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Numerical methods based on spline quasi-interpolation operators for integro-differential equations

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Abstract. In this paper, we propose collocation and Kantorovich methods based on spline quasi-interpolants defined on a bounded interval to solve numerically a class of Fredholm integro-differential equations. We describe the computational aspects for calculating the approximate solutions and give theoretical results corresponding to the convergence order of each method in terms of the degree of the considered spline quasi-interpolant. Finally, we provide some numerical tests that confirm the theoretical results and prove the efficiency of the proposed methods.

Keywords: Integro-differential equations, quasi-interpolants, collocation method, Kantorovich method.
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1 Introduction

Integro-differential equations emerged at the beginning of the twentieth century, notably by the Italian researcher Volterra. This type of equation has attracted much more interest from researchers because they provide efficiency for the description of problems arising in the fields of engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, electromagnetic, control theory and viscoelasticity [1, 4, 18, 23, 25]. Moreover, Integro-differential equations can be founded explicitly in mathematical models of epidemics and spatiotemporal developments [19, 31].

Many numerical methods have been developed for solving integro-differential equations. Each of these methods has its inherent advantages and disadvantages and the search for easier and more accurate

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methods is a continuous and ongoing process. Among the existing methods in the literature, we cite the adomian decomposition [17], homotopy analysis method [18], Chebyshev and Taylor collocation [33], Taylors series expansion [13, 20] and integral mean value [11]. A very popular variational iteration method is considered to solve integral and integro-differential equations [32,34]. Decomposition method was used to solve high-order linear Volterra-Fredholm integro-differential equations in [15]. Moreover, a large number of papers have considered meshless schemes to solve numerically different types of integro-differential equations. For instance, the authors in [9, 10] have solved integro-differential equations arising in oscillating magnetic fields by local multiquadrics collocation and Galerkin methods . In [6], the same type of equation was solved by the thin plate spline collocation method. Local thin plate splines Galerkin scheme was used to solve nonlinear mixed integro-differential equations in [8]. A class of fractional integro-differential equations is solved in [7] by using local radial basis functions.

Recently, many authors have used spline functions for the numerical solution of integro-differential equations, in particular, semiorthogonal spline wavelets approximation method for Fredholm integro-differential equations was proposed in [21]. In [22], authors applied B-spline collocation method to solve numerically linear and nonlinear Fredholm and Volterra integro-differential equations and in [5] a method for solving integro-differential equations using B-spline interpolation was studied.

Spline quasi-interpolants (abbr. QIs) are approximation operators obtained as a linear combination of functions with bounded support (B-splines). These operators are convenient and efficient tools in the approximation of functions since their construction is simple and they provide an optimal convergence order with a uniform bounded norm (see [27]). Recently, it was proved (see [2] and [3]) that the spline QIs work well for approximating the solutions of the linear and nonlinear Fredholm integral equations.

In this paper, we intend to use collocation and Kantorovich schemes based on spline QIs operators to solve numerically the following Fredholm linear integro-differential equation

$$\begin{cases} u'(t) = \int_0^1 k(t,s)u(s)\,ds + a(t)u(t) + g(t)\,, \quad t \in [0\,,1] \\ u(0) = u_0 \end{cases} \tag{1}$$

where u is the function to be determined, a, g, k are continuous functions and $u_0 \in \mathbb{R}$.

The paper is organized as follows. In Section 2, we give some preliminary results on the discrete spline QIs of degree d and we present the explicit formula for the quadratic QI defined on a uniform partition. In Section , we introduce the collocation and Galerkin methods based on spline QIs to solve numerically Fredholm integro-differential equation (1). In Section 4, error estimates are given and precise convergence orders are obtained. Finally, in Section 5, we provide some numerical results that illustrate the approximation properties of the proposed methods.

2 Spline quasi-interpolants

Let $\mathscr{X}_n := \{x_k, 0 \le k \le n\}$ be the uniform partition of the interval $I = [0,1]$ into n equal subintervals, i.e., $x_k := kh$, with $h = 1/n$ and $0 \le k \le n$. We consider the space $\mathscr{S}_d(I, \mathscr{X}_n)$ of splines of degree d and class $\mathscr{C}^{d-1}$ on this partition. Its canonical basis is formed by the $n+d$ normalized B-splines $\{B_k, k \in \mathscr{J}\}$, with $\mathscr{J} := \{1,2,\ldots,n+d\}$. The support of each B_k is the interval $[x_{k-d-1}, x_k]$ if we add multiple knots at the endpoints.

