methods including a direct and two iterative methods for determining outer estimations to the solution set $\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big)$ of the parameterized Sylvester matrix equation (1).

2.1 A direct method

Let D and Q be m-by-m and n-by-n real matrices, respectively, as

$$D_{ij} = \sum_{k=1}^{s} |\alpha_{ijk}| r_k, \qquad Q_{ij} = \sum_{k=1}^{s} |\beta_{ijk}| r_k.$$

Now consider the following interval system

$$V = \mathbf{T_1}V + (\mathbf{T_2}V)\mathbf{T_3} + \mathbf{T_4}, \tag{10}$$

in which $\mathbf{T_i} = [-T_i, T_i]$, $i = 1, 2, 3, 4$, and

$$T_1 = |G|D, \quad T_3 = |B^0| + Q, \quad T_2 = |G|, \quad T_4 = \sum_{k=1}^{s} r_k |F_k|.$$

Let Ξ^I denote the solution set of (10), i.e.

$$\Xi^I = \big\{V \in \mathbb{R}^{m\times n} : V = T_1'V + T_2'VT_3' + T_4', \quad (T_i' \in \mathbf{T}_i, i = 1, 2, 3, 4)\big\}.$$

By these notations, we are ready to prove the following result.

Lemma 1. *We have* $\Xi \subseteq \Xi^I$.

Proof. Consider system (8). For every $u \in \mathbf{u}$ we have

$$|\Delta_{ij}(u)| \leq \sum_{k=1}^{s} |\alpha_{ijk}||u_k| \leq \sum_{k=1}^{s} |\alpha_{ijk}|r_k = D_{ij},$$

$$|\nabla_{ij}(u)| \leq \sum_{k=1}^{s} |\beta_{ijk}||u_k| \leq \sum_{k=1}^{s} |\beta_{ijk}|r_k = Q_{ij},$$

and so we can write

$$\begin{aligned} |G\Delta(u)| &\leq |G||\Delta(u)| \leq |G|D = T_1, \\ |B^0 + \nabla(u)| &\leq |B^0| + |\nabla(u)| \leq |B^0| + Q = T_3, \\ |\sum_{k=1}^{s} u_k F_k| &\leq \sum_{k=1}^{s} |u_k||F_k| \leq \sum_{k=1}^{s} r_k |F_k| = T_4. \end{aligned}$$

Thus by the above relations, we obtain

$$\Xi \subseteq \left\{ V \in \mathbb{R}^{m\times n} : V = T_1' V + T_2' V T_3' + T_4', \quad (T_i' \in \mathbf{T}_i, i = 1,2,3,4) \right\} = \Xi^I,$$

and the proof is completed. □

For the next main theorem, we need to introduce the real matrix equation

$$V = T_1 V + T_2 V T_3 + T_4, \tag{11}$$

and the following iteration

$$\mathbf{V}^{(k+1)} = \mathbf{T_1} \mathbf{V}^{(k)} + (\mathbf{T_2} \mathbf{V}^{(k)}) \mathbf{T_3} + \mathbf{T_4}. \tag{12}$$

The matrix equation (11) is an special case of the generalized Sylvester matrix equations which some methods for solving them can be found e.g., in [6, 7] and references therein.

Theorem 1. *Suppose A^0 is nonsingular. Assume that the interval system (10) has a unique fixed point* $\mathbf{N}$ *and iteration (12) converges for every initial point. If the solution Z^* to the matrix equation (11) exists and is positive, then the interval matrix $\mathbf{X} = X^0 + \mathbf{Z}'$ in which*

$$\mathbf{Z}' = [-Z^*, Z^*],$$

is an enclosure for the solution set of the parameterized Sylvester matrix equation (1).

Proof. Suppose $\tilde{V} \in \Xi^I$, first we show $\tilde{V} \in \mathbf{N}$. For this purpose consider the following iteration

$$\begin{cases} \mathbf{V}^{(0)} := \tilde{V}, \\ \mathbf{V}^{(k+1)} = \mathbf{T_1} \mathbf{V}^{(k)} + (\mathbf{T_2} \mathbf{V}^{(k)}) \mathbf{T_3} + \mathbf{T_4}, \qquad k = 0, 1, \ldots. \end{cases}$$

By induction, we prove that every interval matrix generated by the above process, contains $\tilde{V}$. For $\mathbf{V}^{(0)}$ it is obvious. If $\tilde{V} \in \mathbf{V}^{(k)}$ then since $\tilde{V} \in \Xi^I$ and using monotonicity of the interval arithmetic operations, we have

$$\tilde{V} \in \mathbf{T_1} \tilde{V} + (\mathbf{T_2} \tilde{V}) \mathbf{T_3} + \mathbf{T_4} \subseteq \mathbf{T_1} \mathbf{V}^{(k)} + (\mathbf{T_2} \mathbf{V}^{(k)}) \mathbf{T_3} + \mathbf{T_4} = \mathbf{V}^{(k+1)}.$$

Therefore $\tilde{V} \in \mathbf{V}^{(k)}$ for all k. It is easy to see that the sequence $\{\mathbf{V}^{(k)}\}$ converges to a fixed point of the system (10). Under assumption of the theorem, this sequence is convergent to $\mathbf{N}$. Since $\tilde{V} \in \mathbf{V}^{(k)}$ for all integer k, it is obvious that

$$\tilde{V} \in \lim_{k\to\infty} \mathbf{V}^{(k)} = \mathbf{N}.$$

So $\tilde{V} \in \mathbf{N}$ and we conclude that $\Xi^{I} \subseteq \mathbf{N}$. But by Lemma 1 $\Xi \subseteq \Xi^{I}$ and so

$$\Xi \subseteq \mathbf{N}. \tag{13}$$

Thus $\mathbf{N}$ is an enclosure to the solution set of (8).

Now, we show $\mathbf{N} = [-Z^*, Z^*]$. Since $\mathbf{N}$ is the fixed point of the system (10), we can write

$$\mathbf{N} = \mathbf{T_1 N} + (\mathbf{T_2 N})\mathbf{T_3} + \mathbf{T_4}, \tag{14}$$

and since $\mathbf{T}_i$, $i = 1,2,3,4$, is an interval matrix with zero center, from (14) it follows that $\mathbf{N}$ is also an interval matrix with zero center. On the other hand, by (14) and this fact that Z^* is positive, the radius of $\mathbf{N}$ can be determined by solving the matrix equation $V = T_1V + T_2VT_3 + T_4$ that has the solution Z^*. So $\mathbf{N} = \mathbf{Z}' = [-Z^*, Z^*]$.

