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Numerical solution of space-time variable fractional order advection-dispersion equation using radial basis functions

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Abstract. This paper aims to advance the radial basis function method for solving space-time variable-order fractional partial differential equations. The fractional derivatives for time and space are considered in the Coimbra and the Riemann-Liouville sense, respectively. First, the time-variable fractional derivative is discretized through a finite difference approach. Then, the space-variable fractional derivative is approximated by radial basis functions. Also, we advance the Rippa algorithm to obtain a good value for the shape parameter of the radial basis functions. Results obtained from numerical experiments have been compared to the analytical solutions, which indicate high accuracy and efficiency for the proposed scheme.

Keywords: Advection-dispersion equation, Radial basis functions, Coimbra fractional derivative, Riemann-Liouville fractional derivative.
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1 Introduction

A keen interest in working in the field of fractional calculus has been shown in recent years. Fractional calculus is a branch of applied mathematics that deals with derivatives and integrals of desired order [16, 18]. Fractional differential equations have several applications in various fields of science and engineering [1, 3, 8, 15]. Numerous phenomena, including fluid mechanics, visco-elasticity [10], and physics [5], can be modeled using the mathematical tool of fractional calculus. In this field and since the derivation operator is non-local, furnishing its analytical solutions are much more difficult compared to ordinary differential equations, thus the numerical solution of these equations is fundamental. Plenty of diverse numerical methods, including finite element [14], finite difference [12], and mesh-free [17] approaches have been utilized to obtain numerical solutions for fractional differential equations.

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Different forms of the fractional differential equations are introduced. Furthermore, the derivative order can be fixed or variable in fractional differential equations. In variable-order fractional derivative equations, the order is a function of time, space, or both of them. Several authors have stated various definitions for variable-order differential operators and surveyed their properties and applications based on their purpose, see, e.g. [11] and the references cited therein.

Numerical solution of variable-order fractional differential equations are more complex in comparison to fixed-order fractional differential equations and several numerical approximations have been carried out for such problems. For instance, Zhang et al. in [23] provided a novel numerical method for the time variable fractional order mobile-immobile advection-dispersion model and studied its stability and convergence (see for more [1]). One of the famous variable-order fractional differential equations is the fractional advection-dispersion equation, also known as FADE, [20, 22]. FADE is an extension of the classic advection-dispersion equation (ADE) so that the second-order derivative is replaced with a fractional-order derivative. In this paper, we present an approach based on radial basis functions (RBFs) to approximate the solutions of the space-time variable fractional order FADE of the following form:

$$\frac{\partial^{\alpha(x)}u(x,t)}{\partial t^{\alpha(x)}}=\mu\frac{\partial^{\beta(x,t)}u(x,t)}{\partial x^{\beta(x,t)}}-\vartheta\frac{\partial^{\gamma(x,t)}u(x,t)}{\partial x^{\gamma(x,t)}}+f(x,t),\quad (x,t)\in\Omega=(0,L)\times[0,T],\tag{1}$$

with initial and boundary conditions

$$\begin{aligned}&u(x,0)=g(x),\\&u(0,t)=\psi_1(t),u(L,t)=\psi_2(t),\end{aligned}$$

where $\mu,\vartheta>0$ and $0<\underline{\alpha}\leq\alpha(x)\leq\overline{\alpha}\leq 1$ and $1<\underline{\beta}\leq\beta(x)\leq\overline{\beta}\leq 2$ and $0<\underline{\gamma}\leq\gamma(x)\leq\overline{\gamma}\leq 1$.

The time fractional derivative $\frac{\partial^{\alpha(x)}u(x,t)}{\partial t^{\alpha(x)}}$ in (1) is the Coimbra [2] fractional derivative and is defined by

$$\frac{\partial^{\alpha(x)}u(x,t)}{\partial t^{\alpha(x)}}=\frac{1}{\Gamma(1-\alpha(x))}\int_{0^+}^{t}(t-\sigma)^{-\alpha(x)}\frac{\partial u(x,\sigma)}{\partial\sigma}d\sigma+\frac{(u(x,0^+)-u(x,0^-))t^{-\alpha(x)}}{\Gamma(1-\alpha(x))},$$

where $0<\alpha(x)<1$. The space fractional derivative $\frac{\partial^{\beta(x,t)}u(x,t)}{\partial x^{\beta(x,t)}}$ in equation (1) is the Riemann-Liouville [4] fractional derivative as follows

$$\frac{\partial^{\beta(x,t)}u(x,t)}{\partial x^{\beta(x,t)}}=\left[\frac{1}{\Gamma(m-\beta(x,t))}\frac{d^m}{d\xi^m}\int_0^{\xi}(\xi-\eta)^{m-\beta(x,t)-1}u(\eta,t)d\eta\right]_{\xi=x},$$

where $m-1<\beta(x,t)<m$ with $\Gamma(\cdot)$ being the Euler's gamma function.

