**Journal of Mathematical Modeling**
Vol. 10, No. 3, 2022, pp. 449-465. Research Article

JMM

# A novel approach for modeling system reliability characteristics in an imprecise environment

**Javd Zendehdel[†], Reza Zarei[‡*], Mohammad Ghasem Akbari[§]**

[†]*Department of Statistics, Faculty of Mathematical Sciences and Statistics, University of Birjand, Birjand, Iran*
[‡]*Department of Statistics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
*Email(s): j.zendehdel.stat@gmail.com, rezazarei.r@gmail.com, g_z_akbari@yahoo.com*

**Abstract.** In this paper, we introduce and investigate a new definition for the density function of fuzzy random variables. Then, based on this definition, we give a new viewpoint on aging properties. To do this end, the concepts of the hazard rate function and mean residual function are investigated for the Exponential fuzzy random variable. Also, we obtain the aging properties of new Exponential fuzzy random variables based on existing methods. Finally, using a practical example, we illustrate the proposed approach and show that the performance of proposed approach is better than two other existing methods.

*Keywords*: Exponential lifetime, fuzzy parameter, survival function, Hazard rate function, mean residual life function.
*AMS Subject Classification 2010*: 90B25, 60A86.

## 1 Introduction

The concept of aging plays an important role in reliability analysis. Depending on statistical distribution, a component (system) can have one of the situations No aging, *Positive aging* or *Negative aging*. *No aging* means that the age of a component does not affect on the distribution of the residual lifetime of the component. *Positive aging* (also known as *averse aging*) describes the situation where residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This is a common situation in reliability engineering in which the performence of components tend to become worse with time due to increased wear and tear. On the other hand, negative aging, which is also known as beneficial aging, describes the situation where performance and residual lifetime of components tends to increase. Although this situation is less common, when a system undergoes regular testing and improvement, there are cases for which we have a reliability growth phenomenon.

*Corresponding author.
Received: 11 November 2021 / Revised: 4 February 2022 / Accepted: 6 February 2022
DOI: 10.22124/jmm.2022.20487.1836

The lifetimes of a component are usually assumed to be random variables with probability distributions that have crisp parameters. Also, in classical reliability theory, the parameters of lifetime distributions are assumed as an exact quantity. However, in the real world, our understanding of most physical processes is mainly based on vague quantities. This vagueness is a form of information that can be useful in decision making, prediction, modeling, etc. Examples of imprecise information are such statements as the product quality is "acceptable", the lifetime of a lamp is "approximately 2000(h)", or the necessary dose of a certain chemical in a drug must be "about 20%". In addition, there are situations in which the parameters of the distribution cannot record precisely due to machine errors, experiments, personal judgment or estimation. During the past decades, different approaches and theories have proposed for treating imprecision and uncertainty, among which the fuzzy set theory is a key one, and several researchers have concentrated on applying this theory to various fields [8,28], especially in probability and statistics [21,22,25–27].

As authors known, there are many researchers pay attention to reliability analysis in fuzzy environment [1,3–6,10,15,17,23]. In the following, we review the main studies on fuzzy reliability to display the motivation for this paper. Buckly [2] introduced fuzzy Exponential density using the $\alpha$-cuts of parameters. Guo et al. [9] investigated random fuzzy variable modeling on repairable systems and they proposed a credibility hazard concept associated with fuzzy lifetimes based on the conditional distribution of a fuzzy variable under Liu's non-classical credibility measure theory. Yao et al. [24] considered the reliability of the serial systems and the reliability of parallel systems using triangular fuzzy numbers based on statistical data. Karpíšek et al. [16] defined the fuzzy reliability and its characteristics for the double-state probability model of an object based on the fuzzy probability distribution and its properties. Jamkhaneh [12] considered the problem of evaluation of system reliability, in which the lifetimes of components are described using a fuzzy Exponential distribution. Also, Jamkhaneh et al. [14] proposed a new method for analyzing the fuzzy system reliability of a parallel-series and series-parallel systems using fuzzy confidence intervals, where the reliability of each component of each system is unknown. They proposed a general procedure to construct the reliability characteristics and its $\alpha$-cut set, when the parameters are fuzzy [13]. Pak et al. [18] presented a Bayesian approach to estimate the parameter and reliability function of Rayleigh distribution from fuzzy lifetime data. Recently, El-Damcese and Ramadan [7] investigated some statistical properties of the mixture generalized linear failure rate distribution and mixture generalized linear failure rate distribution with fuzzy parameters and presented formulas of a fuzzy reliability function, fuzzy hazard function and their $\alpha$-cut set. Iqbal and Uduman [11] considered reliability analysis of paper plant using a boolean function with fuzzy logic technique. Shafiq and Viertl [20] generalized parameter estimation, survival functions, and hazard rates for fuzzy lifetime data.

Since the Exponential distribution plays an important role in the reliability of a system, we need to extend the classical methods for the uncertainty situation. However, there are many different definitions of the Exponential random variable in the uncertainty situation which led to various discussions about the aging concepts in this situation. In this paper, we propose a new procedure to construct the reliability characteristics of lifetime random variables, when the parameter is imprecise and represented as a fuzzy number. The rest of the paper is organized as follows. In Section 2, we recall Buckley's approach and propose two new procedures for constructing reliability characteristics in a fuzzy environment. The proposed approach was illustrated and then compared with existing methods using a numerical example in Section 3. A brief conclusion was presented in Section 4.

## 2 Aging concepts

In this section, we want to present the formula (may be closed or opened form) for some aging concepts such as Survival, hazard rate and mean residual life functions based on various approaches in the definition of density function of fuzzy Exponential distribution. Through this section, suppose that $X$ is a random variable with the support $S_X = \{x \in R : f(x|\theta) > 0, \theta \in \Theta\}$, where $f(x;\theta)$ is the density function of $X$ and $\Theta$ is the parameter space.

### 2.1 Buckley's approach

Suppose that due to some conditions in the study of underlying problems, we can not record the parameter $\theta$ as a precise number. In such situations, we can use the fuzzy set theory to formulate $\theta$ as a fuzzy number $\tilde{\theta}$.