A discrete quasi-interpolant (abbr. dQI) of degree $d > 1$ is a spline operator of the form

$$\mathcal{Q}_d f := \sum_{k \in \mathcal{J}} \mu_k(f) B_k \,, \tag{2}$$

where the coefficients $\mu_k(f)$ are linear combinations of values of f on the set $\mathcal{E}_n := \{\xi_i \,, i = 0 \,, \ldots \,, \mathcal{N}\}$, with

$$\begin{cases} \xi_i := t_i \,, \ \mathcal{N} := n+1 \,, & \text{if } d \text{ is even,} \\ \xi_i := x_i \,, \ \mathcal{N} := n \,, & \text{if } d \text{ is odd ,} \end{cases}$$

and $t_0 = x_0$, $t_{n+1} = x_n$, $t_i = (x_{i-1} + x_i)/2 \; i = 1 \,, \ldots \,, n$. More precisely, the functional coefficients μ_k for $d+1 \leq k \leq n$, have the following form

$$\mu_k(f) := \begin{cases} \displaystyle\sum_{i=0}^{d} \alpha_{i,k} f(\xi_{k-d+i}) \,, & \text{if } d \text{ is even ,} \\ \displaystyle\sum_{i=1}^{d} \alpha_{i,k} f(\xi_{k-d+i-1}) \,, & \text{if } d \text{ is odd ,} \end{cases}$$

where $\alpha_{i,k}$ are calculated such that the dQI $\mathcal{Q}_d$ reproduces the space $\mathbb{P}_d$ of all polynomials of total degree at most d, i.e.,

$$\mathcal{Q}_d p = p, \quad \forall p \in \mathbb{P}_d.$$

The extremal coefficients $\mu_k(f)$ have particular expressions. The dQI $\mathcal{Q}_d$ can be written in the following quasi-Lagrange form

$$\mathcal{Q}_d f = \sum_{j=0}^{\mathcal{N}} f(\xi_j) Lj \,,$$

where L_j are linear combinations of finite number of B-splines .

Since μ_k are continuous linear functionals, the operator $\mathcal{Q}_d$ is uniformly bounded on $\mathcal{C}([0,1])$ and using classical results in approximation theory (see [14]) , for any $f \in \mathcal{C}^{d+1}([0\,,1])$, we have

$$\|f - \mathcal{Q}_d f\| \leq C_1 h^{d+1} \|f^{(d+1)}\| \,, \tag{3}$$

where C_1 is a positive constant independent of h.

In what follows, we report an example of a spline dQI of the form (2) for $d = 2$. This operator is defined on the space $\mathcal{S}_2(I, \mathcal{X}_n)$ of $\mathcal{C}^1$ quadratic splines (see e.g. [30]) as follows

$$\mathcal{Q}_2 f := \sum_{k=0}^{n+1} \mu_k(f) B_k \,, \tag{4}$$

where the coefficient functionals $\mu_k(f)$ are given by

$$\begin{aligned} &\mu_0(f) = f_0, \quad \mu_1(f) = -\frac{1}{3} f_0 + \frac{3}{2} f_1 - \frac{1}{3} f_2, \\ &\mu_k(f) = -\frac{1}{8} f_{k-1} + \frac{5}{4} f_k - \frac{1}{8} f_{k+1}, \quad 2 \leq k \leq n-1, \\ &\mu_n(f) = -\frac{1}{3} f_{n-1} + \frac{3}{2} f_n - \frac{1}{3} f_{n+1}, \quad \mu_{n+1}(f) = f_{n+1}. \end{aligned} \tag{5}$$

It is easy to see that the $\mathscr{Q}_2$ is uniformly bounded and its infinity norm is given by

$$\|\mathscr{Q}_2\|_\infty = \frac{305}{207} \approx 1.4734 \,.$$

In the case of even degree, the dQI $\mathscr{Q}_d$ presents an interesting property related to the convergence order of its associated quadrature rule. Indeed, the following theorem holds.

Theorem 1. *Let d be an even integer and let $\mathscr{Q}_d$ be the dQI defined by (2). For any function $f \in \mathscr{C}^{d+2}([0,1])$, and for any weight function $g \in \mathscr{W}^{1,1}$ (i.e. $\|g'\|_1$ is bounded), we have*

$$\int_0^1 g(t)(f(t) - \mathscr{Q}_d f(t))dt = \mathscr{O}(h^{d+2}). \tag{6}$$

Particular cases of quadrature rules based on quadratic ($d = 2$) and quartic ($d = 4$) dQIs are studied in depth in [26] and [28] respectively.

