Journal of Mathematical Modeling
Vol. 10, No. 4, 2022, pp. 463-480. Research Article

JMM

Stable recovery of a space-dependent force function in a one-dimensional wave equation via Ritz collocation method

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Abstract. In this paper, we consider the problem of approximating the displacement and the wave sink or source in a 1D wave equation from various measurements. First, the problem is recast as a certain hyperbolic equation. Then, we propose a Ritz approximation as the solution of the reformulated problem and apply the collocation method to convert the inverse problem to a system of linear equations. Since the problem is not well-posed, the numerical discretization of the problem may produce a system of equations that is not well-conditioned. Therefore, we apply the Tikhonov regularization method to obtain a stable solution. For the contaminated measurements, we take advantage of the mollification method in order to derive stable numerical derivatives. Several test examples are provided to show the effectiveness of the proposed technique for obtaining satisfactory results.

Keywords: Inverse wave problem, Ritz collocation method, Tikhonov regularization, mollification technique.
AMS Subject Classification 2010: 35L05, 35R30, 65M32, 65N35.

1 Introduction

Consider a one-dimensional second-order hyperbolic equation

$$u_{tt} - u_{xx} = f(x) \quad \text{in} \quad (0,L) \times (0,T), \tag{1}$$

with the initial conditions

$$u(x,0) = u_0(x),\; u_t(x,0) = u_1(x) \quad \text{on} \quad (0,L), \tag{2}$$

and the Dirichlet boundary conditions

$$u(0,t) = b_0(t),\; u(L,t) = b_1(t) \quad \text{on} \quad (0,T). \tag{3}$$

*Corresponding author.
Received: 27 April 2022 / Revised: 24 July 2022 / Accepted: 25 July 2022
DOI: 10.22124/JMM.2022.22186.1956

Eqs. (1)-(3) describe the propagation of wave in a uniform string with certain length $L > 0$, which acts on a space-dependent force $f(x)$. If the function $f(x)$ is given, the problem of recovering $u(x,t)$ within Eqs. (1)-(3) is referred to as direct problem.

In contrast to the direct problem, wave source $f(x)$ of the inverse problem is unknown and additional specification is needed to obtain a unique solution [3, 10, 11]. In this paper, we investigate two inverse problems based on different overspecifications as follows:

- Suppose that the additional condition is the normal derivative of u at one of the boundary points, say at $x = 0$, namely,
$$u_x(0,t) = h_0(t) \quad \text{on} \quad (0,T). \tag{4}$$
Then, the problem of finding $(u(x,t), f(x))$ using Eqs. (1)-(4) is called the inverse problem 1 (**IP1**).

- Assume that the extra condition is the displacement measurement in space at final time, i.e.
$$u(x,T) = u_T(x) \quad \text{on} \quad (0,L). \tag{5}$$
Then, the problem of finding $(u(x,t), f(x))$ using Eqs. (1)-(3) and (5) is called the inverse problem 2 (**IP2**).

The outline of this paper is as follows. In Section 2, we propose the reconstruction procedures based on applying the satisfier function and the Ritz collocation method for numerically discretizing the considered problems **IP1** and **IP2**. In Section 3, we discuss the numerical results of solving four test examples. In Section 4, some concluding remarks are presented.

2 Reconstruction algorithm

In this section we propose the specific algorithm for solving problems **IP1** and **IP2**. The key feature of the method is applying the interpolation techniques to construct an auxiliary function, the so-called satisfier function which fulfills all the given initial and boundary conditions [9, 13]. Apart from the initial and boundary conditions, some physical information about the solution such as oscillatory behavior or the descending behavior of the solution can be useful to build the appropriate satisfier function that is close enough to the exact solution [16]. If a product form [1, 23, 24] for the solution of the problem is assumed, then under some restrictions one can find the unique satisfier function otherwise the satisfier functions are not unique globally. Following, we introduce a computational method for obtaining the satisfier function provided that the initial and boundary conditions are continuous in corners [19, 20].

2.1 The solution of inverse problem 1

Assume that the given initial and boundary conditions are smooth on $(0,L) \times (0,T)$ and the following compatibility conditions hold

$$b_0(0) = u_0(0),\ b_0'(0) = u_1(0),\ b_1(0) = u_0(L),\ b_1'(0) = u_1(L),\ u_0'(0) = h_0(0),\ h_0'(0) = u_1'(0), \tag{6}$$

and consider
$$B_1(x,t) = \gamma_1(x) b_0(t) + \gamma_2(x) b_1(t) + \gamma_3(x) h_0(t), \tag{7}$$

where

$$\gamma_i(x) = \sum_{j=0}^{2} \gamma_{ij} x^j, \quad i = 1,2,3,$$

and γ_{ij} are unknown coefficients. To apply the boundary conditions (3) and (4), we take the following equations into account

$$\begin{cases} \gamma_1(0) = 1, \ \gamma_1(L) = 0, \ \gamma_1'(0) = 0, \\ \gamma_2(0) = 0, \ \gamma_2(L) = 1, \ \gamma_2'(0) = 0, \\ \gamma_3(0) = 0, \ \gamma_3(L) = 0, \ \gamma_3'(0) = 1, \end{cases} \tag{8}$$

and obtain

$$\gamma_1(x) = 1 - \frac{x^2}{L^2}, \ \gamma_2(x) = \frac{x^2}{L^2}, \ \gamma_3(x) = x - \frac{x^2}{L}. \tag{9}$$

