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Solvability of the fuzzy integral equations due to road traffic flow

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Abstract. In this research, we investigate the fuzzy integral equations related to traffic flow. Using the Banach fixed point theorem, we prove the existence and uniqueness of the solution for such equations. Using the Picard iterative method, we obtain the upper bound for an accurate and approximate solution. Finally, we obtain an error estimation between the exact solution and the solution of the iterative method. Example shows the applicability of our results.

Keywords: Fuzzy integral equations, traffic Flow, iterative method.
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1 Introduction

Integral and differential equations have many applications in science. Many modellings in different sciences result in integral and differential equations. Traffic flow prediction is one of these applications that many researchers have shown interest in studying in this field and have attempted to model traffic flow (see [18, Page 216], [23, Page 389] and references therein). A topic such as traffic flows is also a topic of substantial research interest to a whole range of academics such as mathematicians, civil engineers, geographers, ecologists and management scientists and has been modelled in a number of ways utilising ideas from areas such as fluid flow, statistical physics and chaos theory [17, Page 115].

It is shown in [10, Page 79] that the integral equation

$$\rho(t) = f\big(t,u(t)\big)\int_0^1 \mathbf{v}\big(t,s,u(s)\big)ds, \tag{1}$$

occurs in traffic-related models.

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Recently, the authors in [7] have generalized Eq. (1) to

$$\rho(t) = g(t) + f\big(t, u(t)\big) \int_0^\infty \mathbf{v}\big(t, s, u(s)\big) ds, \tag{2}$$

in crisp mode. Many researchers have studied the existence and uniqueness of the solution of the fuzzy integral equations of Volterra and Fredholm type (see [11, Page 1], [24, Page 5], [2, Page 2], [1, Page 75] and etc), but few of them have proved the existence and uniqueness of the solution of integral equations of a particular type. The numerical methods for fuzzy integral equations involve various techniques. The authors of this paper have previously used the method of successive approximations and other iterative techniques to solve fuzzy integral equations ([22, Page 6], [21, Page 1773]). In [7, Page 557] the existence of at least one solution of integral Eq. (2) in classic logic is proved. Contrary to Crisp logic, the fuzzy logic takes into account ambiguity as a part of system modelling. So, numerical solution of fuzzy integral equations are more important than solving crisp integral equations [20, Page 13287].

In this paper we prove the existence and uniqueness of the solution for fuzzy integral equations such as

$$\rho(t) = g(t) \oplus f\big(t, \rho(t)\big) \odot (FR) \int_0^1 \mathbf{v}\big(t, s, \rho(s)\big) ds, \quad t \in [0,1], \tag{3}$$

under simpler conditions than in [7, Page 557], where ρ is an unknown fuzzy function and represents the traffic density, g is the generation (or dissipation) rate in vehicle per unit time per length, f is crisp function and $\mathbf{v}$ is traffic flow speed .

The structure of the paper is as follows. In Section 2, we introduce some fuzzy concepts. In Section 3, the existence and uniqueness of the solution of the fuzzy integral equations arising of traffic flow is proved using the Banach fixed point theorem. Also, the error estimation between the exact solution and the solution of the iterative method is obtained . Section 4 includes an example to check the accuracy of the proposed method . Finally, in Section 5 we present our concluding remarks.

2 Some fuzzy concepts

Definition 1. *(See [12, Page 619]). A fuzzy number is a function $\eta : R \to [0,\, 1]$ having the properties:*

(1) η *is normal, that is* $\exists\, x_0 \in R$ *such that* $\eta(x_0) = 1$,

(2) η *is fuzzy convex set*

$$(i.e.\ \eta(\lambda x + (1-\lambda)y) \geq \min\{\eta(x), \eta(y)\} \quad \forall x, y \in R, \lambda \in [0,1]),$$

(3) η *is upper semi-continuous on R,*

(4) the $\overline{\{\, x \in R : \eta(x) > 0\}}$ *is compact set.*

The set of all fuzzy numbers is denoted by R_F. An alternative definition which yields the same R_F is given by [15, Page304].

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Definition 2. *(See [13, Page 33]). An arbitrary fuzzy number is represented, in parametric form, by an ordered pair of functions* $(\underline{u}(r), \overline{u}(r)), 0 \leq r \leq 1$, *which satisfy the following requirements:*

(1) $\underline{u}(r)$ *is a bounded left continuous non-decreasing function over [0,1],*

(2) $\overline{u}(r)$ *is a bounded left continuous non-increasing function over [0,1],*

(3) $\underline{u}(r) \leq \overline{u}(r) \quad , \quad 0 \leq r \leq 1.$

The addition and scaler multiplication of fuzzy numbers in R_F *are defined as follows:*

(4) $(u \oplus v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$

(5) $(\lambda \odot u)(r) = \begin{cases} (\lambda \underline{u}(r), \lambda \overline{u}(r)) & \lambda \geq 0, \\ (\lambda \overline{u}(r), \lambda \underline{u}(r)) & \lambda < 0. \end{cases}$