Using relation (13), we have

$$\mathbf{V}^* \subseteq \mathbf{Z}',$$

but $\mathbf{X}^* \subseteq X^0 + \mathbf{V}^*$ and so $\mathbf{X}^* \subseteq X^0 + \mathbf{Z}'$, i.e., the interval matrix $\mathbf{X}{=}X^0 + \mathbf{Z}'$ is an enclosure for the solution set of the parameterized system (1). □

2.2 The first iterative method

The introduced direct method in Subsection 2.1 needs only to invert a real matrix and solving a real Sylvester matrix equation and so from a computational point of view, it is an efficient method. But on the other hand, utilizing this method is subjected to establishment some certain conditions. Since these conditions may not always hold, we propose another techniques.

As previously mentioned, the problem of finding an enclosure to the solution set of (1) is equivalent to determining an enclosure $\mathbf{V}$ to the solution set of the following system

$$V = G\Delta(u)V + GV(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \qquad u \in \mathbf{u}.$$

The first iteration method for determining $\mathbf{V}$ is based on the following iteration

$$V^{(\theta+1)} = G\Delta(u)V^{(\theta)} + GV^{(\theta)}(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \qquad (u \in \mathbf{u}, V^0 = 0, \theta = 0, 1, \ldots). \tag{15}$$

(It is to be noted that each $V^{(\theta)}$ is in fact a function of u, i.e. $V^{(\theta)} = V^{(\theta)}(u)$, for simplicity of notation, we shall omit the argument u whenever possible.) By (15), we consider the sets

$$\Xi^{(\theta+1)} = \left\{V^{(\theta+1)} : V^{(\theta+1)} = G\Delta(u)V^{(\theta)} + GV^{(\theta)}(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k,\ u \in \mathbf{u}\right\}, \quad \theta = 0, 1, \ldots. \tag{16}$$

It is to be noted that $\Xi^{(\theta)}$, $\theta = 1, 2, \ldots$, is a set so $\{\Xi^{(\theta)}\}$ is a sequence of the sets. The concept of convergence of sequences of points has been extended by several authors to the concept of convergence of sequences of sets, see [2, 3]. We use the introduced norm in [27] for $\Xi^{(\theta)}$, i.e.

$$\|\Xi^{(\theta)}\| = \int_{\mathbf{u}} \|V^{(\theta)}(u)\| du, \tag{17}$$

wherein $\|V\|$ is any desirable norm of V in $\mathbb{R}^{m\times n}$ (note that $V^{(\theta)}$, for $\theta = 0, 1, \ldots$, is in fact a function of u).

Definition 1. *We say sequence $\{x_n\}$ converges to x in the sense of norm $||.||$ if*

$$||x_n - x|| \xrightarrow{n\to\infty} 0.$$

Lemma 2. *If the sequence $\{\Xi^{(\theta)}\}$ converges to Ξ^* in the sense of norm (17) then*

$$\Xi^* = \Xi.$$

Proof. Suppose $\Xi^{(\theta)}$ and $\Xi^{(\theta+1)}$ are two consecutive iterations of the sequence. By (17), we can write

$$\|\Xi^{(\theta+1)} - \Xi^{(\theta)}\| = \int_{\mathbf{u}} \|V^{(\theta+1)}(u) - V^{(\theta)}(u)\| du$$

Since the sequence $\{\Xi^{(\theta)}\}$ is convergent, $\|\Xi^{(\theta+1)} - \Xi^{(\theta)}\|$ tends to zero as θ tends to ∞, which implies convergence of the integral to zero. But since integrand is continuous and non-negative, we conclude that

$$\lim_{\theta\to\infty} \|V^{(\theta+1)}(u) - V^{(\theta)}(u)\| = 0,$$

that yields convergence of the sequence $\{V^{(\theta)}\}$. Let $\lim_{\theta\to\infty} V^{(\theta)} = V$, by (15) and (16) we have

$$\Xi^* = \big\{V : V = G\Delta(u)V + GV(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \;\; u \in \mathbf{u}\big\},$$

which by (9) the proof is completed. □

Now, let us scrutinize the sets $\Xi^{(\theta)}$. For $\theta = 1$ we have

$$\Xi^{(1)} = \big\{V^{(1)} : V^{(1)} = \sum_{k=1}^{s} u_k F_k, \;\; u \in \mathbf{u}\big\}.$$

Let

$$H^{(1)} = 0, \quad \mathbf{S}^{(1)} = 0, \quad F_k^{(1)} = F_k, \quad k = 1, \ldots, s. \tag{18}$$

To achieve a regular style, we write $V^{(1)}$ in the form

$$V^{(1)} = \sum_{k=1}^{s} u_k F_k^{(1)} + H^{(1)} + S^{(1)}, \qquad (u \in \mathbf{u}, S^{(1)} \in \mathbf{S}^{(1)}). \tag{19}$$

Now, for $\theta = 2$ using (19) we obtain

$$V^{(2)} = G\Delta(u)\sum_{k=1}^{s} u_k F_k{}^{(1)} + G\Delta(u)H^{(1)} + G\Delta(u)S^{(1)} + G(\sum_{k=1}^{s} u_k F_k{}^{(1)})(B^0 + \nabla(u))$$
$$+GH^{(1)}(B^0 + \nabla(u)) + GS^{(1)}(B^0 + \nabla(u)) + \sum_{k=1}^{s} u_k F_k, \quad (u \in \mathbf{u}, S^{(1)} \in \mathbf{S}^{(1)}). \tag{20}$$

$V^{(2)}$ contains nonlinear expressions in terms of u, namely $G\Delta(u)\sum_{k=1}^{s} u_k F_k{}^{(1)}$ and $G(\sum_{k=1}^{s} u_k F_k{}^{(1)})(B^0 + \nabla(u))$ and so $\Xi^{(2)}$ has a complicated form. It is obvious that $V^{(\theta)}$ will be more complicated with the growth of θ. By this reason, in the following we approximate them by linear enclosures.