To impose the initial conditions (2), we consider

$$I_1(x,t) = \theta_1(t) u_0(x) + \theta_2(t) u_1(x), \tag{10}$$

where

$$\theta_i(t) = \sum_{j=0}^{1} \theta_{ij} t^j, \quad i = 1,2.$$

The unknown coefficients θ_{ij} are computed by applying the following equations

$$\begin{cases} \theta_1(0) = 1, \ \theta_1'(0) = 0, \\ \theta_2(0) = 0, \ \theta_2'(0) = 1, \end{cases} \tag{11}$$

which results in the following

$$\theta_1(t) = 1, \ \theta_2(t) = t. \tag{12}$$

We establish the satisfier function $s_1(x,t)$ utilizing the following equation

$$s_1(x,t) = I_1(x,t) + B_1(x,t) - \left\{ B_1(x,0) + t \left. \frac{\partial B_1(x,t)}{\partial t} \right|_{t=0} \right\}, \tag{13}$$

and obtain

$$\begin{aligned} s_1(x,t) \ = \ & \theta_1(t) u_0(x) + \theta_2(t) u_1(x) + \gamma_1(x) \Big(b_0(t) - b_0(0) - t b_0'(0) \Big) + \gamma_2(x) \Big(b_1(t) - b_1(0) - t b_1'(0) \Big) \\ & + \gamma_3(x) \Big(h_0(t) - h_0(0) - t h_0'(0) \Big). \end{aligned} \tag{14}$$

To solve the inverse problem **IP1**, we compute the wave source $f(x)$ by setting $t = 0$ in the governing Eq. (1), that is

$$f(x) = u_{tt}(x,t)|_{t=0} - u_0{}''(x). \tag{15}$$

Accordingly, Eq. (1) is rewritten as

$$u_{tt} - u_{xx} = u_{tt}(x,t)|_{t=0} - u_0{}''(x) \quad \text{in} \quad (0,L) \times (0,T), \tag{16}$$

and the Ritz approximation of **IP1** based on polynomial basis functions is sought in the form of the following truncated series

$$u_{N,N'}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij} t^{2+j}(x^{i+3} - Lx^{i+2}) + s_1(x,t). \tag{17}$$

Substituting the approximation $u_{N,N'}(x,t)$ in Eq. (16), the residual function is constructed as follows

$$\begin{aligned} Res_1(x,t) \; := \; & \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}\Big\{(2+j)(1+j)t^j(x^{i+3} - Lx^{i+2}) - t^{2+j}\Big((i+3)(i+2)x^{i+1} - L(i+2)(i+1)x^i\Big)\Big\} \\ & - \sum_{i=0}^{N} 2c_{i0}(x^{i+3} - Lx^{i+2}) + H_1(x,t) = 0, \end{aligned} \tag{18}$$

where

$$H_1(x,t) = \frac{\partial^2 s_1(x,t)}{\partial t^2} - \frac{\partial^2 s_1(x,t)}{\partial x^2} - \frac{\partial^2 s_1(x,t)}{\partial t^2}\bigg|_{t=0} + u_0''(x), \tag{19}$$

and

$$\frac{\partial^2 s_1(x,t)}{\partial t^2} = \gamma_1(x)b_0''(t) + \gamma_2(x)b_1''(t) + \gamma_3(x)h_0''(t), \tag{20}$$

$$\begin{aligned} \frac{\partial^2 s_1(x,t)}{\partial x^2} = \; & \theta_1(t)u_0''(x) + \theta_2(t)u_1''(x) - \frac{2}{L^2}\Big(b_0(t) - b_0(0) - tb_0'(0)\Big) + \frac{2}{L^2}\Big(b_1(t) - b_1(0) - tb_1'(0)\Big) \\ & - \frac{2}{L}\Big(h_0(t) - h_0(0) - th_0'(0)\Big). \end{aligned} \tag{21}$$

By applying the collocation method [2, 17, 19, 20] for $Res_1(x_i,t_j) = 0$ and by using the following collocation points

$$(x_i,t_j) = \Big(\frac{iL}{N+2}, \frac{jT}{N'+2}\Big), \quad i = 1,2,\ldots,N+1, \; j = 1,2,\ldots,N'+1, \tag{22}$$

we form the following system of linear equations

$$Ac = g, \tag{23}$$

where c is the vector of unknown constants c_{ij} that we need to determine to form our approximation. Typically, A is an ill-conditioned matrix, therefore we require using regularization techniques [18, 22] to obtain a stable solution. Hence, instead of (23), according to the Tikhonov regularization method [4–7, 10], we solve the following modified system of equations

$$(A^T A + \lambda I)c = A^T g, \tag{24}$$

where I is the identity matrix and $\lambda > 0$ is the regularization parameter.

2.2 The solution of inverse problem 2

By utilizing the extra condition (5), we derive the following relation

$$f(x) = u_{tt}(x,t)|_{t=T} - u_T''(x), \tag{25}$$

which is used to convert (1) to the following equation

$$u_{tt} - u_{xx} = u_{tt}(x,t)|_{t=T} - u_T''(x). \tag{26}$$

For solving **IP2**, we consider the following Ritz type approximation

$$u_{N,N'}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}(t^{3+j} - Tt^{2+j})(x^{i+2} - Lx^{i+1}) + s_2(x,t), \tag{27}$$

where the notation $s_2(x,t)$ stands for the satisfier function corresponding the initial and boundary conditions (2)-(3) and (5), provided that the given initial and boundary conditions are smooth on $(0,L)\times(0,T)$ and the following consistency conditions are satisfied

$$b_0(0) = u_0(0),\ b_0'(0) = u_1(0),\ b_1(0) = u_0(L),\ b_1'(0) = u_1(L),\ b_0(T) = u_T(0),\ b_1(T) = u_T(L). \tag{28}$$

We construct the satisfier function $s_2(x,t)$ through the following steps:

Step (1): Assume that

$$I_2(x,t) = \alpha_1(t)u_0(x) + \alpha_2(t)u_1(x) + \alpha_3(t)u_T(x),$$

where $\alpha_i(t) = \sum_{j=0}^{2}\alpha_{ij}t^j,\ i = 1,2,3$. By applying the following system of equations

$$\begin{cases} \alpha_1(0) = 1,\ \alpha_1'(0) = 0,\ \alpha_1(T) = 0, \\ \alpha_2(0) = 0,\ \alpha_2'(0) = 1,\ \alpha_2(T) = 0, \\ \alpha_3(0) = 0,\ \alpha_3'(0) = 0,\ \alpha_3(T) = 1, \end{cases} \tag{29}$$

we get the unknown coefficients α_{ij} and distinguish the functions $\alpha_i(t),\ i = 1,2,3$ as

$$\alpha_1(t) = 1 - \frac{t^2}{T^2},\ \alpha_2(t) = t - \frac{t^2}{T},\ \alpha_3(t) = \frac{t^2}{T^2}.$$

Step (2): Consider

$$B_2(x,t) = \beta_1(x)b_0(t) + \beta_2(x)b_1(t),$$

where $\beta_i(x) = \sum_{j=0}^{1}\beta_{ij}x^j,\ i = 1,2$. We use the following equations to get the unknown coefficients β_{ij}

$$\begin{cases} \beta_1(0) = 1,\ \beta_1(L) = 0, \\ \beta_2(0) = 0,\ \beta_2(L) = 1, \end{cases} \tag{30}$$

and specify the functions $\beta_j(x),\ j = 1,2$ as

$$\beta_1(x) = 1 - \frac{x}{L},\ \beta_2(x) = \frac{x}{L}.$$

Step (3): Define

$$s_2(x,t) = I_2(x,t) + B_2(x,t) - \left\{(1-\frac{x}{L})I_2(0,t) + \frac{x}{L}I_2(L,t)\right\},$$

which is equivalent to

$$\begin{aligned} s_2(x,t) \;&=\; \alpha_1(t)\left(u_0(x) - (1-\frac{x}{L})u_0(0) - \frac{x}{L}u_0(L)\right) + \alpha_2(t)\left(u_1(x) - (1-\frac{x}{L})u_1(0) - \frac{x}{L}u_1(L)\right) \\ &+\alpha_3(t)\left(u_T(x) - (1-\frac{x}{L})u_T(0) - \frac{x}{L}u_T(L)\right) + \beta_1(x)b_0(t) + \beta_2(x)b_1(t). \end{aligned} \tag{31}$$

By substituting the approximation (27) in Eq. (26), we get the residual function $Res_2(x,t)$ as

$$\begin{aligned} Res_2(x,t) \;:=& \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}\left\{(3+j)(2+j)t^{j+1} - T(2+j)(1+j)t^j\right\}(x^{i+2} - Lx^{i+1}) \\ &- \sum_{i=0}^{N}\sum_{j=0}^{N'} c_{ij}\left\{(2+i)(1+i)x^i - L(i+1)ix^{i-1}\right\}(t^{3+j} - Tt^{2+j}) + H_2(x,t) = 0, \end{aligned}$$

where

$$H_2(x,t) = \frac{\partial^2 s_2(x,t)}{\partial t^2} - \frac{\partial^2 s_2(x,t)}{\partial x^2} - \frac{\partial^2 s_2(x,t)}{\partial t^2}|_{t=T} + u_T''(x) - \sum_{i=0}^{N}\sum_{j=0}^{N'} 2c_{ij}(2+j)T^{j+1}(x^{i+2} - Lx^{i+1}). \tag{32}$$

It should be noted that the functions $\frac{\partial^2 s_2(x,t)}{\partial x^2}$ and $\frac{\partial^2 s_2(x,t)}{\partial t^2}$ are computed as:

$$\frac{\partial^2 s_2(x,t)}{\partial x^2} \;=\; \alpha_1(t)u_0''(x) + \alpha_2(t)u_1''(x) + \alpha_3(t)u_T''(x), \tag{33}$$

$$\begin{aligned} \frac{\partial^2 s_2(x,t)}{\partial t^2} \;&=\; \frac{-2}{T^2}\left(u_0(x) - (1-\frac{x}{L})u_0(0) - \frac{x}{L}u_0(L)\right) - \frac{2}{T}\left(u_1(x) - (1-\frac{x}{L})u_1(0) - \frac{x}{L}u_1(L)\right) \\ &+\frac{2}{T^2}\left(u_T(x) - (1-\frac{x}{L})u_T(0) - \frac{x}{L}u_T(L)\right) + \beta_1(x)b_0''(t) + \beta_2(x)b_1''(t). \end{aligned} \tag{34}$$

Finally, by applying the collocation points (22) to the equation $Res_2(x_i,t_j) = 0$, we get a system of linear equations for the unknown coefficients c_{ij} which is solved by employing the Tikhonov regularization technique, i.e. similar to Eq. (24).

3 Numerical experiments

In this section, we numerically solve four different problems to demonstrate the effectiveness of our technique. We define

$$E(u) = |u_{exact}(x,t) - u_{M,M'}(x,t)|, \quad E(f) = |f_{exact}(x) - f_{approximation}(x)|,$$

as the absolute error for functions u and f respectively. Moreover, $\Pi(A)$ denotes the condition number of the matrix A. All experiments are performed with Wolfram Mathematica on a PC.

Example 1. The first problem is **IP1** which is described by Eqs. (1)-(4) with

$$u_0(x) = \sin(\pi x),\ u_1(x) = 1,\ h_0(t) = \pi, \tag{35}$$
$$b_0(t) = t + \frac{t^2}{2} = b_1(t) = t + \frac{t^2}{2}.$$

We assume $L = T = 4$ and offer a solution by following the numerical method of Section 2.1, with $N = N' = 3$ and derive the following $s_1(x,t) = \sin(\pi x) + t + \frac{t^2}{2}$. Following the proposed solution and according to Eq. (23), we arrive at a homogenous system of equations $Ac = 0$ where the matrix A is nonsingular with determinant equal to -1.11346×10^{32}. Thus, we have $c_{i,j} = 0$ and so $u(x,t) = s_1(x,t)$, and it can be concluded from Eq. (15) that $f(x) = 1 + \pi^2 \sin(\pi x)$ which is the exact solution. Therefore, in some cases, the satisfier function not only provides the approximation to satisfy the initial and boundary conditions, but also may lead to the exact solution.