3 Quasi-interpolation method

Equation (1) can be written in the form

$$u'(t) = \mathscr{A} u(t) + \mathscr{K} u(t) + g(t)\,, \tag{7}$$

where

$$\begin{cases} \mathscr{A} u(t) = a(t)u(t)\,, & t \in [0\,,1], \\ \mathscr{K} u(t) = \displaystyle\int_0^1 k(t,s)u(s)\,ds\,, & t \in [0\,,1]\,. \end{cases}$$

Let $v \in L^\infty[0,1]$ such that $u'(t) = v(t)$. Then u can be written in the form

$$u(t) = (Jv)(t) + u_0\,, \quad 0 \le t \le 1\,, \tag{8}$$

where

$$(Jv)(t) = \int_0^t v(s)ds\,, \quad 0 \le t \le 1\,.$$

Using the above notations, Eq. (7) takes the form

$$v = \mathscr{T} v + f\,, \tag{9}$$

where

$$\mathscr{T} := (\mathscr{K} + \mathscr{A})J\,, \tag{10}$$

and

$$f(t) := g(t) + u_0 a(t) + u_0 \int_0^1 k(t,s)\,ds\,, \quad t \in [0\,,1]\,. \tag{11}$$

$\mathscr{T}$ is a compact operator, as a linear operator from $L^\infty[0,1]$ into $L^\infty[0,1]$.

3.1 Collocation-type method based on $\mathscr{Q}_d$

In order to solve (1), we construct a collocation-type method for the numerical solution of (8). More precisely, we look for an approximate solution u_n^c of (1) in the form

$$u_n^c(t) = \int_0^t v_n^c(s)ds + u_0 \,, \tag{12}$$

where v_n^c satisfies the following approximate equation

$$v_n^c - \mathscr{Q}_d \mathscr{T} v_n^c = \mathscr{Q}_d f \,. \tag{13}$$

This last equation can be reduced to a linear system of equations. Indeed, from (13) v_n^c is a spline function of the form

$$v_n^c = \sum_{i=0}^{\mathscr{N}} c_i L_i \,. \tag{14}$$

By replacing v_n^c in (13), we obtain

$$\sum_{i=0}^{\mathscr{N}} c_i L_i = \sum_{i=0}^{\mathscr{N}} \mathscr{T} \left(\sum_{j=0}^{\mathscr{N}} c_j L_j \right) (\xi_i) L_i + \sum_{i=0}^{\mathscr{N}} f(\xi_i) L_i \,.$$

Using the linearity of $\mathscr{T}$ and identifying the coefficients of L_j, we obtain the following system

$$c_i - \sum_{j=0}^{\mathscr{N}} \mathscr{T}(L_j)(\xi_i) c_j = f(\xi_i) \,, \quad i = 0, \ldots, \mathscr{N} \,. \tag{15}$$

Let us define the vectors

$$\mathscr{C}_{\mathscr{N}} := (c_0, \ldots, c_{\mathscr{N}})^T \quad \text{and} \quad \mathscr{F}_{\mathscr{N}} := (f(\xi_0), \ldots, f(\xi_{\mathscr{N}}))^T \,,$$

and the matrices

$$\mathscr{A} := (\beta_j(\xi_i))_{0 \leqslant i,j \leqslant \mathscr{N}} \,, \qquad \mathscr{A}_{\mathscr{N}} := \operatorname{diag}(a(\xi_i))_{0 \leqslant i \leqslant \mathscr{N}} \mathscr{A} \,, \tag{16}$$

$$\mathscr{M}_{\mathscr{N}} := \left(\int_0^1 k(\xi_i, s) \beta_j(s) ds \right)_{0 \leqslant i,j \leqslant \mathscr{N}} \tag{17}$$

with

$$\beta_j(s) = \int_0^s L_j(v) dv \,, \quad j = 0, \ldots, \mathscr{N} \,.$$

Then, the system (15) becomes

$$[\mathscr{I}_{\mathscr{N}} - (\mathscr{A}_{\mathscr{N}} + \mathscr{M}_{\mathscr{N}})] \mathscr{C}_{\mathscr{N}} = \mathscr{F}_{\mathscr{N}} \,. \tag{18}$$

Once the solution $\mathscr{C}_{\mathscr{N}}$ of (18) is determined, the approximate solution u_n is given by

$$u_n^c(t) = u_0(t) + \beta^T(t) \mathscr{C}_{\mathscr{N}} \,,$$

where

$$\beta(t) = (\beta_0(t), \beta_1(t), \ldots, \beta_{\mathscr{N}}(t))^T \,.$$

3.2 Kantorovich-type method

In the Kantorovich-type method, the approximate solution u_n^k is given by

$$u_n^k(t) = \int_0^t v_n^k(s)ds + u_0 ,$$

where v_n^k satisfies the following approximate equation

$$v_n^k - \mathcal{Q}_d \mathcal{T} v_n^k = f . \tag{19}$$

Using the expression of $\mathcal{Q}_d$, v_n^k can be written in the form

$$v_n^k = \sum_{i=0}^{\mathcal{N}} \tilde{c}_i L_i + f . \tag{20}$$

Replacing this last expression of v_n^k in (19), it follows that $\tilde{c}_i$, $i = 0, \ldots, \mathcal{N}$ satisfies the linear system given by

$$\tilde{c}_i - \sum_{j=0}^{\mathcal{N}} \mathcal{T}(L_j)(\xi_i)\,\tilde{c}_j = \mathcal{T}(f)(\xi_i) , \quad i = 0, \ldots, \mathcal{N} . \tag{21}$$