According to [2], if we assume that $\theta$ is the fuzzy number, then the fuzzy density function is also a fuzzy number with the following $\alpha-$cuts

$$\tilde{f}(x)[\alpha] = \left[f(x)^l_\alpha, f(x)^u_\alpha\right], \tag{1}$$

where

$$\begin{cases} f(x)^l_\alpha = \min\left\{f(x|\theta)\,|\theta \in \tilde{\theta}[\alpha]\right\}, \\ f(x)^u_\alpha = \max\left\{f(x|\theta)\,|\theta \in \tilde{\theta}[\alpha]\right\}. \end{cases} \tag{2}$$

Now, we can obtain the probability of any arbitrary value in the interval $[c,d]$, with $c > 0$, based on relation (1). If we denote this probability by $\tilde{P}[c,d]$, then the $\alpha$-cuts of $\tilde{P}[c,d]$ are defined as (for all $0 \le \alpha \le 1$)

$$\tilde{P}[c,d][\alpha] = \left\{\int_c^d f(x|\theta)\,dx|\theta \in \tilde{\theta}[\alpha]\right\} = [p_1(\alpha), p_2(\alpha)], \tag{3}$$

where

$$\begin{cases} p_1(\alpha) = \min\left\{\int_c^d f(x|\theta)\,dx|\theta \in \tilde{\theta}[\alpha]\right\}, \\ p_2(\alpha) = \max\left\{\int_c^d f(x|\theta)\,dx|\theta \in \tilde{\theta}[\alpha]\right\}. \end{cases} \tag{4}$$

Then, using relations (3-4), we can obtain the survival, hazard rate and mean residual life functions as the important aging concepts. To this end, first we compute the $\alpha-$cuts of these indexes respectively as (for all $0 \le \alpha \le 1$)

$$\tilde{S}(x)[\alpha] = \left[\min\left\{\int_x^\infty f(t|\theta)\,dt|\theta \in \tilde{\theta}[\alpha]\right\}, \min\left\{\int_x^\infty f(t|\theta)\,dt|\theta \in \tilde{\theta}[\alpha]\right\}\right], \tag{5}$$

$$\begin{aligned} \tilde{h}(x)[\alpha] &= \lim_{\Delta x \to 0} \frac{\tilde{P}(x < X < x+\Delta x|X > x)[\alpha]}{\Delta x} \\ &= \{\lim_{\Delta x \to 0} \frac{S(x) - S(x+\Delta x)}{\Delta x S(x)}|\theta \in \tilde{\theta}[\alpha]\} \\ &= \{\frac{-S'(x)}{S(x)}|\theta \in \tilde{\theta}[\alpha]\} = \{\frac{f(x)}{S(x)}|\theta \in \tilde{\theta}[\alpha]\}, \end{aligned} \tag{6}$$

$$\begin{aligned}\tilde{m}(x)[\alpha] &= \left\{\int_0^\infty S(t|x)dt|\theta \in \tilde{\theta}[\alpha]\right\} \\ &= \left\{\int_0^\infty \frac{S(x+t)}{S(x)}dt|\theta \in \tilde{\theta}[\alpha]\right\} \\ &= \left\{\int_x^\infty \frac{S(u)}{S(x)}du|\theta \in \tilde{\theta}[\alpha]\right\}. \end{aligned} \tag{7}$$

Our purpose is to formulate these indexes based on fuzzy Exponential lifetime data. Let $X$ be an Exponential random variable with the following density function

$$f(x;\theta) = \theta e^{-\theta x}, \qquad x > 0, \quad \theta > 0.$$

It is easy to show that based on this density the survival, hazard rate and mean residual life functions are respectively obtain as follow

$$S(x;\theta) = e^{-\theta x}, \qquad h(x;\theta) = \theta, \qquad m(x;\theta) = \frac{1}{\theta}.$$

In order to model the ambiguity of the parameter $\theta$ in the underlying problem we use the triangular fuzzy number $\tilde{\theta} = (c;l,u)$. It is obvious that

$$\tilde{\theta}[\alpha] = \left[\theta_\alpha^l, \theta_\alpha^u\right] = \left[c + l(\alpha - 1), c - u(\alpha - 1)\right], \qquad \forall \alpha \in [0,1].$$

Now, using the relations (5-7), we will able to obtain the survival, hazard rate and mean residual life functions based on Buckley's approach.

First consider the relations (1) and (2). Then the $\alpha-$cuts of the density function are given as (for all $0 \leq \alpha \leq 1$)

$$\tilde{f}(x)[\alpha] = \left[f(x)_\alpha^l, f(x)_\alpha^u\right], \tag{8}$$

where

$$f(x)_\alpha^l = \begin{cases} \theta_\alpha^l e^{-\theta_\alpha^l x}, & \theta_\alpha^u \leq \frac{1}{x}, \\ \theta_\alpha^u e^{-\theta_\alpha^u x}, & \theta_\alpha^l > \frac{1}{x}, \\ \min\left\{\theta_\alpha^l e^{-\theta_\alpha^l x}, \theta_\alpha^u e^{-\theta_\alpha^u x}\right\}, & \theta_\alpha^l < \frac{1}{x} < \theta_\alpha^u, \end{cases}$$

and

$$f(x)_\alpha^u = \begin{cases} \theta_\alpha^u e^{-\theta_\alpha^u x}, & \theta_\alpha^u \leq \frac{1}{x}, \\ \theta_\alpha^l e^{-\theta_\alpha^l x}, & \theta_\alpha^l > \frac{1}{x}, \\ \frac{e^{-1}}{x}, & \theta_\alpha^l < \frac{1}{x} < \theta_\alpha^u. \end{cases}$$

In the case of survival function, since $S(x;\theta)$ is a decreasing function in terms of $\theta$, the $\alpha-$cuts of survival function are given by

$$\tilde{S}(x)[\alpha] = \left[\exp\left(-(c - u(\alpha-1))x\right), \exp\left(-(c + l(\alpha-1))x\right)\right]. \tag{9}$$

Similarly, the $\alpha-$cuts of the hazard rate and mean residual life functions are obtained as

$$\tilde{h}[\alpha] = \left[c + l(\alpha-1), c - u(\alpha-1)\right], \tag{10}$$

$$\tilde{m}(x)[\alpha] = \left[\frac{1}{c - u(\alpha-1)}, \frac{1}{c + l(\alpha-1)}\right], \tag{11}$$

respectively.

## 2.2 The Saeidi et al.'s approach

Saeidi et al. [19] introduced a fuzzy density function in a fuzzy environment where the parameter of the distribution replaced with a fuzzy number. In the following, we recall this approach briefly.

For every $\theta \in \Theta$, we consider a weighted function, say $\mu(\theta)$, for the probability density function $f(x|\theta)$ and define the desired FPDF, denoted by $f_a(x|\tilde{\theta})$.

The FPDF of the random variable $X$ is defined by

$$f_a(x|\tilde{\theta}) = \frac{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) f(x|\theta) d\theta d\alpha}{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) d\theta d\alpha}, \qquad \text{a} \in [0,1),$$

where $\mu(\theta)$ is the membership function of the fuzzy parameter $\tilde{\theta}$ and $\tilde{\theta}[\alpha]$ is the corresponding $\alpha-$cut.

It is easy to show that the above density function has the following properties

- $f_a(x|\tilde{\theta}) \geq 0$,
- $\int_{x \in S_X} f_a(x|\tilde{\theta}) dx = 1$.

It is clear that the large value of $a$ makes the proposed FPDF $f_a(x|\tilde{\theta})$ more concentrate on the classical density $f(x|\theta)$.