Consider the (i, j)-th component of two sides of (20). We have

$$t_{ij}^{(1)} = (G\Delta(u)H^{(1)})_{ij} = \sum_{k=1}^{s} m_{ijk}^{(1)} u_k, \qquad u \in \mathbf{u},$$

therein

$$m_{ijk}^{(1)} = \sum_{t=1}^{m}\sum_{l=1}^{m} G_{il}\alpha_{ltk} H_{tj}^{(1)}.$$

And

$$t_{ij}^{(2)} = (GH^{(1)}(B^0 + \nabla(u)))_{ij} = \check{B}_{ij} + \sum_{k=1}^{s} m_{ijk}^{(2)} u_k, \qquad u \in \mathbf{u},$$

where

$$\check{B}_{ij} = \sum_{t=1}^{n}\sum_{l=1}^{m} G_{il} H_{lt}^{(1)} B_{tj}^{0}, \qquad m_{ijk}^{(2)} = \sum_{t=1}^{n}\sum_{l=1}^{m} G_{il}\beta_{tjk} H_{lt}^{(1)}.$$

For $(G\Delta(u)S^{(1)})_{ij}$ and $(GS^{(1)}(B^0 + \nabla(u)))_{ij}$, some interval enclosures are found. We have

$$y_{ij}^{(1)} = (G\Delta(u)S^{(1)})_{ij} = \sum_{k=1}^{s}\sum_{t=1}^{m} n_{itk}^{(1)} S_{tj}^{(1)} u_k, \qquad (u \in \mathbf{u}, S^{(1)} \in \mathbf{S}^{(1)}),$$

in which

$$n_{itk}^{(1)} = \sum_{l=1}^{m} G_{il}\alpha_{ltk}.$$

For $y_{ij}^{(1)}$ we get the following interval enclosure

$$\mathbf{y}_{ij}^{(1)} = \sum_{k=1}^{s} (\sum_{t=1}^{m} n_{itk}^{(1)} \mathbf{S}_{tj}^{(1)}) \mathbf{u}_k.$$

Also we can write

$$y_{ij}^{(2)} = (GS^{(1)}(B^0 + \nabla(u)))_{ij} = \hat{B}_{ij} + \sum_{k=1}^{s} n_{ijk}^{(2)} u_k, \qquad (u \in \mathbf{u}, S^{(1)} \in \mathbf{S}^{(1)}),$$

where

$$\hat{B}_{ij} = \sum_{t=1}^{n}\sum_{l=1}^{m} G_{il} S_{lt}^{(1)} B_{tj}^{0}, \qquad n_{ijk}^{(2)} = \sum_{t=1}^{n}\sum_{l=1}^{m} G_{il}\beta_{tjk} S_{lt}^{(1)}.$$

We determine the following interval enclosure for $y_{ij}^{(2)}$

$$\mathbf{y}_{ij}^{(2)} = \hat{\mathbf{B}}_{ij} + \sum_{k=1}^{s} \mathbf{n}_{ijk}^{(2)} \mathbf{u}_k,$$

in which

$$\hat{\mathbf{B}}_{ij} = \sum_{t=1}^{n} \sum_{l=1}^{m} G_{il} \mathbf{S}_{lt}^{(1)} B_{tj}^{0}, \qquad \mathbf{n}_{ijk}^{(2)} = \sum_{t=1}^{n} \sum_{l=1}^{m} G_{il} \beta_{tjk} \mathbf{S}_{lt}^{(1)}.$$

Now, also for nonlinear expressions $(G\Delta(u)\sum_{k=1}^{s} u_k F_k^{(1)})_{ij}$ and $(G(\sum_{k=1}^{s} u_k F_k^{(1)})(B^0 + \nabla(u)))_{ij}$, we should determine interval enclosures. We can write

$$w_{ij}^{(1)} = (G\Delta(u) \sum_{k=1}^{s} u_k F_k^{(1)})_{ij} = \sum_{k=1}^{s} \sum_{k'=1}^{s} \check{T}_{kk'}^{(i,j)} u_{k'} u_k, \qquad u, u' \in \mathbf{u},$$

where $\check{T}^{(i,j)}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, is an s-by-s matrix as

$$\check{T}^{(i,j)}(k,k') = \sum_{t=1}^{m} \sum_{l=1}^{m} G_{il} \alpha_{ltk'} F_{tjk}^{(1)}.$$

We get the following interval enclosure for $w_{ij}^{(1)}$

$$\mathbf{w}_{ij}^{(1)} = \sum_{k=1}^{s} \sum_{\substack{k'=1 \\ k'=k}}^{s} \check{T}_{kk'}^{(i,j)} \mathbf{u}_k \mathbf{u}_{k'} + \sum_{k=1}^{s} \sum_{\substack{k'=1 \\ k<k'}}^{s} (\check{T}_{kk'}^{(i,j)} + \check{T}_{k'k}^{(i,j)}) \mathbf{u}_k \mathbf{u}_{k'}.$$

Also we have

$$w_{ij}^{(2)} = (G(\sum_{k=1}^{s} u_k F_k^{(1)})(B^0 + \nabla(u)))_{ij} = \sum_{k=1}^{s} (\tilde{B}_{ij} + \sum_{k'=1}^{s} \hat{T}_{kk'}^{(i,j)} u_{k'}) u_k, \qquad u, u' \in \mathbf{u},$$

where

$$\tilde{B}_{ij} = \sum_{t=1}^{n} \sum_{l=1}^{m} G_{il} F_{ltk}^{(1)} B_{tj}^{0},$$

and $\hat{T}^{(i,j)}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, is an s-by-s matrix with elements

$$\hat{T}^{(i,j)}(k,k') = \sum_{t=1}^{n} \sum_{l=1}^{m} G_{il} \beta_{tjk'} F_{ltk}^{(1)}.$$

We consider the interval enclosure

$$\mathbf{w}_{ij}^{(2)} = \sum_{k=1}^{s} (\tilde{B}_{ij} + \sum_{k'=1}^{s} \hat{T}_{kk'}^{(i,j)} \mathbf{u}_{k'}) \mathbf{u}_k.$$

As previously mentioned, $w_{ij}^{(1)}$ and $w_{ij}^{(2)}$ are nonlinear functions in terms of $u \in \mathbf{u}$ (in fact they are quadratic forms of $u_k \in \mathbf{u}_k$). In the above we determined linear interval enclosures $\mathbf{w}_{ij}^{(1)}$ and $\mathbf{w}_{ij}^{(2)}$, respectively, for them. Another way for determining linear enclosures, is utilizing the linear interval enclosure of nonlinear functions proposed in [25].