Since the variable-order operator's kernel has a variant power along with the fact that fractional derivative order varies with changes in time and space, then the analytical, as well as the numerical solution of such equations, are cumbersome. A few works have been done about this type of equations. For example, Zhang et al. [24] recently proposed an implicit Euler approximation method for this target. In this paper, we study and advance the radial basis function method for solving these equations. Compared to the classic techniques, the RBF approach has high accuracy, and it can be successfully implemented for integer-order derivative equations. In contrast to the local approximation approaches, global methods (see, e.g. [9]) have particular merits in numerical simulation of space fractional derivative models, such as high accuracy and smaller size of the resulting matrix equation. We utilize the radial basis functions approach to discretize the fractional space derivatives and show that the radial basis function method has considerably better results compared to finite difference approximation.

The rest of the paper is organized as follows. The discretization of the time-fractional derivative is given in Section 2. The fractional derivative of the radial basis function is obtained in Section 3. In Section 4, the discretization of space derivatives using radial basis functions will be discussed. The technique for choosing the shape parameter is given in Section 5. Numerical experiments are brought forward in Section 6 to reveal the effectiveness and accuracy of the proposed approach in comparison with the existing methods. Finally, concluding remarks are given in Section 7.

2 Time-fractional derivative discretization

Assume that the solution $u(x,t) \in c^{(2)}(\Omega)$ is analyzed for $t \geq 0$. Moreover, it is supposed that this property of the function $u(x,t)$ is good enough in $t=0$. Therefore, in the definition of $\frac{\partial^{\alpha(x)}u(x,t)}{\partial t^{\alpha(x)}}$, $u(x,0^+) = u(x,0^-)$.

The finite-difference scheme is utilized for discretizing time-fractional derivative using $\tau = T/M$, $k=0,\ldots,M$, $0 \leq t_k \leq T$, $t_k = k\tau$ as follows:

$$
\begin{aligned}
\frac{\partial^{\alpha(x)}u(x,t_{k+1})}{\partial t^{\alpha(x)}} &= \frac{1}{\Gamma(1-\alpha(x))}\int_0^{(k+1)\tau}(t_{k+1}-\sigma)^{-\alpha(x)}\frac{\partial u(x,\sigma)}{\partial\sigma}d\sigma \\
&\approx \frac{1}{\Gamma(1-\alpha(x))}\sum_{j=0}^{k}\int_{j\tau}^{(j+1)\tau}(t_{k+1}-\sigma)^{-\alpha(x)}\frac{\partial u(x,\sigma_j)}{\partial\sigma}d\sigma,
\end{aligned}
\tag{2}
$$

where the first order time derivative is approximated as follows:

$$
\frac{\partial u(x,\sigma_j)}{\partial\sigma} = \frac{u(x,\sigma_{j+1})-u(x,\sigma_j)}{\tau} + O(\tau). \tag{3}
$$

Then we obtain

$$
\begin{aligned}
\frac{\partial^{\alpha(x)}u(x,t_{k+1})}{\partial t^{\alpha(x)}} &= \frac{1}{\Gamma(1-\alpha(x))}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{1}{(t_{k+1}-\sigma)^{\alpha(x)}}d\sigma + O\left(\tau^{2-\alpha(x)}\right) \\
&= \frac{1}{\Gamma(2-\alpha(x))}\sum_{j=0}^{k}\frac{u\left(x,t_{k+1-j}\right)-u(x,t_{k-j})}{\tau^{\alpha(x)}}b_j + O\left(\tau^{2-\alpha(x)}\right),
\end{aligned}
\tag{4}
$$

where $b_j = (j+1)^{1-\alpha(x)} - j^{1-\alpha(x)}$. Now, we can write

$$
\frac{\partial^{\alpha(x)}u(x,t_{k+1})}{\partial t^{\alpha(x)}} = \begin{cases} \frac{\tau^{-\alpha(x)}}{\Gamma(2-\alpha(x))}\left(u^{k+1}-u^k\right) + \frac{\tau^{-\alpha(x)}}{\Gamma(2-\alpha(x))}\sum_{j=1}^{k}\left(u^{k+1-j}-u^{k-j}\right)b_j, & k \geq 1, \\ \frac{\tau^{-\alpha(x)}}{\Gamma(2-\alpha(x))}\left(u^1-u^0\right), & k=0, \end{cases} \tag{5}
$$

denoting $u^k = u(x,t_k)$. Note that in the discussion about the convergence of the time derivative discretization, Lin and Xu [13] proved that the numerical solution is convergent to the analytical solution with $O\left(\tau^{2-\alpha}\right)$ order.

Our work aims to approximate the solution $u(x,t)$ using RBFs. Therefore, it is needed to furnish the fractional space derivative of the function $u(x,t)$. In order to do this, the fractional derivative of the RBFs should be computed at the beginning of the procedure.