In the following, we formulate the reliability, hazard rate and mean residual life functions based on this approach, respectively.

$$S_a(x|\tilde{\theta}) = \int_x^\infty f_a(t|\tilde{\theta}) dt = \frac{\int_x^\infty \int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) f(t|\theta) d\theta d\alpha dt}{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) d\theta d\alpha}$$

$$= \frac{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) S(x|\theta) d\theta d\alpha}{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) d\theta d\alpha},$$

$$h_a(x|\tilde{\theta}) = \frac{f_a(x|\tilde{\theta})}{S_a(x|\tilde{\theta})} = \frac{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) f(x|\theta) d\theta d\alpha}{\int_x^\infty \int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) f(t|\theta) d\theta d\alpha dt}$$

$$= \frac{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) f(x|\theta) d\theta d\alpha}{\int_a^1 \int_{\theta \in \tilde{\theta}[\alpha]} \mu(\theta) S(x|\theta) d\theta d\alpha},$$

$$m_a(x|\tilde{\theta}) = \frac{\int_x^\infty S_a(t|\tilde{\theta}) dt}{S_a(x|\tilde{\theta})}.$$

Note that sometimes we have to solve these integrals numerically, since the closed form cannot be achieved.

For the Exponential random variable, we have:

$$f_a(x|\tilde{\theta}) = \frac{\left[\begin{array}{l}\left(\begin{array}{l}\exp(-cx)\exp(-ux)(6\exp(ux)-6\exp(aux)\\ -2cx\exp(aux)-2ux\exp(aux)+u^2x^2\exp(ux)\\ +2cx\exp(ux)-4ux\exp(ux)-au^2x^2\exp(ux)\\ +cu^2x^3\exp(ux)+2aux\exp(ux)+au^2x^2\exp(aux)\\ +4aux\exp(aux)-2cux^2\exp(ux)-a^2u^2x^2\exp(aux)\\ +acux^2\exp(ux)-acu^2x^3\exp(ux)+acux^2\exp(aux))\end{array}\right)\Big/(u^2x^4)\\ +\left(\begin{array}{l}\exp(-alx)\exp(-cx)(6\exp(lx)-6\exp(alx)\\ -l^2x^2\exp(alx)-2cx\exp(alx)-4lx\exp(alx)\\ +2cx\exp(lx)-2lx\exp(lx)-al^2x^2\exp(lx)\\ -2clx^2\exp(alx)+4alx\exp(lx)+a^2l^2x^2\exp(lx)\\ +al^2x^2\exp(alx)-cl^2x^3\exp(alx)+2alx\exp(alx)\\ +aclx^2\exp(lx)+aclx^2\exp(alx)+acl^2x^3\exp(alx))\end{array}\right)\Big/(l^2x^4)\end{array}\right]}{\left[(a-1)^2(a+2)(l+u)/6\right]}, \tag{12}$$

$$h_a(x|\tilde{\theta}) = \frac{-\left[\begin{array}{l}\left(\begin{array}{l}\exp(-cx)\exp(-ux)(6\exp(ux)-6\exp(aux)\\ -2cx\exp(aux)-2ux\exp(aux)+u^2x^2\exp(ux)\\ +2cx\exp(ux)-4ux\exp(ux)-au^2x^2\exp(ux)\\ +cu^2x^3\exp(ux)+2aux\exp(ux)+au^2x^2\exp(aux)\\ +4aux\exp(aux)-2cux^2\exp(ux)-a^2u^2x^2\exp(aux)\\ +acux^2\exp(ux)-acu^2x^3\exp(ux)+acux^2\exp(aux))\end{array}\right)\Big/(u^2x^4)\\ +\left(\begin{array}{l}\exp(-alx)\exp(-cx)(6\exp(lx)-6\exp(alx)\\ -l^2x^2\exp(alx)-2cx\exp(alx)-4lx\exp(alx)\\ +2cx\exp(lx)-2lx\exp(lx)-al^2x^2\exp(lx)\\ -2clx^2\exp(alx)+4alx\exp(lx)+a^2l^2x^2\exp(lx)\\ +al^2x^2\exp(alx)-cl^2x^3\exp(alx)+2alx\exp(alx)\\ +aclx^2\exp(lx)+aclx^2\exp(alx)+acl^2x^3\exp(alx))\end{array}\right)\Big/(l^2x^4)\end{array}\right]}{\left[\begin{array}{l}\left(\begin{array}{l}\exp(-alx)\exp(-cx)(2\exp(alx)-2\exp(lx))\\ -l\exp(-alx)\exp(-cx)(ax\exp(alx)\\ -2x\exp(alx)+ax\exp(lx))\end{array}\right)\Big/(l^2x^3)\\ +\left(\begin{array}{l}\exp(-cx-ux)(2\exp(aux)-2\exp(ux))\\ -u\exp(-cx-ux)(ax\exp(aux)\\ -2x\exp(ux)+ax\exp(ux))\end{array}\right)\Big/(u^2x^3)\\ +\left(\begin{array}{l}\exp(-alx)\exp(-cx)(x^2\exp(alx)\\ -ax^2\exp(alx))-(\exp(-cx-ux)(x^2\exp(ux)\\ -ax^2\exp(ux)))\end{array}\right)\Big/x^3\end{array}\right]}, \tag{13}$$

$$S_a(x|\tilde{\theta}) = \frac{-\left[\begin{array}{l}\left(\begin{array}{l}\exp(-alx)\exp(-cx)(2\exp(alx)-2\exp(lx))\\ -l\exp(-alx)\exp(-cx)(ax\exp(alx)\\ -2x\exp(alx)+ax\exp(lx))\end{array}\right)\Big/(l^2x^3)\\ +\left(\begin{array}{l}\exp(-cx-ux)(2\exp(aux)-2\exp(ux))\\ -u\exp(-cx-ux)(ax\exp(aux)\\ -2x\exp(ux)+ax\exp(ux))\end{array}\right)\Big/(u^2x^3)\\ +\left(\begin{array}{l}\exp(-alx)\exp(-cx)(x^2\exp(alx)\\ -ax^2\exp(alx))-(\exp(-cx-ux)(x^2\exp(ux)\\ -ax^2\exp(ux)))\end{array}\right)\Big/x^3\end{array}\right]}{\left[(a-1)^2(a+2)(l+u)/6\right]}, \tag{14}$$

$$m_a(x|\tilde{\theta}) = \frac{\displaystyle\int_x^{\infty}\left(\begin{array}{l}(2\exp(-ct)-2\exp((1-a)lt-ct))/(l^2t^3)\\ +\exp(-ct)(2-a-a\exp((1-a)lt))/(lt^2)\\ +(2\exp((a-1)ut-ct)-2\exp(-ct))/(u^2t^3)\\ +\exp(-ct)(2-a-a\exp((a-1)ut))/(ut^2)\end{array}\right)dt}{\left[\begin{array}{l}(2\exp(-cx)-2\exp((1-a)lx-cx))/(l^2x^3)\\ +\exp(-cx)(2-a-a\exp((1-a)lx))/(lx^2)\\ +(2\exp((a-1)ux-cx)-2\exp(-cx))/(u^2x^3)\\ +\exp(-cx)(2-a-a\exp((a-1)ux))/(ux^2)\end{array}\right]}. \tag{15}$$

### 2.3 The new density-based approach

Under the general assumptions which mentioned in the beginning of the section, we define the new density function for fuzzy random variables.