Now we consider Eq. (20) and define

$$H_{ij}^{(2)} = \check{B}_{ij}, \quad \mathbf{S}_{ij}^{(2)} = \mathbf{w}_{ij}^{(1)} + \mathbf{w}_{ij}^{(2)} + \mathbf{y}_{ij}^{(1)} + \mathbf{y}_{ij}^{(2)}, \quad F_{ijk}^{(2)} = m_{ijk}^{(1)} + m_{ijk}^{(2)} + F_{ijk}^{(1)}, \quad k = 1, \ldots, s, \tag{21}$$

then the following linear enclosure can be considered for $V_{ij}^{(2)}$

$$(V_E^{(2)})_{ij} = \sum_{k=1}^{s} u_k F_{ijk}^{(2)} + H_{ij}^{(2)} + S_{ij}^{(2)}, \qquad (u \in \mathbf{u}, S_{ij}^{(2)} \in \mathbf{S}_{ij}^{(2)}).$$

By the above relation, we construct m-by-n matrix $V_E^{(2)}$ as a linear enclosure for $V^{(2)}$

$$V_E^{(2)} = \sum_{k=1}^{s} u_k F_k^{(2)} + H^{(2)} + S^{(2)}, \qquad (u \in \mathbf{u}, S^{(2)} \in \mathbf{S}^{(2)}), \tag{22}$$

in which using (21), $H^{(2)}$, $\mathbf{S}^{(2)}$, and $F_k^{(2)}$, $k = 1, \ldots, s$, are m-by-n matrices with elements

$$H^{(2)}(i,j) = H_{ij}^{(2)}, \quad \mathbf{S}^{(2)}(i,j) = \mathbf{S}_{ij}^{(2)}, \quad F_k^{(2)}(i,j) = F_{ijk}^{(2)}, \quad k = 1, \ldots, s.$$

If we define

$$\Xi_E^{(2)} = \left\{ V_E^{(2)} : V_E^{(2)} = \sum_{k=1}^{s} u_k F_k^{(2)} + H^{(2)} + S^{(2)}, \quad (u \in \mathbf{u}, S^{(2)} \in \mathbf{S}^{(2)}) \right\},$$

then by construction, it is obvious that the above set encloses $\Xi^{(2)}$, i.e.

$$\Xi^{(2)} \subseteq \Xi_E^{(2)}. \tag{23}$$

In view of the obtained relation (22) and analogy with (19) we can iterate the above procedure to generating new sets $\sum_E^{(\theta)}$ enclosing the sets $\sum^{(\theta)}$ for $\theta \geq 3$, by putting $H^{(1)} = H^{(2)}$, $\mathbf{S}^{(1)} = \mathbf{S}^{(2)}$, and $F_k^{(1)} = F_k^{(2)}$, $k = 1, \ldots, s$. If we put $\Xi_E^{(1)} = \Xi^{(1)}$, then it is easy to see that

$$\Xi^{(\theta)} \subseteq \Xi_E^{(\theta)}, \qquad \theta = 1, 2, \ldots. \tag{24}$$

The computational scheme of the first proposed iterative method for computing enclosure to the solution set $\Xi(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p}))$ can be seen in Algorithm 1. Now, the main theorem of this subsection can be presented.

Theorem 2. *Suppose A^0 is nonsingular. Using the above notations, if the sequence $\{\Xi_E^{(\theta)}\}$ converges to Ξ_E^* in the sense of norm (17), then the interval matrix $\mathbf{X} = X^0 + \mathbf{Z}''$ where $\mathbf{Z}''$ is the interval hull of Ξ_E^*, is an outer estimation to the solution set of the parameterized Sylvester matrix equation (1).*

Proof. Since A^0 is nonsingular, matrix G and so the sets $\Xi^{(\theta)}$ and $\Xi_E^{(\theta)}$ can be constructed. On the other hand, since the sequence $\{\Xi_E^{(\theta)}\}$ is convergent, due to the relation (24) we conclude that the sequence $\{\Xi^{(\theta)}\}$ converges to a limit point Ξ^* that satisfies

$$\Xi^* \subseteq \Xi_E^*, \tag{25}$$

which yields

$$\mathbf{V}^* \subseteq \mathbf{Z}'',$$

and so $\mathbf{X}^* \subseteq X^0 + \mathbf{V}^* \subseteq X^0 + \mathbf{Z}''$ completes the proof. □

Algorithm 1 Algorithm of the First Iterative Method

Require:

Matrices α_k, β_k and γ_k and the interval vector $\mathbf{p}$

Ensure:

An interval matrix $\mathbf{X}$ which encloses the solution set $\sum\left(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p})\right)$

Put (m,n)=size(γ_1); p^0 =mid$(\mathbf{p})$; r =rad$(\mathbf{p})$; $\mathbf{u}$=$[-r,r]$; $A^0=A(p^0)$; $B^0=B(p^0)$; $C^0=C(p^0)$; $G=-\text{inv}(A^0)$;

Compute matrix X^0 from the Sylvester matrix equation $A^0X+XB^0=C^0$

Put $E_k(i,j)=\sum_{t=1}^{m}\alpha_{itk}X_{tj}^0+\sum_{t=1}^{n}X_{it}^0\beta_{tjk}-\gamma_{ijk}; \quad k=1,\ldots,s$

Put $F_k=GE_k; \quad k=1,\ldots,s$

Put $H^{(1)}$=zeros(m,n); $\mathbf{S}^{(1)}$ =[zeros(m,n),zeros(m,n)]; $F_k^{(1)}=F_k, \quad k=1,\ldots,s$

Put $\Xi^{(1)}=\left\{\sum_{k=1}^{s}u_kF_k^{(1)}+H^{(1)}+S^{(1)}, \quad (u\in\mathbf{u},S^{(1)}\in\mathbf{S}^{(1)})\right\}$

Put d=1

Put dis=∞

while dis $>=\varepsilon$ **do**

Put d=d+1

Put $M_k^{(1)}(i,j)=\sum_{t=1}^{m}\sum_{l=1}^{m}G_{il}\alpha_{ltk}H_{tj}^{(1)}, \quad k=1,\ldots,s$

Put $M_k^{(2)}(i,j)=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}\beta_{tjk}H_{lt}^{(1)}, \quad k=1,\ldots,s$

Put $\check{B}(ij)=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}H_{lt}^{(1)}B_{tj}^0$

Put $N^{(1)}(i.j)=\sum_{l=1}^{m}G_{il}\alpha_{ljk}, \quad k=1,\ldots,s$

Put $\check{T}^{(i,j)}(k,k')=\sum_{t=1}^{m}\sum_{l=1}^{m}G_{il}\alpha_{ltk'}F_k^{(1)}(t,j), \quad i=1,\ldots,m, \ j=1,\ldots,n$

Put $\tilde{B}(ij)=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}F_k^{(1)}(l,t)B_{tj}^0$

Put $\hat{T}^{(i,j)}(k,k')=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}\beta_{tjk'}F_k^{(1)}(l,t), \quad i=1,\ldots,m, \ j=1,\ldots,n$