Example 2. The second problem is **IP1** which is described by Eqs. (1)-(4) with

$$u_0(x) = \cos(x) + \sin(\pi x),\ u_1(x) = 1 + e^x - \sin(x),\ h_0(t) = \pi - 1 + e^t - \sin(t), \tag{36}$$
$$b_0(t) = e^t + \cos(t) + t + \frac{t^2}{2} - 1,\ b_1(t) = e^{L+t} + \cos(L+t) - e^L + t + \frac{t^2}{2},$$

where the analytical solution is

$$f(x) = 1 + e^x + \pi^2 \sin(\pi x),\ u(x,t) = e^{x+t} + \cos(x+t) - e^x + \sin(\pi x) + t + \frac{t^2}{2}, \quad (x,t) \in [0,L] \times [0,T].$$

By considering $L = T = 2$, and applying the numerical scheme presented in Subsection 2.1, we obtain the results as shown in Table 3 with different values of $N = N' \in \{2,3,4\}$. Table 3 demonstrates that the approximate solution is close to the analytical solution and errors are decreased as the number of basis functions increases gradually which indicate that our method is convergent.

Next, we assume $L = T = 3$ and study the numerical stability of the approximate solution with respect to the small perturbations of the input data. In this respect, we contaminate the extra condition (4) with artificial errors using the following rule [8]

$$h_0^\sigma(t) = h_0(t) + \sigma \sin(\frac{t}{\sigma^2}), \quad \sigma = r \times 10^{-2},\ r \in \mathbb{N}. \tag{37}$$

Remark 1. *It must be noted that employing Eqs. (20)-(21) in (19) is acceptable as long as we have exact boundary data. Nevertheless, the presence of inaccuracies in the input data in practical applications such as (37), suggests performing the regularization procedure to obtain stable numerical derivatives of the perturbed data such as $h_0'(t)$ and $h_0''(t)$. Therefore, regarding the perturbed boundary data, if $h_0^\sigma(t)$ be perturbation such that $\|h_0(t) - h_0^\sigma(t)\|_\infty \leq \sigma$, then we employ the mollification method of [12] by taking into account the Gaussian mollifier*

$$F_\delta(t) = \frac{\exp(-\frac{t^2}{\delta^2})}{\delta\sqrt{\pi}},$$

where $\delta > 0$ is the radius of mollification. The mollification of the perturbed data $(h_0^\sigma(t))''$ is performed utilizing the following convolution [21]

$$\left\{F_\delta * (h_0^\sigma)''\right\}(t) := \int_{-\infty}^{+\infty} F_\delta(r)(h_0^\sigma)''(t-r)dr.$$

By the property of convolution,

$$\Big\{F_\delta * (h_0^\sigma)''\Big\}(t) = \Big\{F_\delta'' * (h_0^\sigma)\Big\}(t), \tag{38}$$

and for a given $\delta > 0$ we calculate $\Big\{F_\delta'' * (h_0^\sigma)\Big\}(t)$ numerically [14, 15] using the mid-point integration rule, that is

$$\Big\{F_\delta'' * (h_0^\sigma)\Big\}(t) \simeq \frac{\pi}{m_\delta}\sum_{i=0}^{m_\delta-1} Q(t, -\frac{\pi}{2} + \frac{\pi i}{m_\delta} + \frac{\pi}{2m_\delta}),\ Q(t,r) = F_\delta''(t-\tan r)h_0^\sigma(\tan r)\sec^2 r. \tag{39}$$

Then, we consider the following

$$(h_0^\sigma)''(t) = \Big\{F_\delta'' * (h_0^\sigma)\Big\}(t) \simeq \sum_{i=0}^{N'} d_i^{\delta,\sigma} t^i, \tag{40}$$

and consequently

$$(h_0^\sigma)'(t) \simeq \sum_{i=0}^{N'} d_i^{\delta,\sigma}\frac{t^{i+1}}{i+1} + u_1'(0), \quad h_0^\sigma(t) \simeq \sum_{i=0}^{N'} d_i^{\delta,\sigma}\frac{t^{i+2}}{(i+1)(i+2)} + tu_1'(0) + u_0'(0). \tag{41}$$

We call the strategy given by (38)-(41) admissible if for a small value $\varepsilon > 0$, and the appropriate given values δ and m_δ we have

$$\left\|\sum_{i=0}^{N'} d_i^{\delta,\sigma}\frac{t^{i+2}}{(i+1)(i+2)} + tu_1'(0) + u_0'(0) - h_0^\sigma(t)\right\|_\infty \le \varepsilon. \tag{42}$$

By employing the method investigated in Section 2.1, paying attention to Remark 1 to retrieve the regularized values for $(h_0^\sigma)(t)$, $(h_0^\sigma)'(t)$, $(h_0^\sigma)''(t)$ when $N = N' = 3$ and $\sigma \in \{3,6\}\times 10^{-2}$, we obtain the results as shown in Figure 1 and Table 1. In all cases, we set $m_\delta = 500$, $\varepsilon = \sigma$, where finding the appropriate value for δ is accomplished by trial and error. We take $\delta = 0.07$ when $\sigma = 3\times 10^{-2}$ whilst for the case $\sigma = 6\times 10^{-2}$, we find the best results by choosing $\delta = 0.1$. Following the obtained results, it can be seen that by using the regularization method, the errors introduced into the extra measurement of the problem are controlled and acceptable approximations are obtained.