**Definition 1.** *Let X be a random variable with fuzzy parameter $\tilde{\theta}$. We define the density function for this random variable as follows*

$$f_\alpha(x|\tilde{\theta}) = \frac{\displaystyle\int_{\theta\in\tilde{\theta}[\alpha]}\mu(\theta)f(x|\theta)d\theta}{\displaystyle\int_{\theta\in\tilde{\theta}[\alpha]}\mu(\theta)d\theta}, \tag{16}$$

*where $\mu(\theta)$ is the membership function of $\tilde{\theta}$ and $\tilde{\theta}[\alpha]$ is its $\alpha$-cut.*

It is clear that $f_\alpha(x|\tilde{\theta})$ is a weighted average of $f(x|\theta)$ for the value of $\tilde{\theta}$ in the interval $\tilde{\theta}[\alpha]$ and has the following properties

- $f_\alpha(x|\tilde{\theta}) \geq 0$,
- $\displaystyle\int_{x\in S_X} f_\alpha(x|\tilde{\theta})dx = 1$.

In the following, we introduce the reliability function and some aging concepts based on this new definition of fuzzy density function.

According to relation (16), one can obtain the reliability, hazard rate and mean residual life functions respectively as

$$
\begin{aligned}
S_\alpha(x|\tilde{\theta}) &= \int_x^\infty f_\alpha(t|\tilde{\theta})dt = \frac{\int_x^\infty \int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta) f(t|\theta)d\theta dt}{\int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta)d\theta} \\
&= \frac{\int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta)S(x|\theta)d\theta}{\int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta)d\theta}, \\
h_\alpha(x|\tilde{\theta}) &= \frac{f_\alpha(x|\tilde{\theta})}{S_\alpha(x|\tilde{\theta})} = \frac{\int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta) f(x|\theta)d\theta}{\int_{\theta\in\tilde{\theta}[\alpha]} \mu(\theta)S(x|\theta)d\theta}, \\
m_\alpha(x|\tilde{\theta}) &= \frac{\int_x^\infty S_\alpha(t|\tilde{\theta})dt}{S_\alpha(x|\tilde{\theta})},
\end{aligned}
$$

where $f(x|\theta)$ and $S(x|\theta)$ are the classical density and survival functions, respectively. For the Exponential random variable, the above functions are as follow:

$$
h_\alpha(x|\tilde{\theta}) = \frac{-\left[\begin{array}{l}
((\exp(-cx)(c^2x^2+2cx+2))/x^3 \\
-(\exp(-x(c-l+\alpha l))(x^2(c-l+\alpha l)^2 \\
+2x(c-l+\alpha l)+2))/x^3 \\
+(c((\exp(-x(c-l+\alpha l))(x(c-l+\alpha l)+1))/x^2 \\
-(\exp(-cx)(cx+1))/x^2)))/l \\
+((\exp(-cx)(c^2x^2+2cx+2))/x^3 \\
-(\exp(-x(c+u-\alpha u))(x^2(c+u-\alpha u)^2 \\
+2x(c+u-\alpha u)+2))/x^3 \\
-(c((\exp(-cx)(cx+1))/x^2-(\exp(-x(c+u-\alpha u)) \\
(x(c+u-\alpha u)+1))/x^2)))/u \\
+(-(\exp(-x(c-l+\alpha l))(x(c-l+\alpha l)+1)) \\
+(\exp(-x(c+u-\alpha u))(x(c+u-\alpha u)+1)))/x^2
\end{array}\right]}{\left[\begin{array}{l}
\left(\exp(-cx-ux)\left(\begin{array}{c}\exp(\alpha ux)-\exp(ux) \\ +ux\exp(ux)-\alpha ux\exp(\alpha ux)\end{array}\right)\right)\Big/ux^2 \\
-\left(\exp(-\alpha lx)\exp(-cx)\left(\begin{array}{c}\exp(\alpha lx)-\exp(lx) \\ +lx\exp(\alpha lx)-\alpha lx\exp(lx)\end{array}\right)\right)\Big/lx^2
\end{array}\right]}, \tag{17}
$$

$$m_\alpha(x|\tilde{\theta}) = \frac{\int_x^\infty \exp(-ct)\begin{pmatrix} (\exp((\alpha-1)ut)-1)/(ut^2) \\ -(1-\exp((1-\alpha)lt))/(lt^2) \\ -\alpha(\exp((\alpha-1)ut) \\ -\exp(lt(1-\alpha)))/t \end{pmatrix} dt}{\exp(-cx)\begin{pmatrix} (\exp((\alpha-1)ux)-1)/(ux^2) \\ -(1-\exp((1-\alpha)lx))/(lx^2) \\ -\alpha(\exp((\alpha-1)ux) \\ -\exp(lx(1-\alpha)))/x \end{pmatrix}}, \tag{18}$$

$$f_\alpha(x|\tilde{\theta}) = \frac{\begin{bmatrix} ((\exp(-cx)(c^2x^2+2cx+2))/x^3 \\ -(\exp(-x(c-l+\alpha l))(x^2(c-l+\alpha l)^2 \\ +2x(c-l+\alpha l)+2))/x^3 \\ +(c((\exp(-x(c-l+\alpha l))(x(c-l+\alpha l)+1))/x^2 \\ -(\exp(-cx)(cx+1))/x^2)))/l \\ +((\exp(-cx)(c^2x^2+2cx+2))/x^3 \\ -(\exp(-x(c+u-\alpha u))(x^2(c+u-\alpha u)^2 \\ +2x(c+u-\alpha u)+2))/x^3 \\ -(c((\exp(-cx)(cx+1))/x^2-(\exp(-x(c+u-\alpha u)) \\ (x(c+u-\alpha u)+1))/x^2)))/u \\ +(-(\exp(-x(c-l+\alpha l))(x(c-l+\alpha l)+1)) \\ +(\exp(-x(c+u-\alpha u))(x(c+u-\alpha u)+1)))/x^2 \end{bmatrix}}{[(\alpha^2-1)(l+u)/2]}, \tag{19}$$

$$S_\alpha(x|\tilde{\theta}) = \frac{-\begin{bmatrix} \left(\exp(-cx-ux)\begin{pmatrix} \exp(\alpha ux)-\exp(ux) \\ +ux\exp(ux)-\alpha ux\exp(\alpha ux) \end{pmatrix}\right)\Big/ux^2 \\ -\left(\exp(-\alpha lx)\exp(-cx)\begin{pmatrix} \exp(\alpha lx)-\exp(lx) \\ +lx\exp(\alpha lx)-\alpha lx\exp(lx) \end{pmatrix}\right)\Big/lx^2 \end{bmatrix}}{[(\alpha^2-1)(l+u)/2]}. \tag{20}$$

## 3 Practical Example

In this section, we suppose that the components of the underlying system have an Exponential distribution. Then, we compare the aging concepts based on the various definitions presented in the previous section using numerical example.