Put $\mathbf{N}_k^{(2)}(i,j)=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}\beta_{tjk}\mathbf{S}_{lt}^{(1)}, \quad k=1,\ldots,s$

Put $\hat{\mathbf{B}}(i,j)=\sum_{t=1}^{n}\sum_{l=1}^{m}G_{il}\mathbf{S}_{lt}^{(1)}B_{tj}^0$

Put $\mathbf{Y}^{(1)}(i,j)=\sum_{k=1}^{s}\sum_{t=1}^{m}N_k^{(1)}(i,t)\mathbf{S}_{tj}^{(1)}\mathbf{u}_k$

Put $\mathbf{Y}^{(2)}(i,j)=\hat{\mathbf{B}}(i,j)+\sum_{k=1}^{s}\mathbf{N}_k^{(2)}(i,j)\mathbf{u}_k$

Put $\mathbf{W}^{(1)}(i,j)=\sum_{k=1}^{s}\sum_{k'=1}^{s}\check{T}^{(i,j)}(k,k')\mathbf{u}_{k'}\mathbf{u}_k$

Put $\mathbf{W}^{(2)}(i,j)=\sum_{k=1}^{s}(\tilde{B}_{ij}+\sum_{k'=1}^{s}\hat{T}^{(i,j)}(k,k')\mathbf{u}_{k'})\mathbf{u}_k$

Put $H^{(2)}=\check{B}$; $\mathbf{S}^{(2)}=\mathbf{Y}^{(1)}+\mathbf{Y}^{(2)}+\mathbf{W}^{(1)}+\mathbf{W}^{(2)}$; $F_k^{(2)}=M_k^{(1)}+M_k^{(2)}+F_k^{(1)}; \quad k=1,\ldots,s$

Put $\Xi^{(d+1)}=\left\{\sum_{k=1}^{s}u_kF_k^{(2)}+H^{(2)}+S^{(2)}, \quad (u\in\mathbf{u},S^{(1)}\in\mathbf{S}^{(1)})\right\}$

Put dis=$||\Xi^{(d+1)}-\Xi^{(d)}||$

Put $H^{(1)}=H^{(2)}$; $\mathbf{S}^{(1)}=\mathbf{S}^{(2)}$; $F_k^{(1)}=F_k^{(2)}; \quad k=1,\ldots,s$

end while

Put $\mathbf{Z}=[\inf(\Sigma^{(k)}),\sup(\Sigma^{(k)})]$

Put $\mathbf{X}=X^0+\mathbf{Z}$

2.3 The second iterative method

The second iterative method is a modification of the Gauss-Seidel iteration method. First we present a generalized interval Gauss-Seidel method (GIGS) for solving interval Sylvester matrix equations.

Consider the interval Sylvester matrix equation

$$\mathbf{A}X + X\mathbf{B} = \mathbf{C}, \tag{26}$$

in which $\mathbf{A} \in \mathbb{IR}^{m\times m}$, $\mathbf{B} \in \mathbb{IR}^{n\times n}$, and $\mathbf{C} \in \mathbb{IR}^{m\times n}$. The solution set of (26) is defined as

$$\Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big) = \Big\{X \in \mathbb{R}^{m\times n} : AX + XB = C,\ \ (A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C})\Big\}. \tag{27}$$

For arbitrary $\mathbf{X} \in \mathbb{IR}^{m\times n}$, we are interested in good enclosures for the truncated solution set

$$\Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big) \cap \mathbf{X}.$$

Suppose $\tilde{X} \in \Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big)$, so we can write

$$A\tilde{X} + \tilde{X}B = C, \qquad \text{for some } \ A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}. \tag{28}$$

Writing (28) componentwise, we obtain

$$\sum_{t=1}^{m} A_{it}\tilde{X}_{tj} + \sum_{l=1}^{n} \tilde{X}_{il}B_{lj} = C_{ij}, \qquad i = 1,\ldots,m, \ \ j = 1,\ldots,n, \tag{29}$$

which yields

$$\tilde{X}_{ij} = \big(C_{ij} - \sum_{t\neq i} A_{it}\tilde{X}_{tj} - \sum_{l\neq j} \tilde{X}_{il}B_{lj}\big)/\big(A_{ii}+B_{jj}\big) \subseteq \big(\mathbf{C}_{ij} - \sum_{t\neq i} \mathbf{A}_{it}\mathbf{X}_{tj} + \sum_{l\neq j} \mathbf{X}_{il}\mathbf{B}_{lj}\big)/\big(\mathbf{A}_{ii}+\mathbf{B}_{jj}\big) =: \dot{\mathbf{X}}_{ij}, \tag{30}$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, if $0 \notin \mathbf{A}_{ii} + \mathbf{B}_{ii}$ and $\tilde{X} \in \mathbf{X}$. Another enclosure $\dot{\mathbf{X}}$ for $\tilde{X}$ can be obtained by applying (30) for $i = 1,\ldots,m$ and $j = 1,\ldots,n$, provided that $0 \notin \mathbf{A}_{ii} + \mathbf{B}_{ii}$ and since this works for all $\tilde{X} \in \Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big)$, we have

$$\Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big) \cap \mathbf{X} \subseteq \dot{\mathbf{X}} \cap \mathbf{X}.$$

Similar to the interval Gauss-Seidel iteration, we can make the best use of the available information to obtain an improved enclosure $\mathbf{Y}$ for $\Xi\big(\mathbf{A},\mathbf{B},\mathbf{C}\big) \cap \mathbf{X}$ in the following way

$$\mathbf{Y}_{ij} = \Big[\big(\mathbf{C}_{ij} - \sum_{t=1}^{i-1} \mathbf{A}_{it}\mathbf{Y}_{tj} - \sum_{t=i+1}^{m} \mathbf{A}_{it}\mathbf{X}_{tj} - \sum_{l=1}^{j-1} \mathbf{Y}_{il}\mathbf{B}_{lj} - \sum_{l=j+1}^{n} \mathbf{X}_{il}\mathbf{B}_{lj}\big)/\big(\mathbf{A}_{ii}+\mathbf{B}_{jj}\big)\Big] \cap \mathbf{X}_{ij},$$

for $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

Now, we consider a more general case in which we may have $0 \in \mathbf{A}_{ii} + \mathbf{B}_{jj}$ for some $i = 1,\ldots,m$ and $j = 1,\ldots,n$. For this purpose, for interval numbers $\mathbf{a},\mathbf{b},\mathbf{x} \in \mathbb{IR}$, we use the operator $\Gamma(\mathbf{a},\mathbf{b},\mathbf{x})$ as