In the matrix form, it holds

$$\left[\mathcal{I}_{\mathcal{N}} - \left(\mathcal{A}_{\mathcal{N}} + \mathcal{M}_{\mathcal{N}}\right)\right]\tilde{\mathcal{C}}_{\mathcal{N}} = \mathcal{F}_{\mathcal{N}}^K , \tag{22}$$

where $\mathcal{A}_{\mathcal{N}}$, $\mathcal{M}_{\mathcal{N}}$ are as in (16) , (17) and

$$\tilde{\mathcal{C}}_{\mathcal{N}} := (\tilde{c}_0, \ldots, \tilde{c}_{\mathcal{N}})^T \quad , \quad \mathcal{F}_{\mathcal{N}}^K := \left(\mathcal{T}(f)(\xi_0) , \ \ldots \ , \ \mathcal{T}(f)(\xi_{\mathcal{N}})\right)^T .$$

Once (22) is solved, the approximate solution u_n is given by

$$u_n^k(t) = u_0(t) + \beta^T(t)\tilde{\mathcal{C}}_{\mathcal{N}} + \int_0^t f(s)ds .$$

Remark 1. *It is important to note the presence of integrals in systems (18) and (22). When implementing the method, the integrals $\beta_j(\xi_i)$ were calculated exactly, since L_j are given by piecewise polynomial functions. However, the other integrals were calculated numerically using high accuracy quadrature rules, like those defined in [29], to imitate exact integration.*

Remark 2. *It should be noted that the methods introduced above can be extended to the case of nonlinear integro-differential equations. For instance and without loss of generality, we consider the following particular case of nonlinear integro-differential equations*

$$u'(t) = \int_0^1 K(t,s)\psi(s,u(s))ds + g(t), \tag{23}$$

where k, g and ψ are known functions, with $\psi(s,\nu)$ nonlinear in ν, and u is the function to be determined. Thus, the collocation method based on the spline QI $\mathcal{Q}_d$ consists in this case to look for an approximate solution u_n^c given by (12) where v_n^c takes the form

$$v_n^c = \sum_{i=0}^{\mathcal{N}} c_i L_i .$$

and $\{c_i,\ i=0,\dots,\mathscr{N}\}$ *is the solution of the nonlinear system of equations given by*

$$c_i - \int_0^1 K(\xi_i,s)\psi(s,c_jL_j(s))ds = f(\xi_i)\ ,\quad i=0,\dots,\mathscr{N}\ .$$

A more detailed study of the solution of nonlinear integro-differential equations by the methods presented in this work will be the subject of a future paper.

4 Error analysis

For the sake of completeness, we report the theorem of existence and uniqueness of solution for Eq. (1) .

Theorem 2. *Assume that* $a,\ g \in C([0,1])$ *and* $k \in C([0,1]\times[0,1])$*. Then for any initial value* $u_0 \in \mathbb{R}$ *, Eq. (1) possesses a unique solution* $u \in C^1([0,1])$ *satisfying* $u(0)=u_0$*. Moreover, this solution is given by*

$$u(t) = r(t,0)u_0 + \int_0^t r(t,s)g(s)ds,\quad t\in[0,1]\ , \tag{24}$$

where $r \in C^1([0,1]\times[0,1])$ *is a resolvent kernel .*

Proof. See Brunner [12] . □

The following theorem confirms the existence and the uniqueness of the solutions of the linear algebraic systems (18) and (22) .

Theorem 3. *Assume that* $a,\ g\in C([0,1])$ *and* $k\in C([0,1]\times[0,1])$ *. Then for h sufficiently small ,the linear algebraic systems (18) and (22) have unique solutions in* $\mathbb{R}^{\mathscr{N}+1}$*. Hence, the collocation equation (13) and the Kantorovich equation (19) have unique solutions given respectively by* v_n^c *and* v_n^k*.*

Proof. We give only the proof for the collocation method. The proof in the case of Kantorovich method is quite similar.