**Example 1.** Consider two systems for transporting water from place A to place B using equally spaced pressure pumps. Suppose that systems 1 and 2 uses five pressure pumps of type I and each pump can transport water to the next pump. Thus, if anyone of the pumps fails, then the system cannot transport

Table 1: The fuzzy numbers for componnents of underlying systems in Example 1.

| System | Component1 | Component2 | Component3 | Component4 | Component5 |
|---|---|---|---|---|---|
| System1 | $\tilde{\lambda}_1 = (1, 0.6, 0.7)$ | $\tilde{\lambda}_2 = (1, 0.5, 0.6)$ | $\tilde{\lambda}_3 = (1, 0.7, 0.6)$ | $\tilde{\lambda}_4 = (1, 0.7, 0.6)$ | $\tilde{\lambda}_5 = (1, 0.5, 0.5)$ |
| System2 | $\tilde{\lambda}_1 = (1, 0.6, 1.4)$ | $\tilde{\lambda}_2 = (1, 0.7, 1.6)$ | $\tilde{\lambda}_3 = (1, 0.5, 1.5)$ | $\tilde{\lambda}_4 = (1, 0.7, 1.2)$ | $\tilde{\lambda}_5 = (1, 0.5, 1.3)$ |

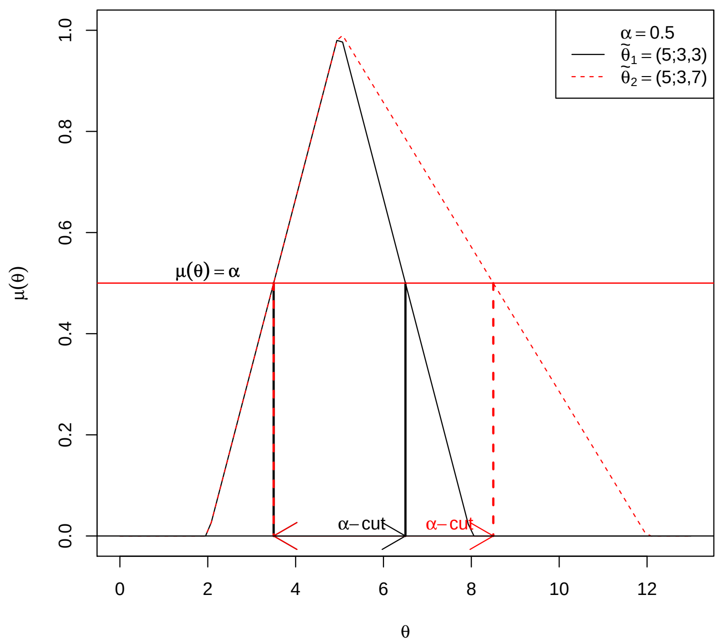


Figure 1: Membeship functions of two triangular fuzzy numbers in Example 1.

the water. Systems 1 and 2 are series systems. Systems 1 and 2 were put on the test, and the parameters value for each component of systems were recorded and presented as fuzzy triangular numbers reported in Table 1.

Since each component of underlying systems has an Exponential distribution, we can conclude that the lifetimes of systems1 and 2 have an Exponential distribution with triangular fuzzy parameters $\tilde{\theta}_1 = (5, 3, 3)$ and $\tilde{\theta}_2 = (5, 3, 7)$, respectively. Hence, we consider two triangular fuzzy numbers $\tilde{\theta}_1$ and $\tilde{\theta}_2$ to model the vagueness in each system and depict density, survival, hazard rate and mean residual life functions.

Figure 1 shows the membership functions of two fuzzy parameters. The density functions of lifetimes for systems 1 and 2 are presented in Figure 2 based on different approaches. It is clear that as the value of $\alpha$ increases, Buckley's density function (8) tends to the classical (crisp) density function. Since $\tilde{\theta}_1$ is a symmetric fuzzy number, there is no difference between the approaches based on $f_\alpha(t|\tilde{\theta})$ and $f_a(t|\tilde{\theta})$. Thus, we can said that when the ambiguity of fuzzy parameters is symmetric, the performance of two new approaches is the same and they are better than the Buckley's approach in modeling the vagueness of underlying systems.

The result of density functions for $\tilde{\theta}_2$ is given in the Figure 3. In this case, the approach based

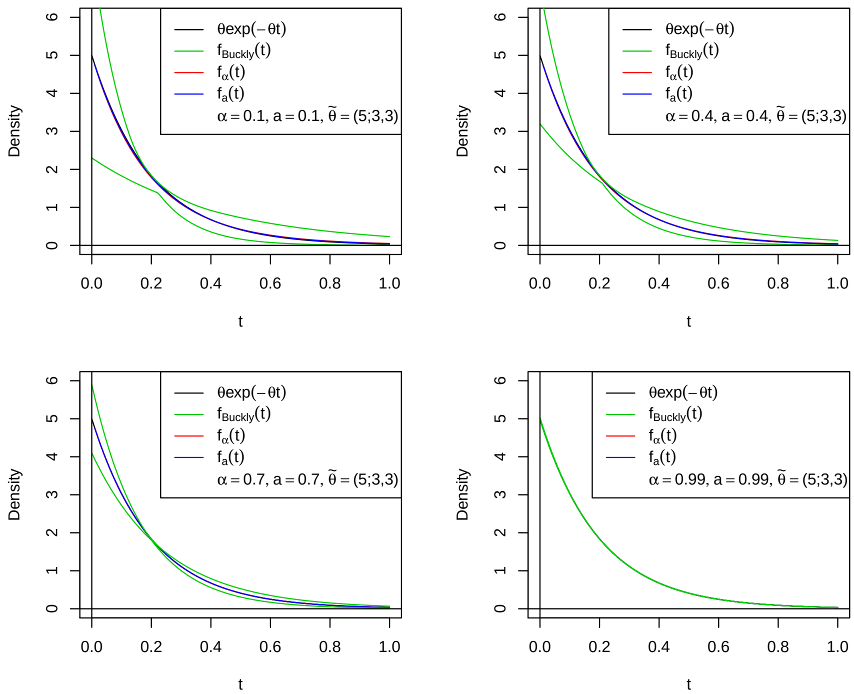


Figure 2: Density function for lifetime of system 1 based on different approaches.

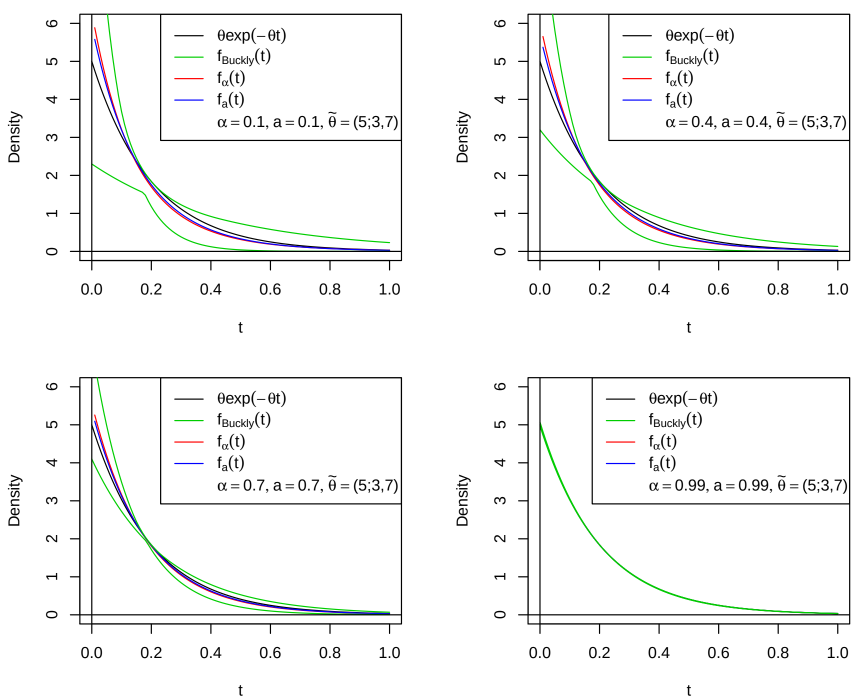


Figure 3: Density function for lifetime of system 2 based on different approaches.