3 Fractional derivatives of radial basis functions

An RBF is a real-valued function whose value depends only on the distance from some other point x_j, called a center, so that $\varphi_j(x) = \phi(\|x-x_j\|)$. The multiquadrics (MQ) RBFs proposed by Hardy $\varphi_j(x) = \sqrt{\|x-x_j\|^2+c^2}$, is one of the most used RBFs [6]. The parameter c in this definition of the RBF is called the shape parameter and plays an essential role in the method's accuracy. Accordingly, a method for finding a suitable shape parameter is needed. Here we use the Rippa algorithm [19] to select a good value for the shape parameter c.

Suppose that, $(N+1)$ different points $\{x_j \,|\, j=0,\dots,N\}$ are given in $\Omega\cup\partial\Omega$, such that x_0 and x_N are the boundary points and the other $(N-1)$ points are internal points. The RBF approximation to the solution $u(x,t)$ in equation (1) at time t_k is considered as follows:

$$u(x,t_k) = \sum_{j=0}^{N} \lambda_j^k \varphi_j(x) + \sum_{l=1}^{2} \lambda_{N+l}^k P_l(x), \tag{6}$$

where $P_l(x)$ $(l=1,2)$ is a base for Π_1, the space of all polynomials on $\mathbb{R}$ with the highest degree one (e.g. $P_1 = x$ and $P_2 = 1$) and λ_j is the unknown coefficient. In order to enforce u to satisfy the equation (1), we need to obtain the fractional derivatives of the RBFs.

Now since finding a fractional derivative in analytical form for a function is slightly challenging, we would limit ourselves to functions Taylor series to implement the fractional derivative operator term by term. Toward this goal, first, the MQ Maclaurin series is derived. Then, the Riemann-Liouville fractional derivative is applied term by term in what follows.

We know that if $f(x) = (x-a)^n$, $n > -1$, and $\beta > 0$, then [4]

$$\frac{\partial^\beta f(x)}{\partial x^\beta} = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\beta)}(x-a)^{n-\beta}, & \beta - n \notin N, \\ 0, & \beta - n \in N. \end{cases} \tag{7}$$

Therefore, by considering relation (7), the fractional derivative of $\varphi_j(x)$ is derived as follows

$$\frac{\partial^\beta \varphi_j(x)}{\partial x^\beta} = \frac{\partial^\beta}{\partial x^\beta}\left(\sum_{n=0}^{\infty} \varphi_j^{(n)}(0)\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{\varphi_j^{(n)}(0)}{n!}\frac{\partial^\beta x^n}{\partial x^\beta} = \sum_{n=0}^{\infty} \frac{\varphi_j^{(n)}(0)}{\Gamma(n+1-\beta)} x^{n-\beta}, \tag{8}$$

and also the fractional derivative of $u(x,t)$ can be obtained as comes next:

$$\begin{aligned} \frac{\partial^\beta u(x,t)}{\partial x^\beta} &= \sum_{j=0}^{N} \lambda_j \frac{\partial^\beta \varphi_j(x)}{\partial x^\beta} + \lambda_{N+1}\frac{\partial^\beta (x)}{\partial x^\beta} + \lambda_{N+2}\frac{\partial^\beta (1)}{\partial x^\beta} \\ &= \sum_{j=0}^{N}\sum_{n=0}^{\infty} \lambda_j \frac{\varphi_j^{(n)}(0)}{\Gamma(n+1-\beta)} x^{n-\beta} + \lambda_{N+1}\frac{\Gamma(2)}{\Gamma(2-\beta)} x^{1-\beta} + \lambda_{N+2}\frac{\partial^\beta (1)}{\partial x^\beta}. \end{aligned} \tag{9}$$

Note that the infinite sum would cut where the terms are smaller in magnitude than the machine precision. Therefore, the MQ RBF fractional derivative and the fractional derivative of $u(x,t)$, mentioned in (6), have been obtained with the help of the MQ function Maclaurin expansion.

4 Spatial derivative discretization by the RBFs

In the time discretization in Section 2, Eq. (1) could be rewritten as follows

$$\frac{\tau^{-\alpha(x)}}{\Gamma(2-\alpha(x))}\sum_{j=0}^{k}(u^{k+1-j}-u^{k-j})b_j=\mu\frac{\partial^{\beta^{k+1}}u^{k+1}}{\partial x^{\beta^{k+1}}}-\vartheta\frac{\partial^{\gamma^{k+1}}u^{k+1}}{\partial x^{\gamma^{k+1}}}+f^{k+1},\quad k=0,\ldots,M-1,$$

where $\zeta^{k+1}=\zeta(x,t_{k+1})$, $\zeta=\beta,\gamma,u,f$. Now, put $r(x)=\tau^{\alpha(x)}\Gamma(2-\alpha(x))$, for $k=0$ and $k\geq 1$, then we have the following equations respectively

$$u^1-r(x)\,\mu\frac{\partial^{\beta^1}u^1}{\partial x^{\beta^1}}+r(x)\,\vartheta\frac{\partial^{\gamma^1}u^1}{\partial x^{\gamma^1}}=u^0+r(x)\,f^1, \tag{10}$$