Example 3. The third problem is **IP2** which is described by Eqs. (1)-(3) and (5) with

$$\begin{gathered} u_0(x) = -e^{4x} - \sec(0.3-x) + \sinh^2(x),\ u_1(x) = 2\sinh(x)\cosh(x), \\ u_T(x) = \sinh^2(x+T) - e^{4x} - T^2 - \sec(x-0.3), \\ b_0(t) = -2.04675 - t^2 + \sinh^2(t),\ b_1(t) = \sinh^2(L+t) - e^{4L} - t^2 - \sec(L-0.3), \end{gathered} \tag{43}$$

where the exact solution of this problem is

$$f(x) = -2 + 16e^{4x} + \sec(x-0.3)\frac{1+\sin^2(x-0.3)}{\cos^2(x-0.3)},\ u(x,t) = \sinh^2(x+t) - e^{4x} - t^2 - \sec(x-0.3).$$

We set $L = T = 1$ and apply the numerical scheme presented in subsection 2.2, with different values of $N = N' \in \{2,3,4\}$, and derive the results tabulated in Table 4. As shown, the accuracy improves as the number of basis functions increases. In other words, our method converges numerically as the number of basis functions increases.

Table 1: Comparison between the infinity norm of absolute errors for the unknown functions u and f, in the presence of the contaminated boundary data discussed in Example 2.

r	$\|E(u)\|_\infty$	$\|E(f)\|_\infty$
0	0.063	0.23
3	0.36	0.39
6	0.44	0.56

Example 4. Consider **IP2** given by Eqs. (1)-(3) and (5) with the following initial and boundary conditions

$$u_0(x) = e^{-2x} - \sin(3x),\ u_1(x) = -2e^{-2x},\ u_T(x) = 2.25 + e^{-3-2x} - \sin(3x), \tag{44}$$
$$b_0(t) = e^{-2t} + t^2,\ b_1(t) = e^{-3-2t} + t^2 - \sin(4.5),$$

and the following exact solution

$$f(x) = 2 - 9\sin(3x),\ u(x,t) = e^{-2x-2t} + t^2 - \sin(3x), \quad (x,t) \in [0,1.5] \times [0,1.5].$$

We examine the performance of the suggested method in the presence of inaccurate additional specification data $u_T(x)$ [8] given by

$$u_T^\sigma(x) = u_T(x) + \sigma \sin(\frac{x}{\sigma^2}), \quad \sigma = r \times 10^{-2},\ r \in \mathbb{N}. \tag{45}$$

We apply the mollification method to obtain stable numerical derivatives of the perturbed data such as $u_T'(x)$ and $u_T''(x)$. Similar to Example 2, we first take

$$F_{\delta_1}(x) = \frac{\exp(-\frac{x^2}{\delta_1^2})}{\delta_1\sqrt{\pi}},$$

and perform the mollification of the perturbed data $(u_T^\sigma(x))'$ using the following convolution

$$\left\{F_{\delta_1} * (u_T^\sigma)'\right\}(x) := \int_{-\infty}^{+\infty} F_{\delta_1}(r)(u_T^\sigma)'(t-r)dr. \tag{46}$$

We take advantage of the following relation

$$\left\{F_{\delta_1} * (u_T^\sigma)'\right\}(x) = \left\{F_{\delta_1}' * (u_T^\sigma)\right\}(x), \tag{47}$$

and for or a given $\delta_1 > 0$, calculate $\left\{F_{\delta_1}' * (u_T^\sigma)\right\}(x)$ numerically using the mid-point integration rule, that is

$$\left\{F_{\delta_1}' * (u_T^\sigma)\right\}(x) \simeq \frac{\pi}{m_{\delta_1}} \sum_{i=0}^{m_{\delta_1}-1} Q_1(x, -\frac{\pi}{2} + \frac{\pi i}{m_{\delta_1}} + \frac{\pi}{2m_{\delta_1}}),\ Q_1(x,r) = F_{\delta_1}'(x - \tan r)u_T^\sigma(\tan r)\sec^2 r. \tag{48}$$

Then, we consider the following

$$(u_T^{\sigma})'(x) = \left\{ F'_{\delta_1} * (u_T^{\sigma}) \right\}(x) \simeq \sum_{i=0}^{N} d_i^{\delta_1,\sigma} x^i, \tag{49}$$

and consequently

$$(u_T^{\sigma})(x) \simeq \sum_{i=0}^{N} d_i^{\delta_1,\sigma} \frac{x^{i+1}}{i+1} + b_0(T). \tag{50}$$

The strategy given by Eqs. (47)-(50) is admissible if for a small value $\varepsilon_1 > 0$, and the appropriate given values δ_1 and m_{δ_1}, we have the following

$$\| \sum_{i=0}^{N} d_i^{\delta_1,\sigma} \frac{x^{i+1}}{(i+1)} + b_0(T) - u_T^{\sigma}(x) \|_{\infty} \leq \varepsilon_1. \tag{51}$$

After recovering $(u_T^{\sigma})'(0) \simeq d_0^{\delta_1,\sigma}$ from (49), we employ the mollification method one more time to get stable value of $(u_T^{\sigma})''(x)$ with the parameters δ_2, m_{δ_2}, N, that is

$$(u_T^{\sigma})''(x) \simeq \frac{\pi}{m_{\delta_2}} \sum_{i=0}^{m_{\delta_2}-1} Q_2(x, -\frac{\pi}{2} + \frac{\pi i}{m_{\delta_2}} + \frac{\pi}{2m_{\delta_2}}) \simeq \sum_{i=0}^{N} d_i^{\delta_2,\sigma} x^i, \tag{52}$$

where $Q_2(x,r) = F''_{\delta_2}(x - \tan r) u_T^{\sigma}(\tan r) \sec^2 r$. The approximation given by Eq. (52) is acceptable if for a small value $\varepsilon > 0$, and the appropriate given values δ_2 and m_{δ_2}, we have the following

$$\| \sum_{i=0}^{N} d_i^{\delta_2,\sigma} \frac{x^{i+2}}{(i+1)(i+2)} + x d_0^{\delta_1,\sigma} + b_0(T) - u_T^{\sigma}(x) \|_{\infty} \leq \varepsilon. \tag{53}$$

The results of this experiment are shown in Figures (2)-(9) and Table 2 to present the agreement between the approximate and exact solutions. In other words, the numerical findings obtained for the wave source $f(x)$ and the displacement $u(x,t)$ deviate from the analytical solution approximately proportionally to the amount of introduced errors. This indicates that we have obtained stable numerical solutions.