on $f_a(t|\tilde{\theta})$ perform more appropriately than two other methods in the sense of closeness to the classical (crisp) mode. In other words, even though the data that occurring in the system are taken as real numbers, the proposed method in this paper can still handle it as we just described above. In fact, the fuzzy approach in the definition of the density function is a kind of extension of the conventional density function. Therefore, the fuzzy density function proposed in this paper has no loss of generality.

Figures 4 and 5 depict the survival functions of different approaches based on $\tilde{\theta}_1$ and $\tilde{\theta}_2$ respectively. According to obtained results, we can conclude that there are no noticeable differences between new approaches (approaches based on $f_\alpha(t|\tilde{\theta})$ and $f_a(t|\tilde{\theta})$) and the classic method when $\tilde{\theta}$ is a symmetric fuzzy number. Also, the performance of two new approaches is close to each other based on $\tilde{\theta}_2$. The performance of the Buckley's approach and the approach based on $f_\alpha(t|\tilde{\theta})$ depends on the size of $\alpha$. It seems that the former is more sensitive to the value of $\alpha$ than the latter.

The hazard rate functions of systems 1 and 2 are shown in Figures 6 and 7, respectively. Based on Figure 6, we can say that when the value of $\alpha$ ($a$ for the approach based on $f_a(t|\tilde{\theta})$) is small, the hazard rate function of the two new methods decreases in time. Also, there is a slight difference between two new approaches. Meanwhile, Figure 7 shows that all three methods tend to the classic mode when $\alpha$ ($a$ for the approach based on $f_a(t|\tilde{\theta})$) tends to 1. It is obvious that the best approach is the new method based on density $f_a(t|\tilde{\theta})$, Because it is the closest to the classic model for each value of $a$.

Finally, the performance of mean residual life functions for different approaches was presented in Figures 8 and 9 for underlying systems. It is seen that the approach based on density $f_\alpha(t|\tilde{\theta})$ fails to perform well in both situations of fuzzy environment. Thus, we can conclude that the best approach is the one based on the density $f_a(t|\tilde{\theta})$.

## 4 Conclusion

In this paper, we provide a new definition for fuzzy density function of fuzzy random variables. The performance of reliability function and aging concepts for fuzzy Exponential distribution were considered carefully and then compared with other existing approaches. An advantage of new method is that all functions are single-valued while the Buckley's one is interval-valued. We apply our method to the situation of Exponential lifetime random variable with fuzzy parameters $\tilde{\theta}_1 = (5;3,3)$ and $\tilde{\theta}_2 = (5;3,7)$. After considering the graphs of density, survival, hazard rate and mean residual life functions, we observe that the performance of the approach based on density $f_a(t|\tilde{\theta})$ is better than two other methods. Thus we can say that the best method is the one based on density $f_a(t|\tilde{\theta})$.

## References

[1] I.M. Aliev, Z. Kara, *Fuzzy system reliability analysis using time dependent fuzzy set*, Control. Cybern. **33** (2004) 653–662.

[2] J.J. Buckley, *Fuzzy Probability and Statistics*, Springer, 2006.

[3] K.Y. Cai, C.Y. Wen, M.L. Zhang, *Fuzzy states as a basis for a theory of fuzzy reliability*, Microelectron. Reliab. **33** (1993) 2253–2263.

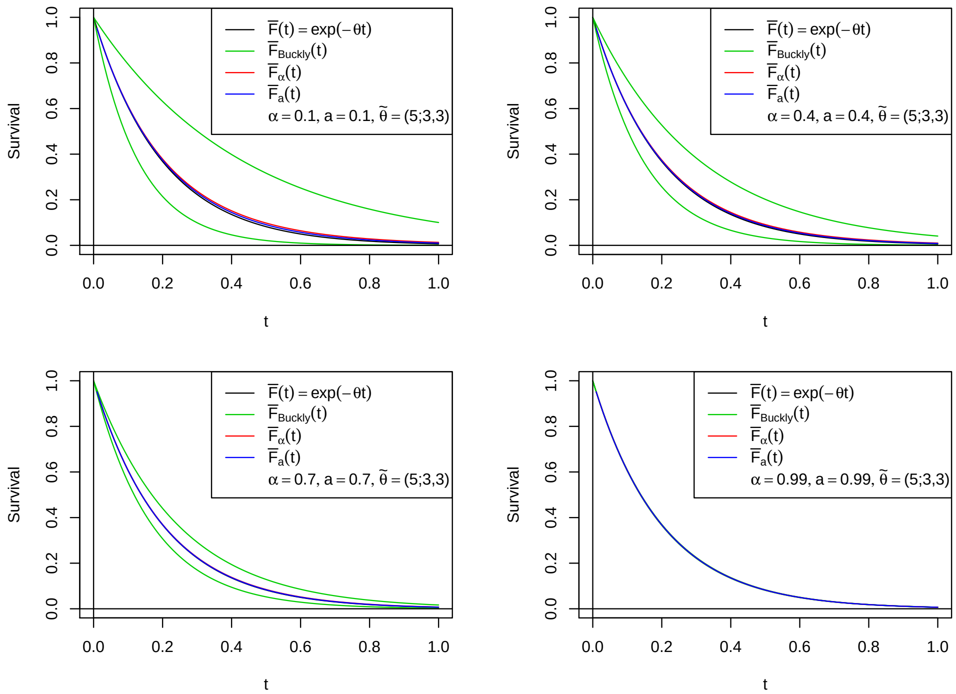


Figure 4: Survival functions for lifetime of system 1 based on different approaches.