The matrices $\mathscr{A}_{\mathscr{N}}$ and $\mathscr{M}_{\mathscr{N}}$ given in (16) and (17) respectivly, can be written as

$$\mathscr{A}_{\mathscr{N}} = h\mathscr{A}^*_{\mathscr{N}} \quad \text{and} \quad \mathscr{M}_{\mathscr{N}} = h^2\mathscr{M}^*_{\mathscr{N}}\ ,$$

with $\mathscr{A}^*_{\mathscr{N}}$ and $\mathscr{M}^*_{\mathscr{N}}$ are matrices with bounded elements that are independent of h . Indeed, using the fact that, the quasi-Lagrange functions L_j have compact supports, that is $\mathrm{supp}(L_j)=[x_{j-p}\,,x_{j+q}]$ for certain p and q , we deduce that for any $0\leqslant i,j\leqslant\mathscr{N}$, it holds

$$\begin{aligned}
\mathscr{A}_{ij} = \beta_j(\xi_i) &= \int_0^{\xi_i} L_j(s)ds \\
&= \begin{cases} 0\ , & \text{if} \quad \xi_i < x_{j-p}, \\ h\displaystyle\int_{j-p}^{\eta_i} L_j(sh)ds\ , & \text{if} \quad x_{j-p}<\xi_i<x_{j+q}, \\ h\displaystyle\int_{j-p}^{j+q} L_j(sh)ds\ , & \text{if} \quad \xi_i > x_{j+q}, \end{cases} \\
&= h\mathscr{A}^*_{ij}\ ,
\end{aligned}$$

where $\xi_i = \eta_i h$ and $\mathscr{A}^*_{ij}$ are bounded and independent of h .

Similarly, we have

$$\mathscr{M}_{ij} = \int_0^1 k(\xi_i\,,s)\beta_j(s)ds = \int_{x_{j-p}}^{x_{j+q}} k(\xi_i\,,s)\left(\int_0^s L_j(t)dt\right)ds$$
$$= h^2\int_{j-p}^{j+q} k(\xi_i\,,sh)\left(\int_{j-p}^s L_j(t)dt\right)ds = h^2\mathscr{M}^*_{ij}$$

where $\mathscr{M}^*_{ij}$ are bounded and independent of h . Hence , the matrix of the system (18) can be written as

$$\mathscr{I}_{\mathscr{N}} - h\,(\mathscr{A}^*_{\mathscr{N}} + h\mathscr{M}^*_{\mathscr{N}})$$

which is invertible with a bounded inverse, whenever h is sufficiently small (from the Neuman lemma, see [24]). So the proof is complete. □

In the sequel, we study the convergence order of the proposed methods.

Theorem 4. *Let $\mathscr{Q}_d$ be the QI operator of degree d given by (2), and v_n be either the collocation solution v_n^c defined by (14) or the Kantorovich solution v_n^k defined by (20). Assume that $a\,,g \in \mathscr{C}^{d+1}\big([0\,,1]\big)$ and $k \in \mathscr{C}^{d+1}\big([0\,,1]\times[0\,,1]\big)$. Then for n large enough, the following error estimations hold*

$$\|v - v_n\|_\infty = \mathscr{O}(h^{d+1})\,, \tag{25}$$

$$\|u - u_n\|_\infty = \mathscr{O}(h^{d+2})\,, \tag{26}$$

where u is the exact solution of (1), $v = u'$ and $v_n = u_n'$.

Proof. We give only the proof for the collocation method. The proof in the case of Kantorovich method is quite similar.

By assumption, $u \in C^{d+2}[0,1]$, then $u' \in C^{d+1}[0,1]$. Using Peano's theorem (see [16], Chapter 3), we get

$$u'(t) = \mathscr{Q}_d u'(t) + \int_0^1 K_{d+1}(t,s)u^{(d+2)}(s)ds\,, \tag{27}$$

where the Peano kernel K_{d+1} is given by

$$K_{d+1}(t,s) = \frac{1}{d!}\left((t-s)_+^d - \sum_{k=0}^{\mathscr{N}}(\xi_k - s)_+^d L_k(t)\right).$$