(6) the product $\eta \odot \mu$ *of fuzzy numbers* η *and* μ*, based on Zadeh's extension principle, is defined by*

$$\underline{(\eta \odot \mu)(r)} = \min\left\{\underline{\eta}(r)\underline{\mu}(r), \underline{\eta}(r)\overline{\mu}(r), \overline{\eta}(r)\underline{\mu}(r), \overline{\eta}(r)\overline{\mu}(r)\right\}$$

$$\overline{(\eta \odot \mu)(r)} = \max\left\{\underline{\eta}(r)\underline{\mu}(r), \underline{\eta}(r)\overline{\mu}(r), \overline{\eta}(r)\underline{\mu}(r), \overline{\eta}(r)\overline{\mu}(r)\right\}$$

Definition 3. *(See [5, Page 126]). A fuzzy number* $u \in R_F$ *is said to be positive if* $\underline{u}(1) \geq 0$*, strict positive if* $\underline{u}(1) > 0$*, negative if* $\overline{u}(1) \leq 0$*, strict negative if* $\overline{u}(1) < 0$*. We say that u and v have the some sign if they are both positive or both negative.*

Definition 4. *(See [6, Page 1102]). Let* $u = \left(\underline{u}(r), \overline{u}(r)\right), v = \left(\underline{v}(r), \overline{v}(r)\right) \in R_F$ *be fuzzy numbers with positive support (i.e.,* $\underline{u}(0) > 0, \underline{v}(0) > 0$*). It is defined the product* $u \odot v = \left(\underline{u \odot v}, \overline{u \odot v}\right) \in C[0,1] \times C[0,1]$ *by* $\underline{(u \odot v)}(r) = \underline{u}(r)\underline{v}(r)$ *and* $\overline{(u \odot v)}(r) = \overline{u}(r)\overline{v}(r)$*,* $\forall r \in [0,1]$*. The power of positive fuzzy number (with positive support) is given as* $(u^2)(r) = \left((\underline{u}(r))^2, (\overline{u}(r))^2\right)$*.*

Definition 5. *(See [3, Page 64]). For arbitrary fuzzy numbers* $u = (\underline{u}(r), \overline{u}(r))$ *,* $v = (\underline{v}(r), \overline{v}(r))$ *the quantity* $D_F(u, v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}$ *is the distance between u and v. The following properties are hold (See [3, Page 67]):*

(1) (R_F, D) *is a complete metric space,*

(2) $D_F(u \oplus w, v \oplus w) = D_F(u, v) \quad \forall\, u, v, w \in R_F,$

(3) $D_F(k \odot u, k \odot v) = |k|\, D_F(u, v) \quad \forall\, u, v \in R_F \quad \forall\, k \in R,$

(4) $D_F(u \oplus v, w \oplus e) \leq D_F(u, w) + D_F(v, e) \quad \forall\, u, v, w, e \in R_F.$

(5) $D_F(a \odot u, b \odot u) \leq |a - b| D_F(u, 0) \quad \forall\, u \in R_F;\ \forall a, b \in R,\ ab > 0,$

Theorem 1. *(1) The pair* $(R_F, \oplus)$ *is a commutative semigroup with* $\tilde{0} = \chi_0$ *zero element.*

(2) *For fuzzy numbers which are not crisp, there is no opposite element (that is, $(R_F, \oplus)$ cannot be a group).*

(3) *For any $a, b \in R$ with $a, b \geq 0$ or $a, b \leq 0$ and for any $u \in R_F$, we have $(a+b) \odot u = a \odot u \oplus b \odot u$. For arbitrary $a, b \in R$, this property is not fulfilled.*

(4) *For any $\lambda, \mu \in R$ and $u \in R_F$, we have $\lambda \odot (u \oplus v) = \lambda \odot u \oplus \lambda \odot u$.*

(5) *For any $\lambda \in R$ and $u, v \in R_F$, we have $\lambda \odot (\mu \odot u) = (\lambda.\mu) \odot u$.*

(6) *The function $\|.\|_F : R_F \to R$ by $\|u\|_F = D_F(u, \tilde{0})$ has the usual properties of the norm, that is, $\|u\|_F = 0$ if and only if $u = \tilde{0}$, $\|\lambda \odot u\|_F = |\lambda| \|u\|_F$ and $\|u \oplus v\|_F \leq \|u\|_F + \|v\|_F$.*

(7) *$| \|u\|_F - \|v\|_F | \leq D_F(u, v)$ and $D_F(u, v) \leq \|u\|_F + \|v\|_F$ for any $u, v \in R_F$.*

Proof. See [9, Page 1283], [4, Page 704]. □

Definition 6. *(See [15, Page 306]). A fuzzy real number valued function $f : [a, b] \to R_F$ is said to be continuous in $x_0 \in [a, b]$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $D_F(f(x), f(x_0)) < \varepsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$, and denote the space of all such functions by $C_F[a, b]$.*

Definition 7. *(See [9, Page 1281]). If $X = \{f : [a, b] \to R_F \,|\, f \text{ is continuous}\}$, then X together with the metric*

$$D^*(f, g) = \sup_{a \leq s \leq b} D_F(f(s), g(s))$$

is complete metric space.