$$u^{k+1}-r(x)\,\mu\frac{\partial^{\beta^{k+1}}u^{k+1}}{\partial x^{\beta^{k+1}}}+r(x)\,\vartheta\frac{\partial^{\gamma^{k+1}}u^{k+1}}{\partial x^{\gamma^{k+1}}}=\sum_{j=0}^{k-1}(b_j-b_{j+1})\,u^{k-j}+b_ku^0+r(x)\,f^{k+1}, \tag{11}$$

for $k=1,\ldots,M-1$. Now $u(x,t_{k+1})$ would be approximated through RBFs as follows:

$$u(x,t_{k+1})=\sum_{j=0}^{N}\lambda_j^{k+1}\varphi_j(x)+\lambda_{N+1}^{k+1}x+\lambda_{N+2}^{k+1},\quad k=0,\ldots,M-1, \tag{12}$$

with the condition that

$$\sum_{j=0}^{N}\lambda_j^{k+1}=\sum_{j=0}^{N}\lambda_j^{k+1}x_j=0,\quad k=0,\ldots,M-1, \tag{13}$$

where $\lambda_0,\ldots,\lambda_{N+2}$ are unknown. Substituting Eq. (12) in Eqs. (10) and (11), a system with $(N+3)$ unknown variables is furnished. Using the collocation method in the $(N+1)$ points as well as the conditions of (13), the unknowns $\lambda_0\ldots,\lambda_{N+2}$ can be calculated.

For $k=0$, substituting (12) in (10) and using $(N-1)$ collocation points, the equation below can be computed:

$$\begin{aligned}\sum_{j=0}^{N}\lambda_j^1\varphi_j(x_i)+\lambda_{N+1}^1x_i+\lambda_{N+2}^1-r_i\mu\left(\sum_{j=0}^{N}\lambda_j^1\frac{\partial^{\beta_i^1}\varphi_j(x_i)}{\partial x^{\beta_i^1}}+\lambda_{N+1}^1\frac{\partial^{\beta_i^1}(x_i)}{\partial x^{\beta_i^1}}+\lambda_{N+2}^1\frac{\partial^{\beta_i^1}(1)}{\partial x^{\beta_i^1}}\right)\\+r_i\vartheta\left(\sum_{j=0}^{N}\lambda_j^1\frac{\partial^{\gamma_i^1}\varphi_j(x_i)}{\partial x^{\gamma_i^1}}+\lambda_{N+1}^1\frac{\partial^{\gamma_i^1}(x_i)}{\partial x^{\gamma_i^1}}+\lambda_{N+2}^1\frac{\partial^{\gamma_i^1}(1)}{\partial x^{\gamma_i^1}}\right)=u_i^0+r_if_i^1,\ \ i=1,\ldots,N-1.\end{aligned} \tag{14}$$

In addition, substituting Eq. (12) in boundary conditions of (2), the followings are obtained:

$$\sum_{j=0}^{N}\lambda_j^1\varphi_j(x_0)+\lambda_{N+1}^1x_0+\lambda_{N+2}^1=\psi_1(t_1), \tag{15}$$

$$\sum_{j=0}^{N}\lambda_j^1\varphi_j(x_N)+\lambda_{N+1}^1x_N+\lambda_{N+2}^1=\psi_2(t_1), \tag{16}$$

where $r_i=r(x_i)$, $u_i^0=g(x_i)$, $\alpha_i=\alpha(x_i)$ and $\zeta_i^k=\zeta(x_i,t_k)$, $\zeta=\beta,\gamma,u,f$, $i=1,\ldots,N-1$.

The $(N+1)$ equations obtained in (14), (15), and (16) along with the condition of

$$\sum_{j=0}^{N}\lambda_j^1=\sum_{j=0}^{N}\lambda_j^1 x_j=0,$$

would create a system of $(N+3)$ equations which matrix form is written as follows:

$$\left[A^{k+1}-\mu W^{k+1}+\vartheta C^{k+1}\right]\lambda^{k+1}=U^k+F^{k+1},\qquad (k=0), \tag{17}$$

where $F^{k+1}=(0,r_1f_1^{k+1},\ldots,r_{N-1}f_{N-1}^{k+1},0,0,0)^T, U^k=(\psi_1(t_{k+1}),u_1^k,\ldots,u_{N-1}^k,\psi_2(t_{k+1}),0,0)^T$, $A^{k+1}=(a_{ij}^{k+1})$, $W^{k+1}=(w_{ij}^{k+1})$ and $C^{k+1}=(c_{ij}^{k+1})$. We have

$$a_{ij}^k=\begin{cases}\varphi_{ij}^k & 0,\leq i,j\leq N,\\ x_i, & 0\leq i\leq N,\ j=N+1,\\ 1, & 0\leq i\leq N,\ j=N+2,\\ x_j, & 0\leq j\leq N,\ i=N+1,\\ 1, & 0\leq j\leq N,\ i=N+2,\\ 0, & \text{otherwise,}\end{cases} \tag{18}$$

$$w_{ij}^k=\begin{cases} r_i\frac{\partial^{\beta_i^k}(a_{ij}^k)}{\partial x^{\beta_i^k}}, & 0\leq j\leq N+2,\ 1\leq i\leq N-1,\\ 0, & 0\leq j\leq N+2,\ i=0,N,N+1,N+2,\end{cases} \tag{19}$$

$$c_{ij}^k=\begin{cases} r_i\frac{\partial^{\gamma_i^k}(a_{ij}^k)}{\partial x^{\gamma_i^k}}, & 0\leq j\leq N+2,\ 1\leq i\leq N-1,\\ 0, & 0\leq j\leq N+2,\ i=0,N,N+1,N+2.\end{cases} \tag{20}$$