$$\Gamma(\mathbf{a},\mathbf{b},\mathbf{x}) = \square\{\tilde{x} \in \mathbf{x} : a\tilde{x} = b, \ \ (a \in \mathbf{a}, b \in \mathbf{b})\}, \tag{31}$$

see [28]. By operator (31), we can consider the general case (29) and have the improved enclosures

$$\mathbf{Y}_{ij} = \Gamma(\mathbf{A}_{ii} + \mathbf{B}_{jj}, \mathbf{C}_{ij} - \sum_{t=1}^{i-1} \mathbf{A}_{it}\mathbf{Y}_{tj} - \sum_{t=i+1}^{m} \mathbf{A}_{it}\mathbf{X}_{tj} - \sum_{l=1}^{j-1} \mathbf{Y}_{il}\mathbf{B}_{lj} - \sum_{l=j+1}^{n} \mathbf{X}_{il}\mathbf{B}_{lj}, \mathbf{X}_{ij}), \tag{32}$$

for $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

By $\Gamma(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{X})$ we denote the interval matrix $\mathbf{Y}$ defined by (32) and is named the generalized interval Gauss-Seidel (GIGS) operator, applied to $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{X}$. Note that the procedure (32) for determining the interval matrix $\mathbf{Y}$ must be done column-by-column or row-by-row.

One of the properties of the operator $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x})$ for interval numbers $\mathbf{a}, \mathbf{b}, \mathbf{x} \in \mathbb{IR}$ is that $\Xi(\mathbf{a}, \mathbf{b}) \cap \mathbf{x} \subseteq \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{x}) \subseteq \mathbf{x}$. By this property and the argument leading to the derivation of relation (32), we have the following theorem.

Theorem 3. *Consider the interval Sylvester matrix equation (26), also let* $\mathbf{X} \in \mathbb{IR}^{m\times n}$*. Then*

$$\Xi\big(\mathbf{A}, \mathbf{B}, \mathbf{C}\big) \cap \mathbf{X} \subseteq \Gamma\big(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{X}\big) \subseteq \mathbf{X}.$$

Now, we return to our main problem, i.e., the parameterized Sylvester matrix equation (1). For arbitrary $\mathbf{X} \in \mathbb{IR}^{m\times n}$, we are interested in good enclosures for the truncated solution set

$$\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big) \cap \mathbf{X}.$$

Similar to the above procedure for GIGS iteration method, first for $p \in \mathbf{p}$ write the system $A(p)X + XB(p) = C(p)$ componentwise

$$\sum_{t=1}^{m} A_{it}(p)X_{tj} + \sum_{l=1}^{n} X_{il}B_{lj}(p) = C_{ij}(p), \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n,$$

this yields

$$\begin{aligned} X_{ij} &= \big(C_{ij}(p) - \sum_{t\neq i} A_{it}(p)X_{tj} - \sum_{l\neq j} X_{il}B_{lj}(p)\big) / \big(A_{ii}(p) + B_{jj}(p)\big) \\ &\subseteq \square\Big\{\big(C_{ij}(p) - \sum_{t\neq i} A_{it}(p)X_{tj} - \sum_{l\neq j} X_{il}B_{lj}(p)\big) / \big(A_{ii}(p) + B_{jj}(p)\big) : \quad (p \in \mathbf{p}, X \in \mathbf{X})\Big\} := \ddot{\mathbf{X}}_{ij}, \end{aligned} \tag{33}$$

provided that $A_{ii}(p) + B_{jj}(p) \neq 0$ and the interval matrix $\mathbf{X}$ contains X. Another enclosure $\ddot{\mathbf{X}}$ for X can be obtained by applying (33) for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, provided that $A_{ii}(p) + B_{jj}(p) \neq 0$ and since this works for all $p \in \mathbf{p}$, we have

$$\sum\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big) \cap \mathbf{X} \subseteq \ddot{\mathbf{X}} \cap \mathbf{X}.$$

Now, for obtaining a compact operator similar to the GIGS operator, for arbitrary interval number $\mathbf{x} \in \mathbb{IR}$ and rational functions $r_1 : \mathbb{R}^s \longrightarrow \mathbb{R}$ and $r_2 : \mathbb{R}^s \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}$ with $\mathbf{p}$ and $\mathbf{p} \times \mathbf{X}$ as their domains, respectively, consider the following operator that for the first time has been used in [30]

$$\Gamma(r_1(p), r_2(p, X), \mathbf{x}, \mathbf{p}) = \square\{x \in \mathbf{x} : r_1(p)x = r_2(p, X), \quad (p \in \mathbf{p}, X \in \mathbf{X})\}. \tag{34}$$

Now, using this operator and available information to obtain an improved enclosure matrix $\mathbf{Z}$ for $\sum\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big)$, we define

$$\mathbf{Z}_{ij} = \Gamma\Big(A_{ii}(p)+B_{jj}(p), C_{ij}(p) - \sum_{t=1}^{i-1} A_{it}(p)Z_{tj} - \sum_{t=i+1}^{m} A_{it}(p)X_{tj} - \sum_{l=1}^{j-1} Z_{il}B_{lj}(p) - \sum_{l=j+1}^{n} X_{il}B_{lj}(p), \mathbf{X}_{ij}, \mathbf{p}\Big), \tag{35}$$

for $i = 1,\dots,m$ and $j = 1,\dots,n$. The interval matrix $\mathbf{Z}$ constructed by (35) is denoted by $\Gamma\big(A(p), B(p), C(P), \mathbf{X}, \mathbf{p}\big)$ and is called the generalized parameterized Gauss-Seidel (GPGS) operator. Note that the procedure (35) for obtaining the interval matrix $\mathbf{Z}$ must be done column-by-column or row-by-row. By the argument leading to the derivation of (35) we have the following theorem.

Theorem 4. *Consider the parameterized Sylvester matrix equation (1), also let* $\mathbf{X} \in \mathbb{IR}^{m\times n}$. *Then*

$$\Xi\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big) \cap \mathbf{X} \subseteq \Gamma\big(A(p), B(p), C(P), \mathbf{X}, \mathbf{p}\big) \subseteq \mathbf{X}.$$

Similar to the Gauss-Seidel method for interval linear systems, if $\Gamma\big(A(p), B(p), C(P), \mathbf{X}, \mathbf{p}\big)$ is strictly contained in $\mathbf{X}$, then we may hope to get more improved interval enclosures for $\sum\big(A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})\big) \cap \mathbf{X}$ by repeating the process, i.e., the iteration

$$\begin{cases} \mathbf{X}^{(0)} := \mathbf{X}, \\ \mathbf{X}^{(k+1)} = \Gamma\big(A(p), B(p), C(P), \mathbf{X}^{(k)}, \mathbf{p}\big), \quad k = 0, 1, \dots. \end{cases} \tag{36}$$

We call iteration (36) the generalized parameterized Gauss-Seidel (GPGS) iteration method for solving parameterized Sylvester matrix equations.