Lemma 1. *(See [9, Page1283]). Let f_1 and f_2 are Riemann integrable functions. If the function given by $D_F\big(f_1(s), f_2(s)\big)$ is Riemann integrable, then*

$$D_F\left((FR)\int_a^b f_1(s)ds, (FR)\int_a^b f_2(s)ds\right) \leq \int_a^b D_F\big(f_1(s), f_2(s)\big)ds. \tag{4}$$

Lemma 2. *(See [20, Page13289]). For given $\alpha, \eta, \mu \in R_F^+$, we have $D_F(\alpha \odot \eta, \alpha \odot \mu) \leq \|\alpha\|_F D_F(\eta, \mu)$*

Lemma 3. *. Let $f : [a, b] \subseteq R \to R_F$ be a bounded and Riemann integrable function, then*

$$\left\|(FR)\int_a^b f(x)dx\right\|_F \leq \int_a^b \|f(x)\|_F\, dx$$

Proof. By Theorem 1-(6), we have

$$\begin{aligned}\left\|(FR)\int_a^b f(x)dx\right\|_F &= D_F\big((FR)\int_a^b f(x)dx, \tilde{0}\big) = D_F\big((FR)\int_a^b f(x)dx, (FR)\int_a^b \tilde{0}dx\big) \\ &\leq \int_a^b D_F\big(f(x), \tilde{0}\big)dx = \int_a^b \|f(x)\|_F dx\end{aligned}$$

□

Lemma 4. *(See [9, Page 1283]). If $H \in C\big([a,b] \times [a,b] \times R_{\mathcal{F}}, E^1\big)$, $g \in C\big([a,b] \times [a,b], R_{\mathcal{F}}\big)$ and $a \in C\big([a,b], R_+\big)$ then the functions $a.g : [a,b] \to R_{\mathcal{F}}$ and $F : [a,b] \to R_{\mathcal{F}}$ given by $(a.g)(t) = a(t).g(t)$, for all $t \in [a,b]$ and $F(t) = (FH)\int_a^b H(t,s,u(s))ds$ are continuous.*

Lemma 5. *(See [8, Page 492]). If $f : [a,b] \to C\big([a,b], R_{\mathcal{F}}\big)$ is continuous then it is bounded and its supremum $sup_{t\in[a,b]} f(t)$ must exist and is determined by $u \in C\big([a,b], R_{\mathcal{F}}\big)$ with $u_-^r = sup_{t\in[a,b]} f_-^r(t)$ and $u_+^r = sup_{t\in[a,b]} f_+^r(t)$. A similar conclusion for the infimum is also true.*

3 Existence and uniqueness theorems

In order to prove the existence and uniqueness of the solution of Eq. (3), we assume that the following conditions are holds:

(1) $g \in C([a,b], R_{\mathcal{F}})$, $f \in C([a,b] \times R_{\mathcal{F}}, R_{\mathcal{F}})$, $f(t,\rho(t)) \geq 0$ and $\mathbf{v} \in C([a,b] \times [a,b] \times R_{\mathcal{F}}, R_{\mathcal{F}})$, $\forall t,s \in [a,b]$;

(2) There exists $\alpha, \beta \geq 0$ such that $D_F\big(\mathbf{v}(t_1,s,\rho_1(s)), \mathbf{v}(t_2,s,\rho_2(s))\big) \leq \alpha|t_1 - t_2| + \beta D_F(\rho_1(s), \rho_2(s))$;

(3) There exists $\beta M_f + \eta M_H < 1$, where $M_f, M_H \geq 0$ are such that $\|f(t,\rho(t))\| \leq M_f$, and $\|\mathbf{v}(t,s,\rho(s))\| \leq M_H$, $\forall t,s \in [a,b]$ according to the continuity of f;

(4) There exists $\gamma \geq 0$, such that $D_F\big(g(t_1), g(t_2)\big) \leq \gamma|t_1 - t_2|$ for all $t_1, t_2 \in [a,b]$;

(5) There exists $\delta \geq 0$, such that $D_F\big(f(t_1,\rho(t_1)), f(t_2,\rho(t_2))\big) \leq \delta|t_1 - t_2|$ for all $t_1, t_2 \in [a,b]$;

(6) There exists $\eta \geq 0$, such that $D_F\big(f(t,\rho_1(t)), f(t,\rho_2(t))\big) \leq \eta D_F\big(\rho_1(t), \rho_2(t)\big)$ for all $t \in [a,b]$.

Theorem 2. *Under conditions (1)-(3), the integral Eq. (3) has a unique solution in $C\big([a,b], R_{\mathcal{F}}\big)$, $\rho^* \in C\big([a,b], R_{\mathcal{F}}\big)$, and the sequence of successive approximations $(\rho_m)_{m\in N} \subset C\big([a,b], R_{\mathcal{F}}\big)$,*

$$\rho_m(t) = g(t) \oplus f\big(t, \rho_{m-1}(t)\big) \odot (FR)\int_0^1 v\big(t,s,\rho_{m-1}(s)\big)ds, \quad t \in [0,1], m \in N^*, \tag{5}$$

converges to ρ^* in $C\big([a,b], R_{\mathcal{F}}\big)$. In addition, the following error estimate hold:

$$D^*\big(\rho^*, \rho_m\big) \leq \frac{(\beta M_f + \eta M_H)^m}{1 - (\beta M_f + \eta M_H)} D^*\big(\rho_1, \rho_0\big), \quad \forall t \in [0,1], m \in N^*, \tag{6}$$

and by choosing $\rho_0 \in C\big([a,b], R_{\mathcal{F}}\big), \rho_0 = g$, the inequality (6) becomes

$$D^*\big(\rho^*, \rho_m\big) \leq \frac{(\beta M_f + \eta M_H)^m}{1 - (\beta M_f + \eta M_H)} M_0 M_f, \quad \forall t \in [0,1], m \in N^*. \tag{7}$$

where $M_0 \geq 0$ such that $D_F\big(v(t,s,\rho(s)), \tilde{0}\big) \leq M_0$. Moreover, the sequence of successive approximations (5) is uniformly bounded, that is, there exists a constant $R \geq 0$ such that $D_F\big(\rho_m(t), \tilde{0}\big) \leq R$, for all $m \in N$, $t \in [0,1]$, and the solution ρ^* is bounded, too.