Using the method mentioned in Section 3 for obtaining the fractional derivative of the RBF, the entries of matrices, called W and C, can be easily obtained. Note that in (8) and for any $1\leq i\leq N-1$, we have:

$$\frac{\partial^{\beta_i^k}(\varphi_{ij})}{\partial x^{\beta_i^k}}=\frac{\partial^{\beta_i^k}(\varphi_j(x_i))}{\partial x^{\beta_i^k}}=\sum_{n=0}^{\infty}\frac{\varphi_j^{(n)}(0)}{\Gamma(n+1-\beta_i^k)}x_i^{n-\beta_i^k},$$
$$\frac{\partial^{\beta_i^k}(x_i)}{\partial x^{\beta_i^k}}=\frac{\Gamma(2)}{\Gamma(2-\beta_i^k)}x_i^{1-\beta_i^k}.$$

The same approach can be applied for γ_i^k order fractional space derivative.

Similarly for $k\geq 1$, this time Eq. (12) is substituted in Eqs. (11) and (2). By considering the condition (13), the matrix form of the new system is as follows:

$$\left[A^{k+1}-\mu W^{k+1}+\vartheta C^{k+1}\right]\lambda^{k+1}=\sum_{j=0}^{k-1}H^{k-j}\lambda^{k-j}+Q^k+F^{k+1},\qquad k\geq 1, \tag{21}$$

where A, W, C, and F have their pervious definitions,

$$Q^k=(\psi_1(t_{k+1}),b_{k,1}u_1^0,\ldots,b_{k,N-1}u_{N-1}^0,\psi_2(t_{k+1}),0,0)^T$$

and $H^{k-j} = (h_{is}^{k-j})$, whereas

$$h_{is}^{k-j} = \begin{cases} (b_{j,i} - b_{j+1,i})a_{is}^{k-j}, & 0 \le s \le N+2,\ 1 \le i \le N-1, \\ 0, & 0 \le s \le N+2,\ i = 0, N, N+1, N+2, \end{cases} \tag{22}$$

so that $b_{j,i} = (j+1)^{1-\alpha_i} - j^{1-\alpha_i}$.

In order to solve systems (17) and (21), the shape parameter should be chosen and it needs to be furnished according to the Rippa algorithm [19] to minimize error. This will be done for each time step and $\{\lambda^{k+1}\}$s furnished from the optimal shape parameter will be taken into account.

5 Choosing the shape parameter

In this section we advance the Rippa algorithm [19] to obtain a good value for shape parameter c of the radial basis functions. The algorithm proposed by Rippa in 1999 corresponds to one of its variables called Leave-One Out Cross Validation (LOOCV). Systems (17) and (21) can be considered as $G^{k+1}\lambda^{k+1} = B^{k+1}$, so that

$$\begin{cases} G^{k+1} = \left[A^{k+1} - \mu W^{k+1} + \vartheta C^{k+1}\right], & 0 \le k \le M-1, \\ B^{k+1} = \sum_{j=0}^{k-1} H^{k-j}\lambda^{k-j} + Q^k + F^{k+1}, & 1 \le k \le M-1, \\ B^{k+1} = U^k + F^{k+1,} & k = 0, \end{cases} \tag{23}$$

Entries of the matrices G^{k+1} are dependent on the shape parameter c, which we are seeking to optimize.

Remark 1. *Unlike the interpolation problems, the RBFs method could result in singular matrix for some special centers arrangements. Numerical examples indicate that the RBFs method is robust for a set of distinct centers and the case that the resultant matrix G is singular is rare [7].*

According to [19], the cost function is given by the norm of an error vector $e^{k+1}(c) = (e_0^{k+1}, \dots, e_N^{k+1})^T$ with components

$$e_i^{k+1}(c) = \frac{\lambda_i^{k+1}}{(G^{-1})_{ii}^{k+1}}, \quad k = 0, \dots, M-1, \tag{24}$$

where λ_i^{k+1} is the ith component of the vector λ^{k+1} and $(G^{-1})_{ii}^{k+1}$ is the ith diagonal element of the inverse of matrix G^{k+1}. Thus, the optimal value of the shape parameter is considered as the one which minimizes the cost function $e(c)$ and for each time step t_{k+1}, the optimal shape parameter of c is separately computed: $\left\|e(c_*^{k+1})\right\| = \underset{c}{\mathrm{Min}}\left\|e^{k+1}(c)\right\|$. Now, the algorithm below can be used for selecting a good value for c parameter at each time level t_{k+1}, in RBF approximation method for FPDEs [21]:

1. fix c
2. for $i = 0 : N$
   $e_i^{k+1}(c) = \frac{\lambda_i^{k+1}}{(G^{-1})_{ii}^{k+1}}$
   end
3. $e^{k+1} = (e_0^{k+1}, \dots, e_N^{k+1})^T$
4. $\underset{c}{\mathrm{Min}}\left\|e^{k+1}\right\|$

6 Computational aspects

In this section, some tests will be carried out to show the method accuracy and the validity of the relations obtained in Sections 2-5. In addition, it is our goal to compare the computational findings with the results in [24]. in [24] proposed an implicit Euler method for advection-dispersion equation in which the time-fractional derivative is discretized similar to what is mentioned in Section 2, and the fractional space derivative is stated using Grunwald [4] approximation. The proposed method is applied for two FADEs. The computations are carried out using MATLAB R2016a. In order to show the accuracy of our method, we compute E_∞ and E_2, which are the errors corresponding to the infinity and Euclidean norms, respectively. In order to select a good value for the shape parameter c with good speed, MATLAB *fminbnd* function is used to minimize the cost function $\|e\|$ relative to c. The algorithm returns a good value of $c \in (c_{\min}, c_{\max})$. $c_{\min}$ and $c_{\max}$ are the initial choices.

Example 1. Consider the following FADE

$$\frac{\partial^{\alpha(x)} u(x,t)}{\partial t^{\alpha(x)}} = 2\frac{\partial^{\beta(x,t)} u(x,t)}{\partial x^{\beta(x,t)}} - \frac{\partial^{\gamma(x,t)} u(x,t)}{\partial x^{\gamma(x,t)}} + f(x,t), \quad (x,t) \in \Omega = (0,1) \times [0,T], \tag{25}$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0) = 5x^2(1-x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \end{cases}$$

where

$$\begin{aligned} f(x,t) &= \frac{10x^2(1-x)t^{2-\alpha(x)}}{\Gamma(3-\alpha(x))} - 10(t^2+1)\left[\frac{2x^{2-\beta(x,t)}}{\Gamma(3-\beta(x,t))} - \frac{6x^{3-\beta(x,t)}}{\Gamma(4-\beta(x,t))}\right] \\ &\quad + 5(t^2+1)\left[\frac{2x^{2-\gamma(x,t)}}{\Gamma(3-\gamma(x,t))} - \frac{6x^{3-\gamma(x,t)}}{\Gamma(4-\gamma(x,t))}\right], \\ \alpha(x) &= 0.8 + 0.01\ln(5x), \\ \beta(x,t) &= 1.8 + 0.01x^2t^2, \\ \gamma(x,t) &= 0.8 + 0.01x^2\sin(t). \end{aligned}$$

The exact solution of this problem is $u(x,t) = 5(t^2+1)x^2(1-x)$.

This example is solved for different values of M, and N. The results are compared to finite difference method (FDM) [24]. (M is the number of time steps). Notice that an optimal value of c (shape parameter) would be chosen for each time step and the selection is done through Rippa algorithm. In Rippa algorithm, $c_{\min} = 0$ and $c_{\max} = 10$ (for more details see [21]).

In this example, Maclaurin series of ϕ radial basis function is cut from the 20th order where the terms are smaller in magnitude than 10^{-15}. Table 1 represents point-to-point error at $T = 1$ for $M = 60$ and $N = 60$. Table 2 shows the optimal shape parameter furnished in each time step. According to the results obtained in Table 3, it can be seen that this method can reach higher accuracy with far less number of points compared to implicit Euler approximation method [24]. The implicit Euler approximation method with 200 points has lower accuracy compared to the proposed method with 10 points. Figures 1 and 2 represent numerical solution in 2D and 3D states in various times, respectively.

Table 1: Numerical solution, exact solution and absolute error ($M = 60$, $N = 60$, $c = 7.0820$) at $T = 1$ in Example 1.

x	Exact solution	Numerical solution	$\lvert u_{exact} - u_{approximate}\rvert$
0.1	0.0900	0.089959624432524	4.0376e-05
0.2	0.3200	0.319997409861813	2.5901e-06
0.3	0.6300	0.630035146095551	3.5146e-05
0.4	0.9600	0.960063656416509	6.3656e-05
0.5	1.2500	1.250080209614623	8.0210e-05
0.6	1.4400	1.440080623173316	8.0623e-05
0.7	1.4700	1.470069451603195	6.9452e-05
0.8	1.2800	1.280039618588432	3.9619e-05
0.9	0.8100	0.809991189508622	8.8105e-06

Table 2: Shape parameter obtained at each time step with its corresponding error ($N = 60, M = 60$) in Example 1.

t	Shape parameter (c)	E_2	E_∞
0.1	4.9578	6.2581e-04	1.2328e-04
0.2	7.6370	5.1246e-04	1.0279e-04
0.3	7.0823	4.4055e-04	9.0138e-05
0.4	7.4265	5.1051e-04	1.0256e-04
0.5	6.1803	6.2063e-04	1.1409e-04
0.6	5.4702	6.8184e-04	1.1865e-04
0.7	7.2602	3.7615e-04	7.1783e-05
0.8	5.0643	3.5607e-04	6.7964e-05
0.9	7.6416	3.7071e-04	7.2795e-05
1.0	7.0820	4.1851e-04	8.1786e-05

Table 3: Error behavior for different values of M, N at $T = 1$ in Example 1.