Table 2: Comparison between the infinity norm of absolute errors for the unknown functions u and f, in the presence of the contaminated boundary data discussed in Example 4.

r	$\|E(u)\|_{\infty}$	$\|E(f)\|_{\infty}$
0	0.0004	0.001
4	0.076	0.12
7	0.12	0.14
10	0.166	0.17

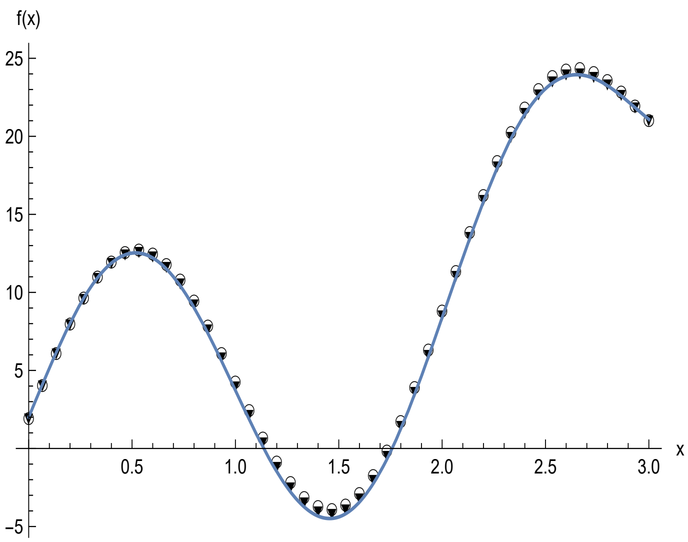

Figure 1: In this figure, the blue curve shows the exact values. The approximation of f is derived by our method with $N = N' = 3$ where the perturbed boundary data depends on the values of σ, i.e. ○○○: corresponding to $\sigma = 6\times 10^{-2}$, ▼▼▼: corresponding to $\sigma = 3\times 10^{-2}$ as discussed in Example 2.

Table 3: Comparison between the infinity norm of errors for the approximations of unknown functions u and f, the condition number $\Pi(A)$ and regularization parameter λ for approximations of unknown functions u and f in the presence of precise boundary data as discussed in Example 2.

Dimensions of A	$\|E(f)\|_\infty$	$\|E(u)\|_\infty$	$\Pi(A)$	λ
9×9	0.12	0.092	1.5×10^{3}	10^{-1}
16×16	0.031	0.022	7.39×10^{5}	10^{-2}
25×25	0.0026	0.00061	1.072×10^{7}	10^{-5}
49×49	7×10^{-4}	1.1×10^{-4}	2.96×10^{11}	10^{-4}
81×81	1×10^{-4}	6.3×10^{-5}	3.9×10^{15}	10^{-5}

Table 4: Comparison between the infinity norm of errors for the approximations of unknown functions u and f, the condition number $\Pi(A)$ and regularization parameter λ for approximations of unknown functions u and f in the presence of precise boundary data as discussed in Example 3.

Dimensions of A	$\|E(f)\|_\infty$	$\|E(u)\|_\infty$	$\Pi(A)$	λ
9×9	0.019	0.0014	1.38×10^{3}	10^{-3}
16×16	0.0041	0.00065	3.59×10^{17}	10^{-4}
25×25	0.001	0.000034	8.89×10^{6}	10^{-5}
49×49	1.3×10^{-4}	7.12×10^{-6}	1.21×10^{10}	10^{-8}
81×81	1×10^{-5}	3×10^{-6}	3.3×10^{13}	10^{-10}

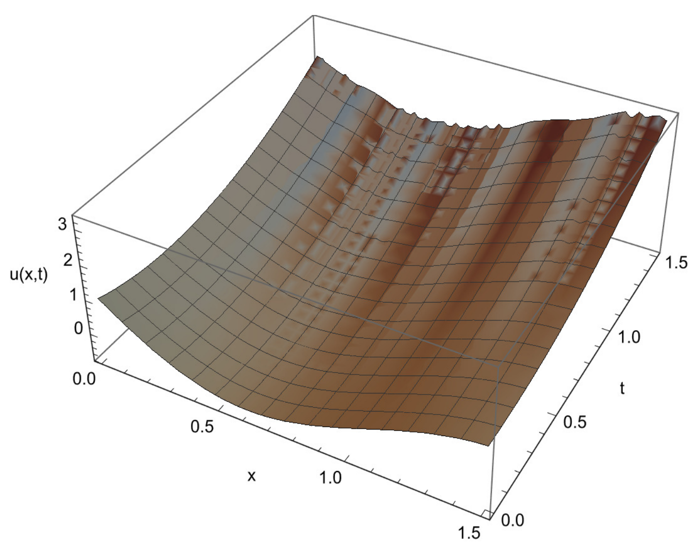

Figure 2: In this figure, the graphs of the approximate values of u and the exact values of u are sketched according to our method with $N = N' = 5,\ \delta_1 = \delta_2 = 0.06,\ m_{\delta_1} = 600,\ m_{\delta_2} = 700$ and $\lambda = 0.01$. Plots for the perturbed boundary data with $\sigma = 10\times 10^{-2}$ as discussed in Example 4.