Proof. In order to prove the theorem, we investigate the conditions of the Banach fixed point principle. Let $X = C\big([0,1],R_{\mathcal{F}}\big) = \{$h: $[a,b] \longrightarrow R_{\mathcal{F}}\,|$h is continuous$\}$ be the space of fuzzy continuous functions with the metric $D^*(h_1,h_2) = \sup_{a\le t\le b} D_F\big(h_1(t),h_2(t)\big)$. Now, we define the operator $T : X \longrightarrow X$ as follows

$$T(\rho(t)) = g(t) \oplus f(t,\rho(t)) \odot (FR)\int_0^1 \mathbf{v}\big(t,s,\rho(s)\big)ds.$$

Firstly, we demonstrate that T maps X into X (i.e. $T(X) \subset X$). For this purpose we show that the operator T is uniformly continuous.

Let arbitrary $\rho \in X, s_0 \in [0,1]$ and $\varepsilon > 0$. Since ρ is continuous, for $\delta(\varepsilon) = \frac{\varepsilon}{2\alpha} > 0$, it follows that $D_F(\rho(s),\rho(s0)) < \frac{\varepsilon}{2\beta}$ for any $s \in [0,1]$ with $|s-s_0| < \delta(\varepsilon)$. Thus,

$$D_F\big(\mathbf{v}(t,s,\rho(s)),\mathbf{v}(t,s_0,\rho(s_0))\big) \le \alpha|s-s_0| + \beta D_F(\rho(s),\rho(s_0)) \le \alpha\frac{\varepsilon}{2\alpha} + \beta\frac{\varepsilon}{2\beta} = \varepsilon,$$

therefor the function $\mathbf{v}(t,s,\rho(s))$ is continuous in s_0. We infer that $\mathbf{v}$ is continuous on $[0,1]$ for any $\rho \in C\big([0,1],R_{\mathcal{F}}\big)$. Using Lemma 4 it follows that the function $f(t,\rho(t)) \odot (FR)\int_0^1 \mathbf{v}(t,s,\rho(s))ds$ is continuous on $C\big([0,1],R_{\mathcal{F}}\big)$ for any $\rho \in C\big([0,1],R_{\mathcal{F}}\big)$. we conclude that $T(\rho)$ is continuous on $[0,1]$ for any $\rho \in C\big([0,1],R_{\mathcal{F}}\big)$, and then $T\big(C([0,1],R_{\mathcal{F}})\big) \subset C\big([0,1],R_{\mathcal{F}}\big)$. Now, we show that the operator T is a contraction. For arbitrary $\rho_1,\rho_2 \in C\big([0,1],R_{\mathcal{F}}\big)$ and $t \in [0,1]$, we can write

$$\begin{aligned}
&D_F\Bigg(T(\rho_1(t)),T(\rho_2(t))\Bigg)\\
&= D_F\Bigg(g(t)\oplus f(t,\rho_1(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_1(s)\big)ds, g(t)\oplus f(t,\rho_2(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_2(s)\big)ds\Bigg)\\
&\le D_F\Bigg(f(t,\rho_1(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_1(s)\big)ds, f(t,\rho_2(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_2(s)\big)ds\Bigg)\\
&= D_F\Bigg((FR)\int_0^1 f(t,\rho_1(t))\odot\mathbf{v}\big(t,s,\rho_1(s)\big)ds, (FR)\int_0^1 f(t,\rho_2(t))\odot\mathbf{v}\big(t,s,\rho_2(s)\big)ds\Bigg)\\
&\le \int_0^1 D_F\Bigg(f(t,\rho_1(t))\odot\mathbf{v}\big(t,s,\rho_1(s)\big), f(t,\rho_2(t))\odot\mathbf{v}\big(t,s,\rho_2(s)\big)\Bigg)ds\\
&\le \int_0^1 \Bigg(D_F\big(f(t,\rho_1(t))\odot\mathbf{v}\big(t,s,\rho_1(s)\big), f(t,\rho_1(t))\odot\mathbf{v}\big(t,s,\rho_2(s)\big)\big)\\
&\quad + D_F\big(f(t,\rho_1(t))\odot\mathbf{v}\big(t,s,\rho_2(s)\big), f(t,\rho_2(t))\odot v\big(t,s,\rho_2(s)\big)\big)\Bigg)ds\\
&\le \int_0^1 \Bigg(\|f(t,\rho_1(t))\|_F D_F\big(\mathbf{v}(t,s,\rho_1(s)),\mathbf{v}(t,s,\rho_2(s))\big) + \big\|\mathbf{v}\big(t,s,\rho_2(s)\big)\big\|_F D_F\big(f(t,\rho_1(t))\big),f(t,\rho_2(t))\big)\Bigg)ds\\
&\le \int_0^1 \big(M_f\beta D_F(\rho_1(s),\rho_2(s)) + M_H\eta D_F(\rho_1(t),\rho_2(t))\big)ds\\
&\le \int_0^1 \big(M_f\beta D^*(\rho_1,\rho_2) + M_H\eta D^*(\rho_1,\rho_2)\big)ds = (\beta M_f + \eta M_H)D^*(\rho_1,\rho_2)
\end{aligned}$$