M	N	E_∞ (RBF method)	E_2 (RBF method)	E_∞ (FDM, Ref. [24])	E_2 (FDM, Ref. [24])
5	5	3.2442e-03	4.9727e-03	0.1748	0.2861
10	10	7.5047e-04	1.5446e-03	0.0917	0.2117
15	15	6.7470e-04	1.6235e-03	0.0620	0.1751
20	20	4.3175e-04	1.2230e-03	0.0469	0.1526
40	40	1.2957e-04	5.2798e-04	0.0238	0.1088
60	60	8.1786e-05	4.1851e-04	0.0162	0.0899

Example 2. In this example, we consider the following equation

$$\frac{\partial^{\alpha(x)} u(x,t)}{\partial t^{\alpha(x)}} = \frac{\partial^{\beta(x,t)} u(x,t)}{\partial x^{\beta(x,t)}} - \frac{\partial^{\gamma(x,t)} u(x,t)}{\partial x^{\gamma(x,t)}} + f(x,t), \qquad (x,t) \in \Omega = (0,1) \times [0,T], \tag{26}$$

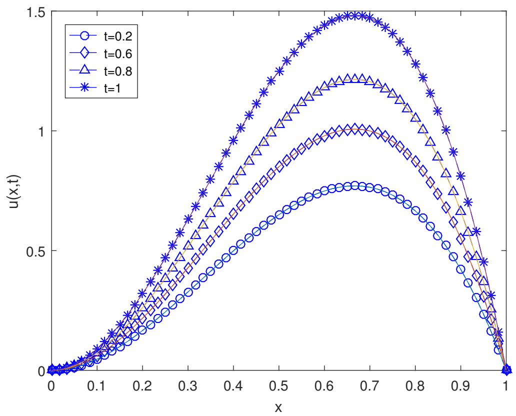

Figure 1: The solution behavior of Example 1 at $t = 0.2, 0.6, 0.8, 1$ ($N = 60, M = 60$)

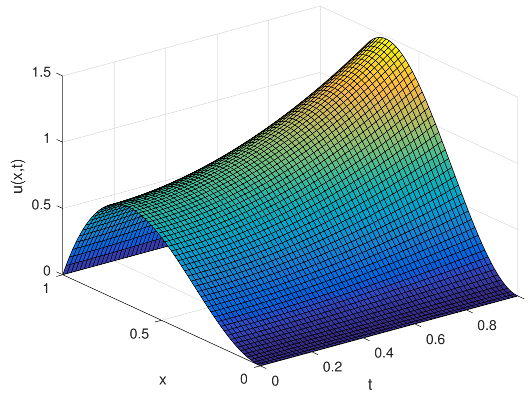

Figure 2: The 3-dimension numerical solution of (25) ($N = 60$, $M = 60$)

with the following initial and boundary conditions

$$\begin{cases} u(x,0) = 10x^2(1-x), & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \end{cases}$$

where

$$\begin{aligned} f(x,t) &= \frac{10x^2(1-x)\,t^{1-\alpha(x)}}{\Gamma(2-\alpha(x))} - 10(t+1)\left[\frac{2x^{2-\beta(x,t)}}{\Gamma(3-\beta(x,t))} - \frac{6x^{3-\beta(x,t)}}{\Gamma(4-\beta(x,t))}\right] \\ &\quad + 10(t+1)\left[\frac{2x^{2-\gamma(x,t)}}{\Gamma(3-\gamma(x,t))} - \frac{6x^{3-\gamma(x,t)}}{\Gamma(4-\gamma(x,t))}\right], \\ \alpha(x) &= 1 - 0.5e^{-x}, \\ \beta(x,t) &= 1.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}, \\ \gamma(x,t) &= 0.7 + 0.5e^{-\frac{x^2}{1000} - \frac{t}{50} - 1}. \end{aligned}$$

The exact solution of this problem is: $u(x,t) = 10(t+1)x^2(1-x)$.

Results of this example are presented in Tables 4-6. In this example, Maclaurin expansion of the radial basis function ϕ is cut from the 20th order and in Rippa algorithm, $c_{\min} = 0$ and $c_{\max} = 10$. A comparison of the numerical solution and the exact solution of (26) at $T = 1$ are listed in Table 4. Table 5 shows the optimal shape parameter furnished in each time step. As it can be seen in Table 6, using much less points, the proposed method has considerably better results compared to finite difference approximation and the implicit Euler approximation method with 200 points has lower accuracy compared to the proposed method with 8 points [24]. Figures 3 and 4 represent numerical solution in 2D and 3D states in various times, respectively. In this example the order of convergence q is calculated using the following equation

$$q \approx \frac{log\frac{E_1}{E_2}}{log\frac{N_2}{N_1}}$$

Where E_1 and E_2 are the errors corresponding to N_1 and N_2. The results are given in Table 6.