Definition 2. *Suppose f is a real-valued function. We say F is an interval extension of f, if for degenerate interval arguments $[x,x]$, F agrees with f, i.e., $F([x,x]) = f(x)$.*

Usually, the interval extension of a function is displayed with its corresponding English capital letter.

Definition 3. *Let $\leq$ denotes the partial order relation of any partially ordered set. The function f is called $\leq$-isotone if it satisfies the following property*

$$x \leq y \Rightarrow f(x) \leq f(y),$$

for all x and y in its domain.

The dual notion of the above definition is often called $\leq$-antitone. Hence, an $\leq$-antitone function f satisfies the property

$$x \leq y \Rightarrow f(x) \geq f(y),$$

for all x and y in its domain.

Now, only thing that remains is the computation of $\mathbf{Z}_{ij}$ from (35), i.e., the interval hull of the set

$$\Big\{(C_{ij}(p) - \sum_{t=1}^{i-1} A_{it}(p)Z_{tj} - \sum_{t=i+1}^{m} A_{it}(p)X_{tj} - \sum_{l=1}^{j-1} Z_{il}B_{lj}(p) - \sum_{l=j+1}^{n} X_{il}B_{lj}(p))/(A_{ii}(p)+B_{jj}(p)) : \ (p \in \mathbf{p}, X \in \mathbf{X})\Big\}.$$

To obtain the interval hull of the above set, we can apply the next theorem to the rational function $f_{ij} : \mathbf{X} \times \mathbf{p} \longrightarrow \mathbb{R}$

$$f_{ij}(X,p) = (C_{ij}(p) - \sum_{t=1}^{i-1} A_{it}(p)Z_{tj} - \sum_{t=i+1}^{m} A_{it}(p)X_{tj} - \sum_{l=1}^{j-1} Z_{il}B_{lj}(p) - \sum_{l=j+1}^{n} X_{il}B_{lj}(p))/(A_{ii}(p) + B_{jj}(p)). \quad (37)$$

This theorem helps us to eliminate the dependency problem by using generalized interval arithmetic. First let $f(x_1,\dots,x_m)$ be a rational function and $f(\mathbf{x}) = \{f(x) : x \in \mathbf{x}\}$ denotes the range of f over $\mathbf{x} \in \mathbb{IR}^m$.

Theorem 5. *[13, 30] Let $f(x,a)$ be a rational function multi-incident on a and there exists a splitting $a' = (a'_1,\dots,a'_p)$ and $a'' = (a''_1,\dots,a''_q)$ of the incidents of a. Let $g(x,a',a'')$ corresponds to the expression of f with explicit reference to the incidents of a and $g(x,a',a'')$ is continuous on $\mathbf{x} \times \mathbf{a}' \times \mathbf{a}''$. Suppose that $g(x,a',a'')$ is unconditionally $\leq$-isotone for any component of a' and unconditionally $\leq$-antitone for any component of a'' on $\mathbf{x} \times \mathbf{a}' \times \mathbf{a}''$, then*

- *if $f(x,a)$ is unconditionally $\leq$-isotone for a on $\mathbf{x} \times \mathbf{a}$,*

$$f(\mathbf{x},\mathbf{a}) = G(\mathbf{x},\mathbf{a}', dual(\mathbf{a}'')) \subseteq F(\mathbf{x},\mathbf{a}),$$

- *if $f(x,a)$ is unconditionally $\leq$-antitone for a on $\mathbf{x} \times \mathbf{a}$,*

$$f(\mathbf{x},\mathbf{a}) = G(\mathbf{x}, dual(\mathbf{a}'),\mathbf{a}'') \subseteq F(\mathbf{x},\mathbf{a}).$$

Now, using Theorem 5 and a conclusion of Theorem 3 in [30], we have the following theorem. Before it, corresponding to the parameterized Sylvester matrix equation (1) we consider the interval Sylvester matrix equation

$$A^{\mathbf{p}}X + XB^{\mathbf{p}} = C^{\mathbf{p}}, \quad (38)$$

therein

$$A^{\mathbf{p}} = \square\{A(p) \in \mathbb{R}^{m\times m} : p \in \mathbf{p}\}, \quad B^{\mathbf{p}} = \square\{B(p) \in \mathbb{R}^{n\times n} : p \in \mathbf{p}\}, \quad C^{\mathbf{p}} = \square\{C(p) \in \mathbb{R}^{m\times n} : p \in \mathbf{p}\}.$$

Theorem 6. *$\mathbf{X} \in \mathbb{IR}^{m\times n}$ is given. Let $\Gamma(A^{\mathbf{p}},B^{\mathbf{p}},C^{\mathbf{p}},\mathbf{X})$ be the GIGS operator related to the interval Sylvester matrix equation (38) corresponding to the parametric system (1). If there exist pairs (i,j), $i = 1,\dots,m$ and $j = 1,\dots,n$, such that the function $f_{ij}(X,p)$ defined by (37) satisfies Theorem 5, then for $\Gamma\big(A(p),B(p),C(P),\mathbf{X},\mathbf{p}\big)$ computed by Theorem 5, we have*

$$\Gamma\big(A(p),B(p),C(P),\mathbf{X},\mathbf{p}\big) \subsetneq \Gamma(A^{\mathbf{p}},B^{\mathbf{p}},C^{\mathbf{p}},\mathbf{X}).$$

Note that because the initial interval matrix $\mathbf{X}$ is arbitrary, so it may happen that during the procedure (35) some $\mathbf{Z}_{ij}$ becomes empty, in this case using the convention that an arithmetic expression involving an empty set has the value of empty set, we break the process. If $\Xi\big(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p})\big) \cap \mathbf{X} \neq \emptyset$ then the result of the execution of the GPGS method is a nested sequence of m-by-n interval matrices which has a limit in $\mathbb{IR}^{m\times n}$ and this limit point contains $\sum\big(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p})\big) \cap \mathbf{X}$.