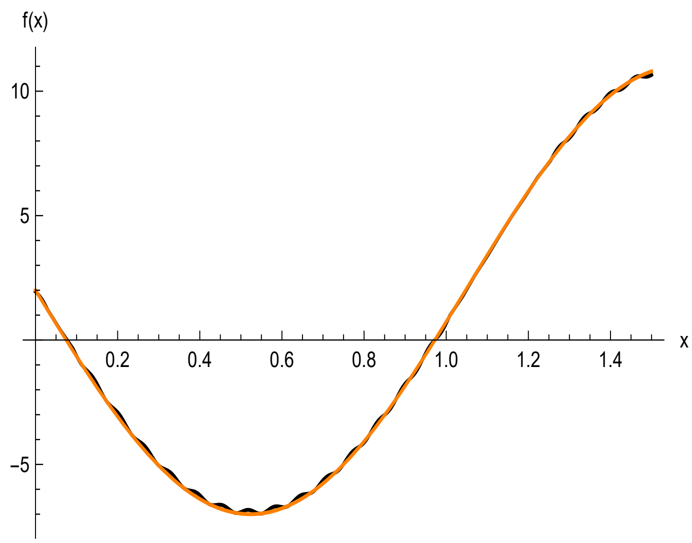

Figure 3: In this figure, the orange curve shows the exact solution of f while the black curve shows the approximate solution of f which is derived by our method with $N = N' = 5$, $\delta_1 = \delta_2 = 0.06$, $m_{\delta_1} = 600$, $m_{\delta_2} = 700$ and $\lambda = 0.01$. Plots for the perturbed boundary data with $\sigma = 10 \times 10^{-2}$ as discussed in Example 4.

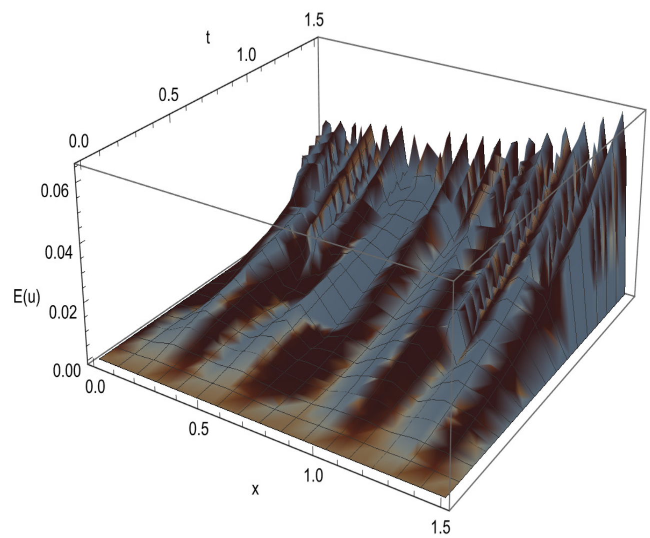

Figure 4: In this figure, the graph of the absolute error for function u is derived by our method with $N = N' = 5$, $\delta_1 = 0.01$, $\delta_2 = 0.02$, $m_{\delta_1} = m_{\delta_2} = 600$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 4 \times 10^{-2}$ as discussed in Example 4.

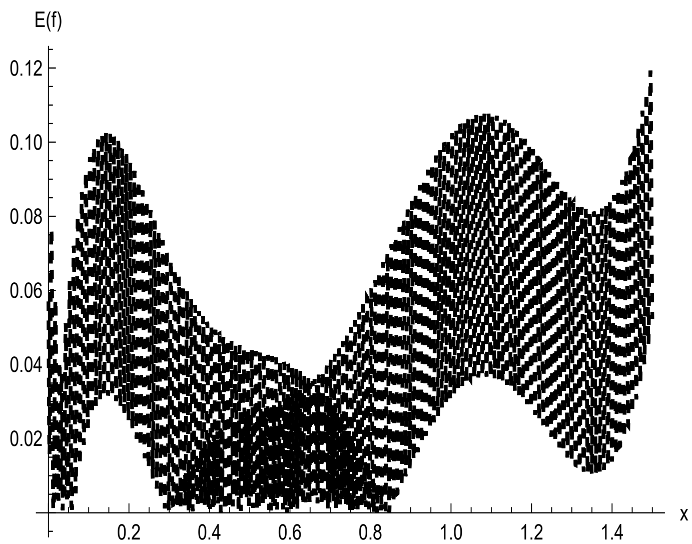

Figure 5: In this figure, the graph of the absolute error for function f is derived by our method with $N = N' = 5,\ \delta_1 = 0.01,\ \delta_2 = 0.02,\ m_{\delta_1} = m_{\delta_2} = 600$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 4 \times 10^{-2}$ as discussed in Example 4.

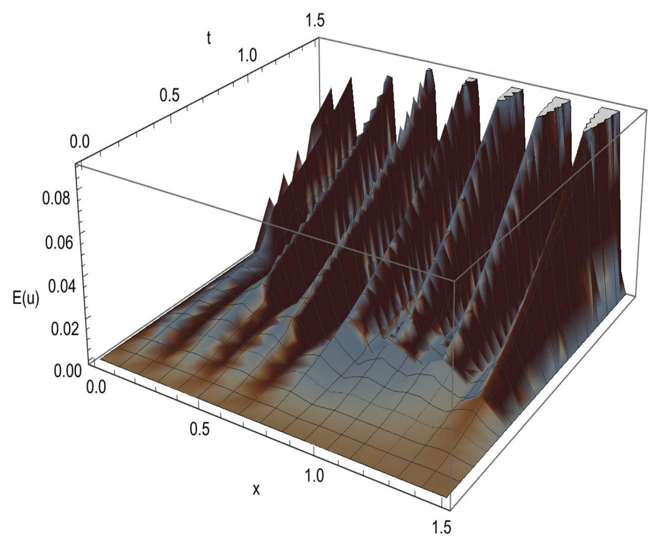

Figure 6: In this figure, the graph of the absolute error for function u is derived by our method with $N = N' = 5,\ \delta_1 = 0.03,\ \delta_2 = 0.045,\ m_{\delta_1} = m_{\delta_2} = 600$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 7 \times 10^{-2}$ as discussed in Example 4.