for all $t \in [0,1]$. So

$$D^*\big(T(\rho_1),T(\rho_2)\big) \le (\beta M_f + \eta M_H)D^*(\rho_1,\rho_2), \quad \forall \rho_1,\rho_1 \in C\big([0,1],R_{\mathcal{F}}\big).$$

According to the condition (3), T is a contraction. by using of the Banach fixed point theorem, Eq. (3) has a unique solution $\rho^* \in C\big([0,1],R_{\mathcal{F}}\big)$.

Choosing $\rho_0 = g$, we infer that $\mathbf{v}(t,s,\rho_0) = \mathbf{v}(t,s,g)$ is continuous since $\mathbf{v} \in C\big([0,1]\times[0,1]\times R_{\mathcal{F}}\big),R_{\mathcal{F}}\big)$and $g \in C\big([0,1],R_{\mathcal{F}}\big)$. Using Lemma 5, there are $M_g, M_0 \geq 0$ such that $D_F\big(\rho_0(s),\tilde{0}\big) = D_F\big(g(s),\tilde{0}\big) \leq M_g$ and

$$D_F\big(\mathbf{v}(t,s,\rho_0(s)),\tilde{0}\big) = D_F\big(\mathbf{v}(t,s,g(s)),\tilde{0}\big) \leq M_0, \quad \forall s \in [0,1].$$

So,

$$\begin{aligned}
D_F\big(\rho_1(t),(\rho_0(t)\big) &= D_F\bigg(g(t)\oplus f(t,\rho_0(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_0(s)\big)ds, g(t)+\tilde{0}\bigg)\\
&= D_F\bigg(g(t)\oplus f(t,\mathbf{g}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\mathbf{g}(s)\big)ds, g(t)+\tilde{0}\bigg)\\
&\leq D_F(g(t),g(t)) + D_F\big(f(t,\mathbf{g}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\mathbf{g}(s)\big)ds,\tilde{0}\big)\\
&= D_F\bigg((FR)\int_0^1 f(t,\mathbf{g}(t))\odot\mathbf{v}\big(t,s,\mathbf{g}(s)\big)ds,(FR)\int_0^1 f(t,\mathbf{g}(t))\odot\tilde{0}ds\bigg)\\
&\leq \int_0^1 D_F\bigg(f(t,\mathbf{g}(t))\odot\mathbf{v}\big(t,s,\mathbf{g}(s)\big), f(t,\mathbf{g}(t))\odot\tilde{0}\bigg)ds\\
&\leq \int_0^1 \|f(t,\rho(t))\|_F D_F\big(\mathbf{v}\big(t,s,\mathbf{g}(s)\big),\tilde{0}\big)ds \leq M_0M_f, \quad \forall t\in[0,1],\\
D^*\big(\rho_1,\rho_0\big) &\leq M_0M_f, \quad \forall t\in[0,1]. \qquad (8)
\end{aligned}$$

For arbitrary $t \in [0,1]$ and $m \in N^*$, we have

$$\begin{aligned}
D_F\big(\rho_m(t),\rho_{m-1}(t)\big) &\leq D_F\bigg(f(t,\rho_{m-1}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds,\\
&\qquad f(t,\rho_{m-2}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-2}(s)\big)ds\bigg)\\
&= D_F\bigg((FR)\int_0^1 f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds,\\
&\qquad (FR)\int_0^1 f(t,\rho_{m-2}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big)ds\bigg)\\
&\leq \int_0^1 D_F\bigg(f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big), f(t,\rho_{m-2}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big)\bigg)ds\\
&\leq \int_0^1 D_F\bigg(f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big), f(t,\rho_{m-2}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big)\bigg)ds\\
&\leq \int_0^1 D_F\bigg(f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big), f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big)\bigg)ds\\
&\quad + \int_0^1 D_F\bigg(f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big), f(t,\rho_{m-2}(t))\odot\mathbf{v}\big(t,s,\rho_{m-2}(s)\big)\bigg)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 \|f(t,\rho(t))\|_F D_F\big(\mathbf{v}(t,s,\rho_{m-1}(s)),\mathbf{v}(t,s,\rho_{m-2}(s))\big)ds \\
&\quad + \int_0^1 \big\|\mathbf{v}(t,s,\rho_{m-2}(s))\big\|_F D_F\big(f(t,\rho_{m-1}(t)),f(t,\rho_{m-2}(t))\big)ds \\
&\leq \int_0^1 \beta M_f D^*\big(\rho_{m-1},\rho_{m-2}\big)ds + \int_0^1 \eta M_H D^*\big(\rho_{m-1},\rho_{m-2}\big)ds \\
&\leq (\beta M_f + \eta M_H) D^*\big(\rho_{m-1},\rho_{m-2}\big).
\end{aligned}
$$