By taking $t = \tau h$, $s = zh$ and $\xi_k = \eta_k h$, the kernel K_{d+1} takes the form

$$K_{d+1}(t,s) = \frac{h^d}{d!}\left((\tau - z)_+^d - \sum_{k=0}^{\mathscr{N}}(\eta_k - z)_+^d L_k(t)\right),$$

and (27) can be written as

$$u'(t) = \mathscr{Q}_d u'(t) + h^{d+1}\int_0^1 K_{d+1}(\tau h, zh)u^{(d+2)}(zh)dz = \sum_{j=0}^{\mathscr{N}} u_j L_j(t) + h^{d+1}R_{d+1}^{(1)}(\tau)\,, \tag{28}$$

where $u_j = u'(\xi_j),\ j = 0,\ldots,\mathscr{N}$, and

$$R^{(1)}_{d+1}(\tau) = \int_0^1 K_{d+1}(\tau h, zh) u^{(d+2)}(zh)dz.$$

Using the fact that the Lagrange function has local support, we can show that $R^{(1)}_{d+1}(\tau)$ is bounded and independent of n. Integration of (28) leads to

$$u(t) = u_0 + \sum_{j=0}^{\mathscr{N}} u_j \beta_j(t) + h^{d+2} R_{d+2}(\tau),$$

where

$$\beta_j = \int_0^t L_j(s)ds \quad \text{and} \quad R_{d+2}(\tau) = \int_0^\tau R^{(1)}_{d+1}(s)ds.$$

For $t = \tau h$, the representation (13) of u_n implies that

$$\begin{cases} e_n = u(t) - u_n(t) = \displaystyle\sum_{j=0}^{\mathscr{N}} \varepsilon_j \beta_j(t) + h^{d+2} R_{d+2}(\tau), \\ e'_n = u'(t) - u'_n(t) = \displaystyle\sum_{j=0}^{\mathscr{N}} \varepsilon_j L_j(t) + h^{d+1} R^{(1)}_{d+1}(\tau), \end{cases} \tag{29}$$

with $\varepsilon_j := u_j - c_j,\ j = 0,\ldots,\mathscr{N}$. From Eqs. (1) and (15) , we have

$$\begin{aligned}\varepsilon_i &= \int_0^1 k(\xi_i,s)e_n(s)ds + a(\xi_i)e_n(\xi_i) \\ &= \sum_{j=0}^{\mathscr{N}} \left(\int_0^1 k(\xi_i,s)\beta_j(s)ds + a(\xi_i)\beta_j(\xi_i) \right) \varepsilon_j + h^{d+3} \int_0^{\mathscr{N}} R_{d+2}(\tau)k(\xi_i,\tau h)d\tau + h^{d+2} a(\xi_i) R_{d+2}(\eta_i).\end{aligned}$$

It follows that $\varepsilon := (\varepsilon_0\ ,\ \ldots\ ,\ \varepsilon_{\mathscr{N}})$ satisfies the linear algebraic system

$$\left[\mathscr{I}_{\mathscr{N}} - \left(\mathscr{A}_{\mathscr{N}} + \mathscr{M}_{\mathscr{N}}\right)\right] \varepsilon = h^{d+2} \varphi_{\mathscr{N}},$$

where $\mathscr{A}_{\mathscr{N}}$ and $\mathscr{M}_{\mathscr{N}}$ are given in (16) and (17) , and

$$\varphi_{\mathscr{N}} = \left(h \int_0^{\mathscr{N}} k(\xi_i,\tau h) R_{d+2}(\tau) d\tau + a(\xi_i) R_{d+2}(\eta_i) \right)_{0 \leqslant i \leqslant \mathscr{N}}$$

From the proof of Theorem 3 , we get

$$\left\| \left(\mathscr{I}_{\mathscr{N}} - \left(\mathscr{A}_{\mathscr{N}} + \mathscr{M}_{\mathscr{N}}\right) \right)^{-1} \right\| \leqslant D_0,$$

where D_0 is a positive constant independent of h.

Moreover, we have

$$\|\varphi_{\mathscr{N}}\|_\infty \leqslant h\mathscr{N} \|k\|_\infty \left\|u^{(d+2)}\right\|_\infty M_K + A_0 \left\|u^{(d+2)}\right\|_\infty M_K \leqslant \left(\|k\|_\infty + A_0\right) \left\|u^{(d+2)}\right\|_\infty M_K,$$

where

$$M_K = \max_{t\in[0,1]} \int_0^1 \left|K_{d+1}(t,s)\right| ds \quad \text{and} \quad A_0 = \max_{t\in[0,1]} |a(t)| \,.$$

Thus , we deduce that $\|\varepsilon\| \leqslant Ch^{d+2}$ where

$$C = M_k D_0 \Big(\|k\|_\infty + A_0 \Big) \left\|u^{(d+2)}\right\|_\infty .$$

Using (29) , we deduce that

$$\left|e_n'(t)\right| \leqslant \Lambda_{d+1} C h^{d+2} + h^{d+1} \left\|u^{(d+2)}\right\|_\infty M_K \,, \tag{30}$$

and

$$\left|e_n(t)\right| \leqslant \tilde{\Lambda}_{d+1} C h^{d+3} + h^{d+2} \left\|u^{(d+2)}\right\|_\infty M_K \,, \tag{31}$$

where

$$\Lambda_{d+1} = \max_{t\in[0,1]} \sum_{j=0}^{\mathcal{N}} \left|L_j(t)\right| \quad \text{and} \quad \tilde{\Lambda}_{d+1} = \max_{t\in[0,1]} \sum_{j=0}^{\mathcal{N}} \left|\beta_j(t)\right| \,.$$