Table 4: Numerical solution, exact solution and absolute error ($M = 14$, $N = 14$, $c = 6.7258$) at $T = 1$ of Example 2.

x	Exact solution	Numerical solution	$\lvert u_{exact} - u_{approximate}\rvert$
0.1429	0.3499	0.349849695023295	5.0305e-05
0.2143	0.7216	0.721572870072123	2.7130e-05
0.3571	1.6399	1.639939302262064	3.9302e-05
0.4286	2.0991	2.099115365799662	1.5366e-05
0.5000	2.5000	2.499985688980814	1.4311e-05
0.6429	2.9519	2.951885217484232	1.4783e-05
0.7143	2.9155	2.915451997574564	4.8002e-05
0.8571	2.0991	2.099128717240092	2.8717e-05
0.9286	1.2318	1.231772416886088	2.7583e-05

Table 5: Shape parameter obtained at each time step with its corresponding error ($N = 14$, $M = 14$) in Example 2.

t	Shape parameter (c)	E_2	E_∞
0.1429	6.3894	1.9885e-05	1.0954e-05
0.2143	6.3929	1.8234e-05	9.3436e-06
0.3571	6.7737	2.2064e-05	1.1692e-05
0.4286	6.2481	2.3342e-05	1.2448e-05
0.5000	6.6822	2.1830e-05	1.2120e-05
0.6429	6.3364	3.9601e-05	2.0812e-05
0.7143	3.7384	1.9879e-05	8.8813e-06
0.8571	6.0846	3.5095e-05	2.0369e-05
0.9286	6.6825	2.4100e-05	1.1746e-05
1.0	6.7258	2.6566e-05	1.4311e-05

7 Conclusions

In this paper, the radial basis function method was advanced for obtaining numerical solutions of a variable-order fractional advection-dispersion equation with a Coimbra time variable fractional derivative and Riemann-Liouville space variable fractional derivatives. The time-fractional derivative was discretized through a finite-difference approach and the fractional space derivatives were approximated

Table 6: Error behavior for different values of M,N at $T=1$ in Example 2.

M	N	E_∞ (RBF method)	E_2 (RBF method)	E_∞ (FDM, Ref. [24])	E_2 (FDM, Ref. [24])	q-RBF	q-FDM
4	4	1.8239e-03	2.1497e-03	0.3273	0.4570	-	-
8	8	2.1763e-04	3.3667e-04	0.2016	0.3896	3.0671	0.6991
12	12	1.8888e-05	3.4423e-05	0.1430	0.3366	4.1600	0.7537
14	14	1.4311e-05	2.6566e-05	0.1243	0.3166	3.8696	0.7728

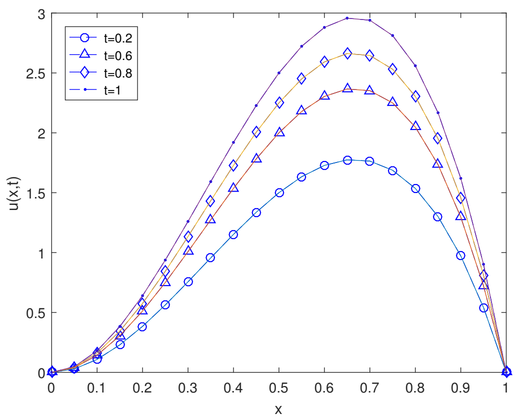

Figure 3: The solution behavior of (26) at $t=0.2$, $0.6, 0.8, 1$ ($N=20$, $M=20$).

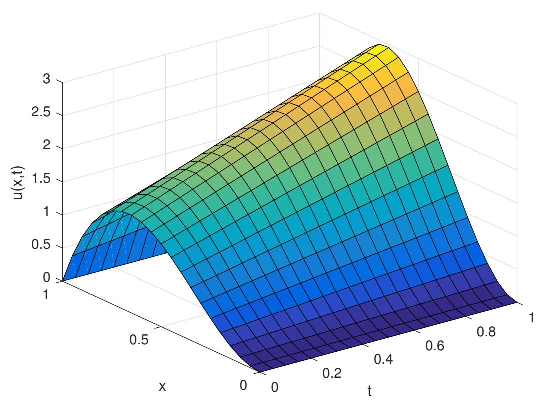

Figure 4: The 3-dimension numerical solution of Example 2 ($N=20$, $M=20$).

by MQ radial basis functions. The MQ radial basis functions are defined by a shape parameter which has an important role in the accuracy of the results and the numerical stability of the method. The shape parameter c in MQ radial basis functions was selected with the Rippa algorithm. The Numerical examples indicated that the radial basis function method has considerably better results compared to finite difference approximation for variable-order fractional differential equation.

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