By induction it follows that

$$
D^*\big(\rho_m,\rho_{m-1}\big) \leq \big(\beta M_f + \eta M_H\big)^{m-1} D^*\big(\rho_1,\rho_0\big). \tag{9}
$$

Consequently,

$$
\begin{aligned}
D_F\big(\rho_m(t),\rho_0(t)\big) &\leq D_F\big(\rho_{m-1}(t),\rho_{m-2}(t)\big) + D_F\big(\rho_{m-2}(t),\rho_{m-3}(t)\big) + \cdots + D_F\big(\rho_1(t),\rho_0(t)\big) \\
&\leq \Big(\big(\beta M_f+\eta M_H\big)^{m-1} + \big(\beta M_f+\eta M_H\big)^{m-2} + \cdots + 1\Big) D^*\big(\rho_1,\rho_0\big) \\
&\leq \frac{1-\big(\beta M_f+\eta M_H\big)^m}{1-(\beta M_f+\eta M_H)} M_0 M_f \leq \frac{M_0 M_f}{1-(\beta M_f+\eta M_H)}, \quad \forall t \in [0,1], m \in N^*.
\end{aligned}
$$

for any $t \in [a,b]$ and for all $m \in N$, that is the uniformly boundedness of the sequence $(\rho_m)_{m\in N}$ in $C([a,b],R_F)$. For $m \in N^*$, we see that

$$
\begin{aligned}
D_F\big(\mathbf{v}(t,s,\rho_m(s)),\tilde{0}\big) &\leq D_F\big(\mathbf{v}(t,s,\rho_m(s)),\mathbf{v}(t,s,\rho_0(s))\big) + D_F\big(H(t,s,\rho_0(s)),\tilde{0}\big) \\
&\leq \beta D_F\big(\rho_m(s),\rho_0(s)\big) + M_0 \leq \frac{\beta M_0 M_f}{1-(\beta M_f+\eta M_H)} + M_0, \quad \forall s \in [0,1].
\end{aligned}
$$

So, the sequence of functions $a_m = \mathbf{v}(t,s,\rho_m)$ is uniformly bounded in $C([0,1],R_F)$.

Now, to prove Eq. (6), we can write

$$
\begin{aligned}
D_F\big(\rho^*(t),\rho_{m-1}(t)\big) &\leq D_F\Big(f(t,\rho^*(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho^*(s)\big)ds, \\
&\qquad f(t,\rho_{m-1}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds\Big) \\
&\leq D_F\Big(f(t,\rho^*(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho^*(s)\big)ds, \\
&\qquad f(t,\rho^*(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds\Big) \\
&\quad + D_F\Big(f(t,\rho^*(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds, \\
&\qquad f(t,\rho_{m-1}(t))\odot(FR)\int_0^1 \mathbf{v}\big(t,s,\rho_{m-1}(s)\big)ds\Big) \\
&\leq \int_0^1 D_F\Big(f(t,\rho^*(t))\odot\mathbf{v}\big(t,s,\rho^*(s)\big), f(t,\rho^*(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big)\Big)ds \\
&\quad + \int_0^1 D_F\Big(f(t,\rho^*(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big), f(t,\rho_{m-1}(t))\odot\mathbf{v}\big(t,s,\rho_{m-1}(s)\big)\Big)ds
\end{aligned}
$$

$$\leq \int_0^1 \|f(t,\rho^*(t))\|_F D_F\big(\mathbf{v}(t,s,\rho^*(s)),\mathbf{v}(t,s,\rho_{m-1}(s))\big)ds$$
$$+\int_0^1 \big\|\mathbf{v}\big(t,s,\rho_{m-1}(s)\big)\big\|_F D_F\big(f(t,\rho^*(t)),f(t,\rho_{m-1}(t))\big)ds$$
$$\leq \int_0^1 M_f\beta D^*\big(\rho^*,\rho_{m-1}\big)ds+\int_0^1 M_H\eta D^*\big(\rho^*,\rho_{m-1}\big)ds.$$

Thus

$$D^*\big(\rho^*,\rho_m\big)\leq(\beta M_f+\eta M_H)D^*\big(\rho^*,\rho_{m-1}\big). \tag{10}$$

By using Defination 5, we have

$$D^*\big(\rho^*,\rho_{m-1}\big)\leq D^*\big(\rho^*,\rho_m\big)+D^*\big(\rho_m,\rho_{m-1}\big). \tag{11}$$

Combaning Eqs. (10) and (11) yields

$$D^*\big(\rho^*,\rho_m\big)\leq\frac{(\beta M_f+\eta M_H)}{1-(\beta M_f+\eta M_H)}D^*\big(\rho_m,\rho_{m-1}\big). \tag{12}$$

From Eqs. (9) and (12) we conclude that

$$D^*\big(\rho^*,\rho_m\big)\leq\frac{(\beta M_f+\eta M_H)^m}{1-(\beta M_f+\eta M_H)}D^*\big(\rho_1,\rho_0\big). \tag{13}$$

From Eqs. (8) and (13), relation (7) is obtained.