Finally, from (30) and (31) , we have

$$\left\|e_n'\right\|_\infty = \mathscr{O}\big(h^{d+1}\big) \quad \text{and} \quad \|e_n\|_\infty = \mathscr{O}\big(h^{d+2}\big),$$

which completes the proof . □

5 Numerical results

To illustrate the theoretical results established in the previous sections, we consider four examples of FIDE that we solve numerically by collocation and Kantorovich methods based on quadratic QI given by (4) and defined on the interval $[0,1]$ endowed with a uniform partition of length $h = 1/n$. For different values of n, we compute the maximum absolute errors

$$E^c_\infty := \|u - u^c_n\|_\infty \,;\; \tilde{E}^c_\infty := \|v - v^c_n\|_\infty \,;\; E^k_\infty := \|u - u^k_n\|_\infty \,;\; \tilde{E}^k_\infty := \|v - v^k_n\|_\infty \,,$$

where the u^c_n and u^k_n are the approximate solutions obtained by the collocation and Kantorovich methods respectively. Moreover, we present the corresponding numerical convergence orders denoted by **NCO** and calculated as the logarithm to base 2 of the ratio between two consecutive errors. We note that the numerical algorithm was run on a PC with Intel Pentium 2.16GHz CPU, 4GB RAM, and the programs were compiled by using Mathematica.

In the following table, we give the data associated with the considered examples.

The obtained results for Examples 1,2, and 3 are reported in Tables 2, 3, and 4 respectively. It can be seen from these tables that the approximate solutions gradually converge to the exact ones along with the increase of nodes. Moreover, it also confirms that the numerical convergence orders match well with the expected values given in Theorem 4.

Example 4 is given in [5], we consider it here to give a comparison with the results presented in [5] and those obtained by the collocation and Kantorovich method based on quadratic QI given by (4). We notice that, the method presented in [5] is based on B-splines interpolation and the unknown solution

Table 1: Numerical examples

	Kernel K	*Function a*	Function g	Exact solution u
Example 1	$\pi t\sin(\pi s)$	$\sin(\pi t)$	$\pi\sin(\pi t)-\cos(\pi t)\sin(\pi t)$	$\cos(\pi t)$
Example 2	ts	$2t^2$	$-\frac{(-1+e)t}{e}-2t^2\cosh(t)+\sinh(t)$	$\cosh(t)$
Example 3	$\exp(t+s)$	$\exp(-2t)$	$-\exp(-t)+\frac{3\exp t-\exp(2+t)}{2}$	$\exp(t)$
Example 4	$\exp(t+s)$	1	$-\frac{1}{2}\exp(x)(\exp(2)-1)$	$\exp(t)$

Table 2: The absolute errors E_∞^c , $\tilde{E}_\infty^c$, E_∞^k , $\tilde{E}_\infty^k$ and the corresponding **NCO** .

Example 1								
n	E_∞^c	**NCO**	$\tilde{E}_\infty^c$	**NCO**	E_∞^k	**NCO**	$\tilde{E}_\infty^k$	**NCO**
8	$5.86(-04)$	$--$	$3.00(-03)$	$--$	$3.31(-04)$	$--$	$2.83(-03)$	$--$
16	$4.24(-05)$	3.78	$3.90(-04)$	2.94	$1.98(-05)$	4.06	$3.14(-04)$	3.17
32	$2.77(-06)$	3.93	$4.93(-05)$	2.98	$1.23(-06)$	4.00	$3.53(-05)$	3.15
64	$1.76(-07)$	3.97	$4.73(-06)$	3.38	$8.17(-08)$	3.91	$4.58(-06)$	2.94
128	$1.11(-08)$	3.98	$5.93(-07)$	2.99	$4.66(-09)$	4.13	$5.52(-07)$	3.05
Theoretical order	$-$	04	$-$	03	$-$	04	$-$	03

Table 3: The absolute errors E_∞^c , $\tilde{E}_\infty^c$, E_∞^k , $\tilde{E}_\infty^k$ and the corresponding **NCO** .