Example 1. Consider the parameterized Lyaponov matrix equation $A(\mathbf{p})X + XA^T(\mathbf{p}) = C$ with

$$A(p) = \begin{pmatrix} p_1+p_4 & -p_4 & 0 \\ -p_4 & p_2+p_4+p_5 & -p_5 \\ 0 & -p_5 & p_3+p_5 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix}, \quad (39)$$

where $\mathbf{p} = (\mathbf{p}_1,\dots,\mathbf{p}_5)^T$ with $\mathbf{p}_k = [0.9, 1.1]$ for $k = 1,\dots,5$. The coefficient matrix $A(p)$ is taken from a physical problem similar to the one presented in [29], i.e., a linear resistive network. The considered resistive network consists of five resistors and when the voltage of each conductance p_k, $k = 1,\dots,5$, varies in intervals $\mathbf{p}_k$, $k =$

Table 1: The range of $\mathbf{Z}_{ij}$ for Example 1 using Theorem 5

Range of $\mathbf{Z}_{ij}$	Monotonicity p_4			Monotonicity p_5		
$\mathbf{Z}_{ij}$	Total	p_4n	p_4d	Total	p_5n	p_5d
$\mathbf{Z}_{11}=\frac{20+\text{dual}(\mathbf{p}_4)(\mathbf{X}_{12}+\mathbf{X}_{21})}{2(\mathbf{p}_1+\mathbf{p}_4)}$	↓	↑	↓	-	-	-
$\mathbf{Z}_{21}=\frac{\mathbf{p}_4(\mathbf{X}_{22}+\mathbf{Z}_{11})+\mathbf{p}_5\mathbf{X}_{31}}{2(\mathbf{p}_2+\text{dual}(\mathbf{p}_4)+\text{dual}(\mathbf{p}_5))}$	↑	↑	↓	↑	↑	↓
$\mathbf{Z}_{31}=\frac{\mathbf{p}_5\mathbf{Z}_{21}+\mathbf{p}_4\mathbf{X}_{32}}{2(\mathbf{p}_5+\text{dual}(\mathbf{p}_5))}$	-	-	-	↑	↑	↓
$\mathbf{Z}_{12}=\frac{\mathbf{p}_4(\mathbf{X}_{22}+\mathbf{Z}_{11})+\mathbf{p}_5\mathbf{X}_{13}}{2(\mathbf{p}_1+\text{dual}(\mathbf{p}_4))}$	↑	↑	↓	-	-	-
$\mathbf{Z}_{22}=\frac{20+\text{dual}(\mathbf{p}_4)(\mathbf{Z}_{12}+\mathbf{Z}_{21})+\text{dual}(\mathbf{p}_5)(\mathbf{X}_{23}+\mathbf{X}_{32})}{2(\mathbf{p}_2+\mathbf{p}_4+\mathbf{p}_5)}$	↓	↑	↓	↓	↑	↓
$\mathbf{Z}_{32}=\frac{\mathbf{p}_5(\mathbf{X}_{33}+\mathbf{Z}_{22})+\mathbf{p}_4\mathbf{Z}_{31}}{2(\mathbf{p}_3+\text{dual}(\mathbf{p}_5))}$	-	-	-	↑	↑	↓
$\mathbf{Z}_{13}=\frac{\text{dual}(\mathbf{p}_4)\mathbf{X}_{23}+\mathbf{p}_5\mathbf{Z}_{12}}{2(\mathbf{p}_1+\mathbf{p}_4)}$	↓	↑	↓	-	-	-
$\mathbf{Z}_{23}=\frac{\text{dual}(\mathbf{p}_4)\mathbf{Z}_{13}+\mathbf{p}_5(\mathbf{X}_{33}+\mathbf{Z}_{22})}{2(\mathbf{p}_2+\mathbf{p}_4+\text{dual}(\mathbf{p}_5))}$	↓	↑	↓	↑	↑	↓
$\mathbf{Z}_{33}=\frac{20+\text{dual}(\mathbf{p}_5)(\mathbf{Z}_{23}+\mathbf{Z}_{32})}{2(\mathbf{p}_3+\mathbf{p}_5)}$	-	-	-	↓	↑	↓

$1,\dots,5$, then the problem of finding its voltages, leads to a parameterized linear system with coefficient matrix $A(p)$ in (39).

Now, we want to enclose the ranges of the voltages. We compare the enclosures obtained by our new approach GPGS and `Verifylss.m` code of `INTLAB` when applied to the interval transformed parametric linear system (3).

`Verifylss.m` yields the following enclosure for the solution of the problem

$$\begin{pmatrix} [\,4.1894, 8.3106] & [\,0.4917, 4.5083] & [\,-0.2551, 2.7551] \\ [\,0.4917, 4.5083] & [\,2.6229, 7.3771] & [\,0.4917, 4.5083] \\ [\,-0.2551, 2.7551] & [\,0.4917, 4.5083] & [\,4.1894, 8.3106] \end{pmatrix},$$

and the new approach GPGS gives

$$\begin{pmatrix} [\,4.5454, 6.0556] & [\,1.0227, 2.2153] & [\,0.2301, 0.8343] \\ [\,0.7053, 1.4293] & [\,3.3183, 4.6445] & [\,0.5225, 1.2000] \\ [\,0.1586, 0.6681] & [\,0.7823, 1.7360] & [\,4.8716, 6.2896] \end{pmatrix}.$$

As you can see, GPGS approach gives tighter enclosure than the one obtained by `Verifylss.m` which shows advantage of our approach. On the other hand, dealing with the transformed system (3) requires much higher computational costs.

GPGS operator in each step involves nine parameter dependent functions that for elimination of the dependency problem, we apply Theorem 5. The first column in Table 1 consists the solution components $\mathbf{Z}_{ij}$ as a result of application of Theorem 5. In other columns, by arrows, we denote the total monotonicity of the function Z_{ij} with respect to the corresponding multi-incident parameter and the monotonicity with respect to each of its incidents. Note that by p_4n and p_4d, we denote the occurrence of p_4 in the numerator and denominator, respectively. For the first time, Popova in [30] efficiently used Theorem 5 for elimination of the dependency problem that was appeared in her method for solving parameterized linear systems. Note that the presented GPGS method can be applied for improvement of the obtained outer solutions from another methods.

3 Conclusion

In this paper, the Sylvester matrix equations depending linearly on interval parameters were considered. Besides the frequently appearance of Sylvester matrix equations in real problems, the possibility of dependency of their components to interval parameters, leads to the parameterized Sylvester matrix equations. A direct method for solving parametric system (1) was proposed. We proposed also two iterative methods, the first is based on fixed point iterations and the second is based on the well-known Gauss-Seidel method for solving linear systems. It is to be noted that the introduced GPGS method can be used for improving the obtained enclosures from another methods for the solution set of the parameterized system (1).

Acknowledgment

The author would like to thank the Shahid Chamran University of Ahvaz for financial support under the grant number SCU.MM1400.33518.

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