On the other hand,

$$D^*\big(\rho^*,\tilde{0}\big)\leq D^*\big(\rho^*,\rho_m\big)+D^*\big(\rho_m,\tilde{0}\big),$$

and hence

$$D^*\big(\rho^*,\tilde{0}\big)\leq\frac{(\beta M_f+\eta M_H)^m}{1-(\beta M_f+\eta M_H)}M_0M_f+\frac{M_0M_f}{1-(\beta M_f+\eta M_H)}+M_g.$$

Therefore, from $(\beta M_f+\eta M_H)<1$, we see that

$$D^*\big(\rho^*,\tilde{0}\big)\leq\frac{M_0M_f}{1-(\beta M_f+\eta M_H)}+\frac{M_0M_f}{1-(\beta M_f+\eta M_H)}+M_g=\frac{2M_0M_f}{1-(\beta M_f+\eta M_H)}+M_g$$

we conclud that the solution of Eq. (3), ρ^*, is bounded. □

Theorem 3. *Under the conditions (1)-(6), the sequence of successive approximations (5) is uniformly Lipschitz, that is, there exist a constant L such that $D_F\big(\rho_m(t_1),\rho_m(t_2)\big)\leq L|t_1-t_2|$ for all $m\in N^*$ and $t_1,t_2\in[0,1]$.*

Proof. From the condition (4) it is clear that $D_F\big(\rho_0(t_1),\rho_0(t_2)\big) \le \gamma|t_1-t_2|$ for all $t_1,t_2 \in [0,1]$. For $m \in N^*$ it results that

$$\begin{aligned}
&D_F\big(\rho(t_1),\rho(t_2)\big)\\
&\le D_F\big(g(t_1),g(t_2)\big) + D_F\bigg(f(t_1,\rho(t_1))\odot(FR)\int_0^1 \mathbf{v}\big(t_1,s,\rho(s)\big)ds, f(t_2,\rho(t_2))\odot(FR)\int_0^1 \mathbf{v}\big(t_2,s,\rho(s)\big)ds\bigg)\\
&\le \gamma|t_1-t_2| + D_F\bigg(f(t_1,\rho(t_1))\odot(FR)\int_0^1 \mathbf{v}\big(t_1,s,\rho(s)\big)ds, f(t_1,\rho(t_1))\odot(FR)\int_0^1 \mathbf{v}\big(t_2,s,\rho(s)\big)ds\bigg)\\
&\quad + D_F\bigg(f(t_1,\rho(t_1))\odot(FR)\int_0^1 \mathbf{v}\big(t_2,s,\rho(s)\big)ds, f(t_2,\rho(t_2))\odot(FR)\int_0^1 \mathbf{v}\big(t_2,s,\rho(s)\big)ds\bigg)\\
&\le \gamma|t_1-t_2| + D_F\bigg((FR)\int_0^1 f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_1,s,\rho(s)\big)ds, (FR)\int_0^1 f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_2,s,\rho(s)\big)ds\bigg)\\
&\quad + D_F\bigg((FR)\int_0^1 f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_2,s,\rho(s)\big)ds, (FR)\int_0^1 f(t_2,\rho(t_2))\odot\mathbf{v}\big(t_2,s,\rho(s)\big)ds\bigg)\\
&\le \gamma|t_1-t_2| + \int_0^1 D_F\bigg(f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_1,s,\rho(s)\big), f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_2,s,\rho(s)\big)\bigg)ds\\
&\quad + \int_0^1 D_F\bigg(f(t_1,\rho(t_1))\odot\mathbf{v}\big(t_2,s,\rho(s)\big), f(t_2,\rho(t_2))\odot\mathbf{v}\big(t_2,s,\rho(s)\big)\bigg)ds\\
&\le \gamma|t_1-t_2| + \int_0^1 \|f(t_1,\rho(t_1))\|_F\, D_F\bigg(\mathbf{v}\big(t_1,s,\rho(s)\big),\mathbf{v}\big(t_2,s,\rho(s)\big)\bigg)ds\\
&\quad + \int_0^1 \big\|\mathbf{v}\big(t_2,s,\rho(s)\big)\big\|_F\, D_F\bigg(f(t_1,\rho(t_1)), f(t_2,\rho(t_2))\bigg)ds\\
&\le \gamma|t_1-t_2| + M_f\alpha|t_1-t_2| + M_H\delta|t_1-t_2| = (\gamma+\alpha M_f+\delta M_H)|t_1-t_2|\\
&= L|t_1-t_2|, \quad \forall t_1,t_2\in[0,1], m\in N^*.
\end{aligned}$$

Thus the sequence $(\rho_m)_{m\in N}$ is uniformly Lipschitz with the Lipschitz constant $L=(\gamma+\alpha M_f+\delta M_H)$, which completes the proof. □

Remark 1. *Since* $(\beta M_f+\eta M_H)<1$*, it is easy to show that* $\lim_{m\to\infty} D^*(u^*,\rho_m)=0$*. Thus, the proposed method is convergent.*

In the next section we present a numerical example.