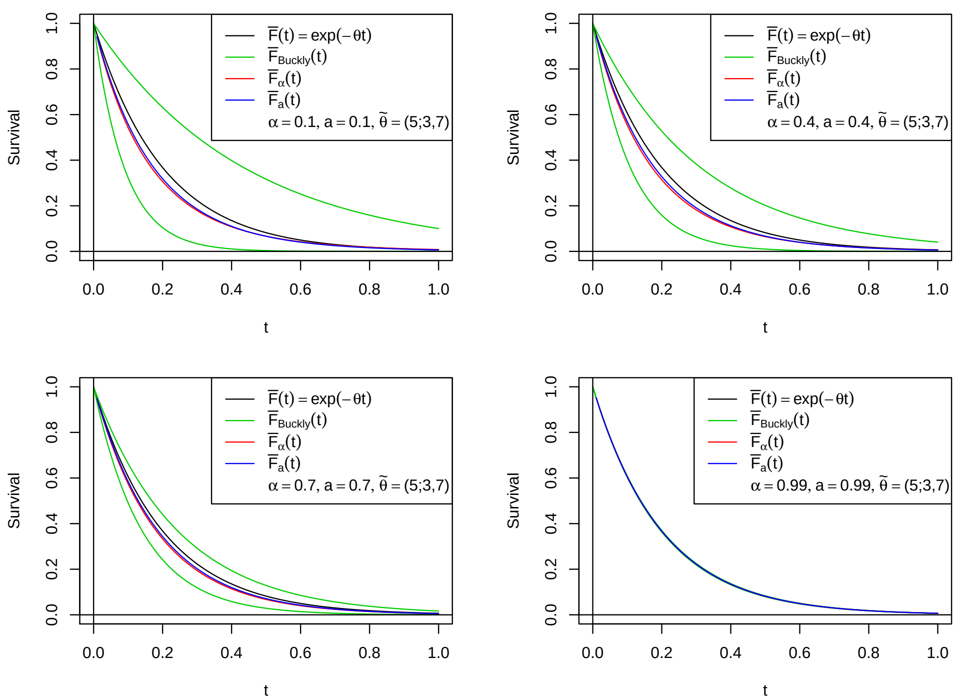


Figure 5: Survival functions for lifetime of system 2 based on different approaches.

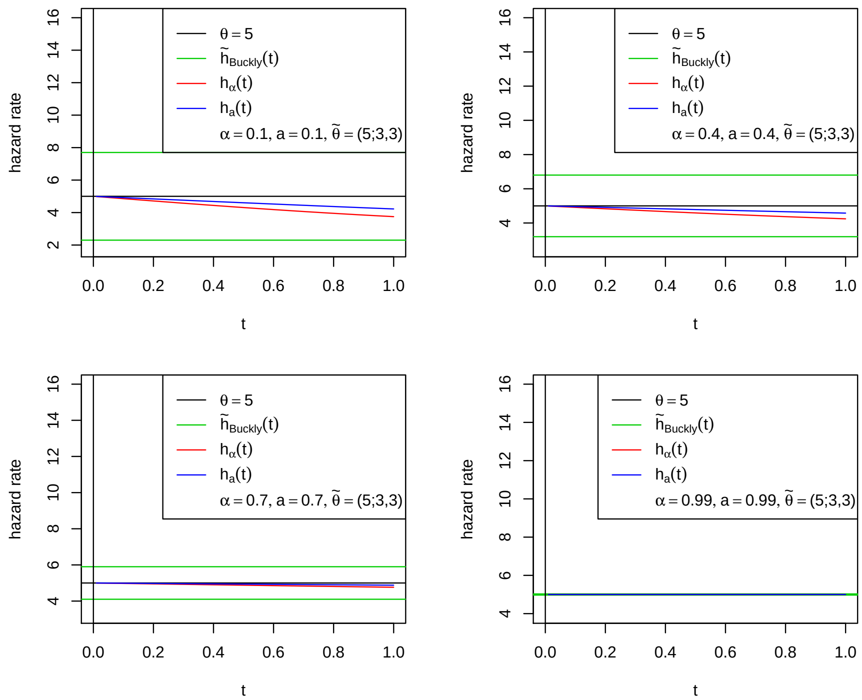


Figure 6: Hazard rate functions for lifetime of system 1 based on different approaches.

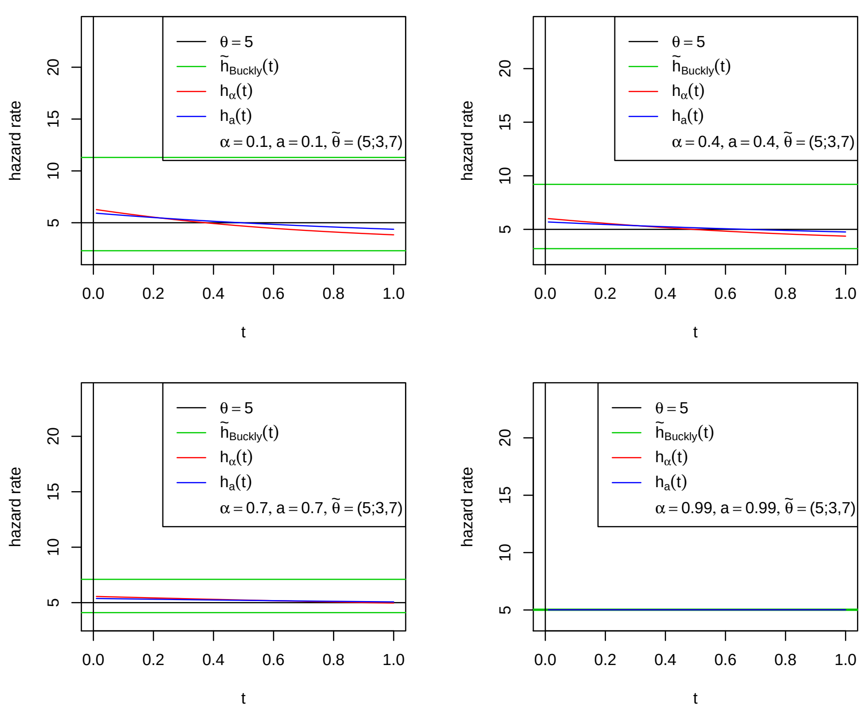


Figure 7: Hazard rate functions for lifetime of system 2 based on different approaches.

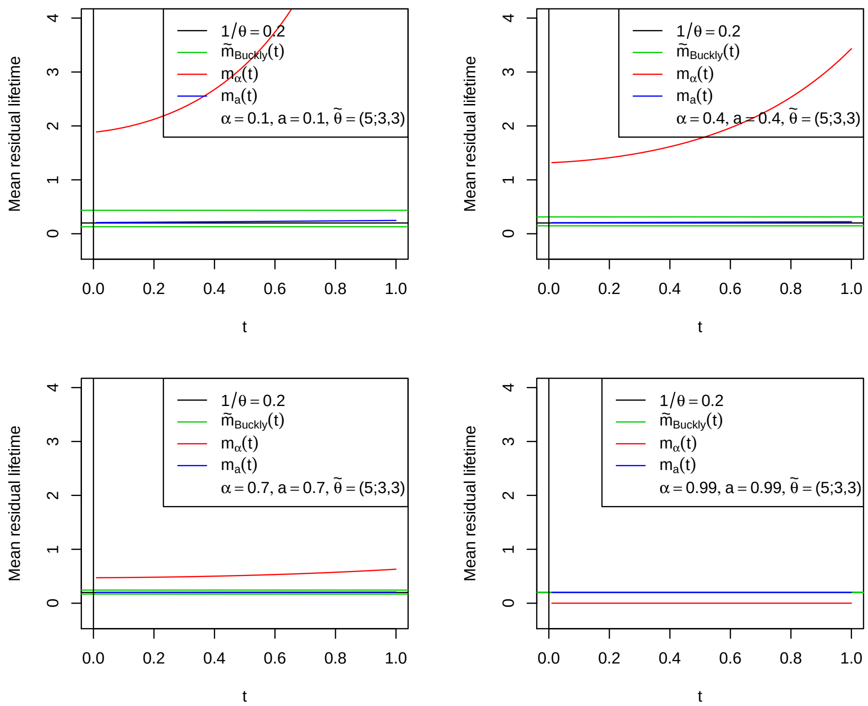


Figure 8: Mean residual life functions for lifetime of system 1 based on different approaches.