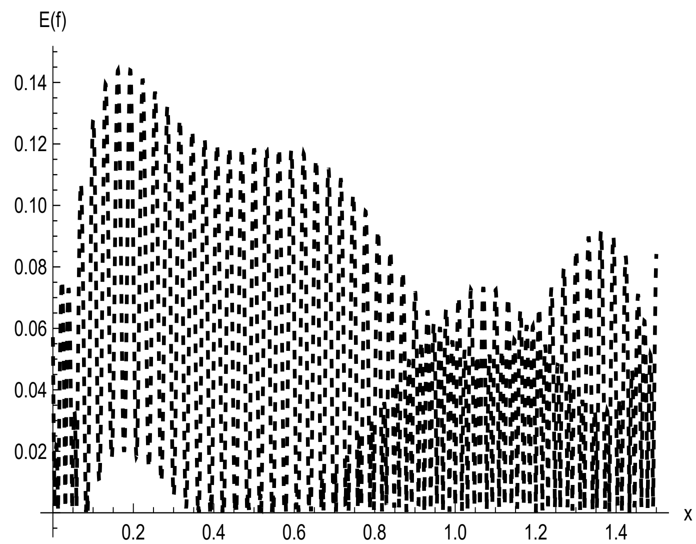

Figure 7: In this figure, the graph of the absolute error for function f is derived by our method with $N = N' = 5,\ \delta_1 = 0.03,\ \delta_2 = 0.045,\ m_{\delta_1} =\ m_{\delta_2} = 600$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 7 \times 10^{-2}$ as discussed in Example 4.

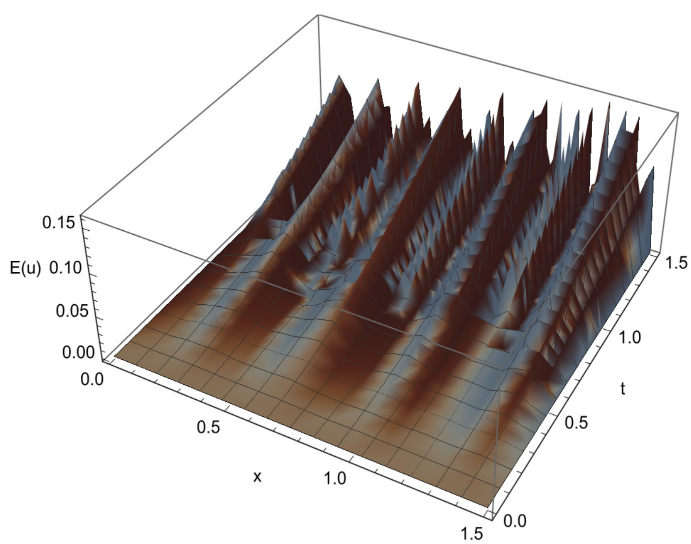

Figure 8: In this figure, the graph of the absolute error for function u is derived by our method with $N = N' = 5,\ \delta_1 = 0.03,\ \delta_2 = 0.045,\ m_{\delta_1} =\ m_{\delta_2} = 600$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 10 \times 10^{-2}$ as discussed in Example 4.

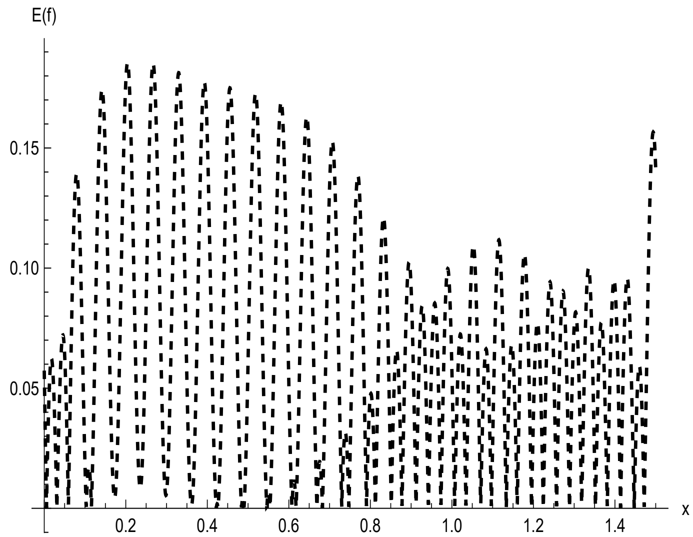

Figure 9: In this figure, the graph of the absolute error for function f is derived by our method with $N = N' = 5$, $\delta_1 = \delta_2 = 0.06$, $m_{\delta_1} = 600$, $m_{\delta_2} = 700$ and $\lambda = 0.01$. Plot for the perturbed boundary data with $\sigma = \varepsilon = \varepsilon_1 = 10 \times 10^{-2}$ as discussed in Example 4.

4 Conclusion

This article presents an approximation for the solution of the inverse problem of a one-dimensional wave equation from two additional measurements. We recast the problem as a certain hyperbolic equation and consider the Ritz approximation as the solution of the unknown displacement. Then, the collocation technique is employed to convert the problem to a system of linear equations. We take advantage of the mollification method to derive stable numerical derivatives and solve the system of equations that is not well-conditioned by employing the Tikhonov regularization method. Following the numerical simulations, it is confirmed that our method is a robust approach in dealing with introduced artificial errors in the input boundary data. Furthermore, our method performs quite well in the presence of exact boundary data since the approximate solutions converge to the exact solutions numerically. Compared to the results presented in [3], it can be observed that the algorithms proposed in this paper yield better results. In fact, the proposed algorithms provide higher accuracy with lower computational cost due to the use of a satisfier function. Moreover, in some cases, we arrive at the exact solution (see example 1). The proposed technique can be adapted to solve similar problems in higher dimensions.

Acknowledgements

We would like to thank the anonymous referees for their valuable comments and helpful suggestions to improve the earlier version of this article.

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