4 Numerical experiments

Example 1. In ([19, Page 47]) the author showed that a relationship between three fundamental traffic variables is as follows

$$q(x,t)=\rho(x,t)\mathbf{v}(x,t). \tag{14}$$

where q, ρ and $\mathbf{v}$ represents the traffic flow, traffic density and the speed of the traffic flow, respectively. This relationship represents the fundamental equation of traffic flow. Lighthill and Whitham and also Richards observed that the average equilibrium speed of the vehicles is a function of the traffic density ([16, Page 321], [14, Page 286])

$$\mathbf{v}(x,t) = F\big(\rho(x,t)\big),$$

If there is a gap or depression in part of the road, the conservation equation is as follows

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = g(x,t) \tag{15}$$

where $g(x,t)$ is the generation (or dissipation) rate in vehicle per unit time per length. Therefore for vehicle traffic flow, the flow is given by Eq. (14), and so we can rewrite the $\frac{\partial q}{\partial x}$ as

$$\frac{\partial q}{\partial x} = \frac{\partial(\rho \mathbf{v})}{\partial x}. \tag{16}$$

Eq. (15) then becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \mathbf{v})}{\partial x} = g(x,t). \tag{17}$$

This is another form of the conservation equation that shows the relationship between speed and traffic density.

On the other hand, in [10, Page 85] author showed that some models for "concrete" problems in vehicular traffic, biology, queuing theory, etc. lead to integro differential equations involving terms such as $f(t,\rho(t))\int_0^1 \mathbf{v}\big(t,s,\rho(s)\big)ds$.

Given the need for fuzzy information in analyzing real world problems, we can present the following nonlinear fuzzy integral equation to compute the density of traffic

$$\rho(x,t) = g(x,t) \oplus f(x,t,\rho(t)) \odot (FR)\int_0^1 \mathbf{v}\big(x,t,s,\rho(s)\big)ds,$$

where

$$\begin{aligned} g(x,t) &= \big(\overline{g}(x,t,r), \underline{g}(x,t,r)\big) \\ &= \bigg(x(r^2+r)\big(1-\frac{(r^2+r)t^2}{\pi}\big)\sin \pi t, x(4-r^3-r)\big(1-\frac{(4-r^3-r)t^2}{\pi}\big)\sin \pi t\bigg), \quad t,r \in [0,1], \end{aligned}$$

and kernels

$$f\big(x,t,\rho(t)\big) = xt\rho(t), \quad \mathbf{v}\big(x,t,s,\rho(s)\big) = \frac{ts\rho(s)}{x}, \quad t,r \in [0,1].$$

Since the solution of the integral equation, ρ, is a continuous function, therefore it is uniform continuous and bounded function in [0,1]. Hence, the functions g, f and $\mathbf{v}$ satisfy all the conditions of Theorem 2. So the aforementioned integral equation has a unique solution. The exact solution of this example is

$$\rho(x,t) = \big(\overline{\rho}(x,t,r), \underline{\rho}(x,t,r)\big) = \big(x(r^2+r)\sin \pi t, x(4-r^3-r)\sin \pi t\big).$$

To compare the exact and iterative solutions with $m = 20$, see Table 1.

Table 1: Numerical results for Example 1.

	t=0.25		t=0.5		t=0.75	
r-level	$\lvert\underline{\rho}-\underline{\rho}_m\rvert$	$\lvert\overline{\rho}-\overline{\rho}_m\rvert$	$\lvert\underline{\rho}-\underline{\rho}_m\rvert$	$\lvert\overline{\rho}-\overline{\rho}_m\rvert$	$\lvert\underline{\rho}-\underline{\rho}_m\rvert$	$\lvert\overline{\rho}-\overline{\rho}_m\rvert$
0.0	0	3.43852e-8	0	2.42751e-7	0	6.68776e-7
0.1	0	3.75784e-8	0	4.36718e-7	0	5.23589e-7
0.2	0	2.24772e-8	2.77556e-17	6.74589e-7	1.38778e-16	4.37159e-7
0.3	0	2.61437e-8	4.92347e-15	8.39624e-7	4.27496e-15	3.94238e-7
0.4	5.16254e-15	2.88822e-8	3.69704e-14	9.09434e-8	1.07582e-13	2.50537e-7
0.5	7.23742e-14	5.48536e-9	5.32641e-14	6.42791e-8	7.57912e-13	2.39874e-7
0.6	3.81332e-13	5.58662e-9	2.74669e-12	3.94391e-8	7.56528e-12	1.08646e-7
0.7	5.34912e-12	3.12874e-9	9.77556e-11	2.29613e-8	8.22602e-11	1.00534e-7
0.8	9.93228e-12	3.44815e-9	6.01157e-11	1.02232e-8	1.93125e-10	2.81613e-8
0.9	6.74521e-11	3.53852e-9	3.17255e-11	7.32187e-9	6.84639e-10	6.39541e-8
1.0	1.36869e-10	1.36869e-10	9.66202e-10	9.66202e-10	2.66141e-9	2.66141e-9

5 Conclusion

In this paper, fuzzy integral equations that are derived from the road traffic flow are discussed. In this research, Banach fixed point theorem was used to prove the existence and uniqueness of the solution of fuzzy integral equations arising of traffic flow. Finaly, we got the error estimation between the exact solution and the solution of the iterative method. Comparison of the results of the iteration method and the exact answer of the integral equations arising of traffic flow in the one example presented showed that the use of the iterative method in finding the solution of such integral equations has high accuracy. Given the widespread use of this kind of integral equations in traffic engineering, in future, the authors plan to work on numerical solution of these integral equations.

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