Example 2								
n	E_∞^c	**NCO**	$\tilde{E}_\infty^c$	**NCO**	E_∞^k	**NCO**	$\tilde{E}_\infty^k$	**NCO**
8	$9.22(-06)$	$--$	$3.61(-05)$	$--$	$6.47(-05)$	$--$	$6.54(-04)$	$--$
16	$7.18(-07)$	3.72	$4.74(-06)$	2.96	$4.41(-06)$	3.87	$7.29(-05)$	3.08
32	$5.04(-08)$	3.86	$6.81(-07)$	2.82	$2.73(-07)$	4.01	$7.25(-06)$	3.40
64	$3.30(-09)$	3.93	$6.59(-08)$	3.36	$1.82(-08)$	3.90	$9.68(-07)$	2.90
128	$2.12(-10)$	3.97	$8.81(-09)$	2.91	$1.05(-09)$	4.10	$1.23(-07)$	2.96
Theoretical order	$-$	04	$-$	03	$-$	04	$-$	03

Table 4: The absolute errors E^c_∞ , $\tilde{E}^c_\infty$, E^k_∞ , $\tilde{E}^k_\infty$ and the corresponding **NCO** .

Example 3								
n	E^c_∞	**NCO**	$\tilde{E}^c_\infty$	**NCO**	E^k_∞	**NCO**	$\tilde{E}^k_\infty$	**NCO**
8	$3.52(-05)$	$--$	$1.37(-04)$	$--$	$7.22(-06)$	$--$	$1.28(-04)$	$--$
16	$2.46(-06)$	3.83	$1.46(-05)$	3.23	$4.04(-07)$	4.15	$1.62(-05)$	2.98
32	$1.63(-07)$	3.91	$1.62(-06)$	3.17	$2.52(-08)$	4.00	$1.79(-06)$	3.18
64	$1.11(-08)$	3.87	$1.45(-07)$	3.47	$3.24(-09)$	2.96	$2.35(-07)$	2.93
128	$6.16(-10)$	4.17	$1.73(-08)$	3.07	$8.60(-11)$	5.23	$2.95(-08)$	2.99
Theoretical order	–	04	–	03	–	04	–	03

is approximated by a cubic B-spline defined on the interval $[0,1]$ endowed with a uniform partition of length $h = 1/n$. For $n = 4$, we compute the punctual errors

$$E^c_j = |u(x_j) - u^c_n(x_j)| \quad \text{and} \quad E^k_j = |u(x_j) - u^k_n(x_j)| \quad \text{for} \quad x_j = \frac{j}{10},\ j = 0, \cdots, 10$$

and we compare them with the errors $E_{Sp,j}$ given in [5].

Table 5: Comparison with results given in [5].

		Collocation and Kantorovich methods				Method [5]	
x_j	$u(x_j)$	$u^c_n(x_j)$	E^c_j	$u^K_n(x_j)$	E^K_j	app values in [5]	$E_{Sp,j}$
0	1	1	0	1	0	1	2.22×10^{-16}
0.1	1.10517	1.10516	1.04×10^{-6}	1.10515	1.29×10^{-5}	1.10518	1.00×10^{-5}
0.2	1.22140	1.22140	6.87×10^{-7}	1.22141	1.29×10^{-5}	1.22137	3.39×10^{-5}
0.3	1.34985	1.34985	1.36×10^{-6}	1.34987	1.72×10^{-5}	1.34979	6.92×10^{-5}
0.4	1.49182	1.49183	5.76×10^{-6}	1.49183	1.51×10^{-5}	1.49171	1.14×10^{-4}
0.5	1.64872	1.64876	4.16×10^{-5}	1.64878	6.53×10^{-5}	1.64855	1.16×10^{-4}
0.6	1.82211	1.82217	5.47×10^{-5}	1.82216	4.30×10^{-5}	1.82189	2.30×10^{-4}
0.7	2.01375	2.01386	1.16×10^{-4}	2.01384	9.29×10^{-5}	2.01345	3.30×10^{-4}
0.8	2.22554	2.22573	1.97×10^{-4}	2.22564	1.05×10^{-4}	2.22515	3.28×10^{-4}
0.9	2.45960	2.45990	3.01×10^{-4}	2.45971	1.08×10^{-4}	2.45909	5.15×10^{-4}
1	2.71828	2.71872	4.39×10^{-4}	2.71849	2.13×10^{-4}	2.71773	5.51×10^{-4}

The results from Table 5 shows that our methods, based on quadratic QI, give very good approximations similar to those obtained in [5] by using cubic B-splines.

6 Conclusions

In this paper, we have proposed collocation and Kantorovich methods based on the QI $\mathscr{Q}_d$ to solve the Fredholm linear integro-differential equations. The theorems on the convergence and error estimates of the methods have been stated and proved. Some numerical examples are provided to illustrate the efficiency and effectiveness of the proposed approach.

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