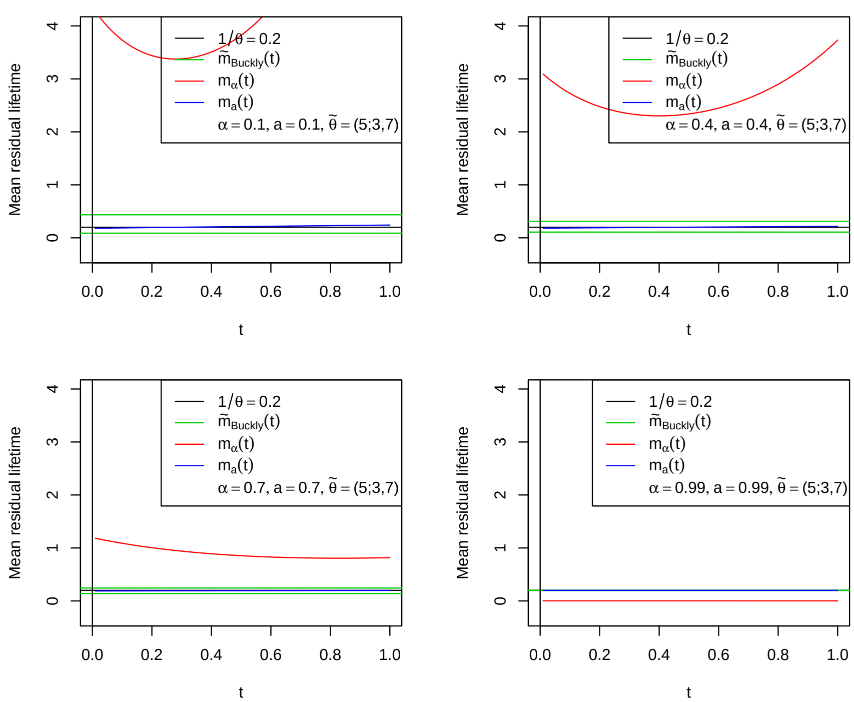


Figure 9: Mean residual life functions for lifetime of system 2 based on different approaches.

[4] S.M. Chen, *Analyzing fuzzy system reliability using vague set theory*, Int. J. Appl. Sci. Eng. **1** (2003) 82–88.

[5] S.M. Chen, *Fuzzy system reliability analysis using fuzzy number arithmetic operations*, Fuzzy Sets Syst. **64** (1994) 31–38.

[6] C.H. Cheng, D.L. Mon, *Fuzzy system reliability analysis by interval of confidence*, Fuzzy Sets Syst. **56** (1993) 29–35.

[7] M.A. El-Damcese, D.A. Ramadan, *Analyzing system reliability using fuzzy mixture generalized linear failure rate distribution*, Am. J. Math. Stat. **5** (2015) 43–51.

[8] A. Fakharzadeh Jahromi, Z. Ebrahimi Mimand, *A new outlier detection method for high dimensional fuzzy databases based on LOF*, J. Math. Model. **6** (2018) 123–136.

[9] R. Guo, R.Q. Zhao, X. Li, *Reliability analysis based on scalar fuzzy variables*, Econ. Qual. Cont. **22** (2007) 55–70.

[10] H.Z. Huang, M.J. Zuo, Z.Q. Sun, *Bayesian reliability analysis for fuzzy lifetime data*, Fuzzy Sets Syst. **157** (2006) 1674–1686.

[11] P. Iqbal, P.S. Uduman, *Reliability analysis of paper plant using boolean function with fuzzy logic technique*, Int. J. Appl. Eng. Res. **11** (2016) 573–577.

[12] E.B. Jamkhaneh, *An evaluation of the systems reliability using fuzzy lifetime distribution*, J. Appl. Math. **7** (2011) 73–80.

[13] E.B. Jamkhaneh, *Analyzing system reliability using fuzzy weibull lifetime distribution*, Int. J. Appl. Oper. Res. **4** (2014) 93–102.

[14] E.B. Jamkhaneh, A. Nozari, *Fuzzy system reliability analysis based on confidence interval*, Adv. Mat. Res. **433** (2012) 4908–4914.

[15] J. Kacprzyk, T. Onisawa, *Reliability and Safety Analyses under Fuzziness*, Springer, 1995.

[16] Z. Karpisek, P. Stepanek and P. Jurak, *Weibull fuzzy probability distribution for reliability of concrete structures*, Engi. Mech. **17** (2010) 363–372.

[17] D.L. Mon, C.H. Cheng, *Fuzzy system reliability analysis for components with different membership functions*, Fuzzy Sets Syst. **64** (1994) 145–157.

[18] A. Pak, G.A. Parham, M. Saraj, *Reliability estimation in rayleigh distribution based on fuzzy lifetime data*, Int. J. Syst. Assur. Eng. Manag. **5** (2014) 487–494.

[19] A.R. Saeidi, M.G. Akbari, M. Doostparast, *Hypotheses testing with the two-parameter pareto distribution on the basis of records in fuzzy environment*, Kybernetika, **50** (2014) 744–757.

[20] M. Shafiq, R. Viertl, *On the estimation of parameters, survival functions, and hazard rates based on fuzzy life time data,* Commun. Stat. Theory Methods, **46** (2017) 5035–5055.

[21] S.M. Taheri, *Trends in fuzzy statistics*, Aust. J. Stat. **32** (2003) 239–257.

[22] S.M. Taheri, R. Zarei, *Bayesian system reliability assessment under the vague environment*, Appl. Soft Comput. **11** (2011) 1614–1622.

[23] L.V. Utkin, *Knowledge based fuzzy reliability assessment*, Microelectron. Reliab. **34** (1994) 863–874.

[24] J.S. Yao, J.S. Su, T.S. Shih, *Fuzzy system reliability analysis using triangular fuzzy numbers based on statistical data*, J. Inf. Sci. Eng. **24** (2008) 1521–1535.

[25] R. Zarei, M.G. Akbari, J. Chachi, *Modeling autoregressive fuzzy time series data based on semi-parametric methods*, Soft Comput. **24** (2020) 7295–7304.

[26] R. Zarei, M. Amini, A.H. Rezaei Roknabadi, *Fuzzy stochastic ordering for C-Fuzzy random variables and its applications*, Soft Comput. **19** (2015) 179–188.

[27] R. Zarei, M. Amini, A.H. Rezaei Roknabadi, *Some fuzzy stochastic orderings for fuzzy random variables*, Fuzzy Optim. Decis, **11** (2012) 209–225.

[28] H.J. Zimmermann, *Fuzzy set theory*, Wiley Interdisciplinary Reviews: Computational Statistics, **2** (2010) 317–332.