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Bound-preserving interpolation using quadratic splines

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Abstract. In this work, we study a data visualization problem which is classified in the field of shape-preserving interpolation. When a function is known to be bounded, then it is natural to expect its interpolant to adhere boundedness. Two spline-based techniques are proposed to handle this kind of problem. The proposed methods use quadratic splines as basis and involve solving a linear programming or a mixed integer linear programming problem which gives C^1 interpolants. An energy minimization technique is employed to gain the optimal smooth solution. The reliability and applicability of the proposed techniques have been illustrated through examples.

Keywords: Shape preserving interpolation, boundedness, quadratic splines, linear programming.
AMS Subject Classification 2020: 65D05, 65D07, 65D17.

1 Introduction

Suppose a data set, say $\{(x_i, f_i)\}_{i=1}^n$, is given. These data may be arising from a physical phenomenon or be a sampling of a function. One wishes to gain more information about the original phenomenon by constructing a suitable approximant for the given data. This is the main goal of many areas of research such as data visualization, geometric design and approximation theory. Once we find a suitable approximant, then it is possible to understand the original physical phenomenon or function with more insight and detail. There are various approaches to handle this kind of problem and among them "Interpolation" is the most popular one. Interpolation seeks for an approximant which passes through the data points. Hence, it is assumed here that the data are sufficiently accurate to warrant interpolation.

The original function and thus the data set at hand may have some important features and properties such as monotonicity, positivity and convexity. It is very natural to require the interpolant to preserve these features. Therefore, we are generally faced with a "Shape-preserving interpolation" problem. This

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is a topic which arises in various fields of science such as computer graphics, geometric modeling, numerical analysis, image processing, etc. It gives an insight and guide to understand some physical phenomenon pertaining to the data which one would otherwise only have partial information about. It is an effective way of communication as it helps to reflect the numeric data to a quickly understandable pictorial display.

Various shape-preserving interpolation methods have been proposed and every approach has its own advantages and drawbacks. More insight into the subject could be traced in the literature [5–7], and some recent advances could be found in [1, 3, 4, 10] and in the references therein. The shape-preserving interpolation techniques may be classified according to

- the base functions used to represent the closed form of the interpolant,
- the feature which is preserved by the technique,
- the degree of smoothness,
- whether there is a knot-insertion or not.

In this paper, we focus on a feature which has not been broadly studied in the literature: the "*boundedness*". Suppose we have a data set generated from a sampling of a bounded function, this happens, for example, when the data reflect the probability or efficiency of a process. To be precise, suppose a data set $\{(x_i, f_i)\}_{i=1}^{n}$, generated from a bounded function with m and M as lower and upper bounds respectively, is given. When we try to approximate this data set by interpolation techniques, we need to ensure that the interpolating curve adheres to these known properties. The bounded interpolation problem seeks for a function g which interpolates the data set and is also bounded into $[m, M]$ [9].

Actually, any monotonicity-preserving approach [5] could be employed to solve a bounded interpolation problem. Especially when we know that the original function or phenomenon has a monotone behavior. However, our data set could be a monotone sampling of an oscillatory phenomenon (function). In these cases, by imposing monotonicity, the interpolant would be bounded into $[\min f_i, \max f_i]$, whilst the original bound of the function is $[m, M]$. A bounded interpolation technique must allow the approximant to attain values outside the range $[\min f_i, \max f_i]$.

Generally, a bounded interpolation problem could be considered as a generalization of a positivity-preserving interpolation. Once we have a positivity-preserving interpolation technique, one can apply it to the data sets $\{f_i - m\}$ and $\{M - f_i\}$, simultaneously.

There are a number of positivity-preserving techniques that could be used to handle the bounded case as well. Here we propose a method based on quadratic splines. Quadratic visualization has the drawback that the maximum smoothness supported by these functions is C^1. However, it also has advantages; one is that it is computationally cheap and simple. It should be noticed that, although smoothness is one of the very important requirements for a pleasing visual display, the base functions providing C^2 continuity, for example cubic splines, usually result in semi-linear approximations [13].

The structure of the paper is as follows. In Section 2, quadratic splines with unknown derivative parameters is used to provide a C^1 bounded interpolant. Section 3 studies a weighted quadratic spline method where the weights are suitably chosen to yield boundedness and maximum smoothness. An energy minimization criterion is presented in Section 4 to gain approximations with higher order of smoothness. Section 5 is devoted to examples and finally Section 6 concludes by summarizing the highlights.

2 Quadratic Hermite spline for bounded interpolation

Let $[a,b]$ be an interval containing a mesh $\{x_i\}_{i=1}^n$ such that $x_1 < x_2 < \ldots < x_n$ and let $\{f_i\}_{i=1}^n$ and $\{m_i\}_{i=1}^n$ be real numbers. For each sub-interval $[x_i, x_{i+1}]$, $i = 1, \ldots, n-1$, we define $h_i = x_{i+1} - x_i$ and $\delta_i = \frac{f_{i+1} - f_i}{h_i}$. The following is called the Hermite interpolation problem (HIP):

Problem 1. *(HIP) Find a quadratic spline S with the fewest number of breakpoints such that*

$$S(x_i) := f_i, \qquad S'(x_i) = m_i, \qquad i = 1, \ldots, n.$$

Schumaker has given a solution to this problem [12]. Schumaker's quadratic spline is defined based on the m_i-values, the construction is done separately to each sub-interval $[x_i, x_{i+1}], i = 1, \ldots, n-1$.

- If $m_i + m_{i+1} = 2\delta_i$, then restriction of S on $[x_i, x_{i+1}]$ is a parabola

$$S(x) \equiv S_i(x) := f_i + m_i(x - x_i) + \frac{m_{i+1} - m_i}{2h_i}(x - x_i)^2. \tag{1}$$

- If $m_i + m_{i+1} \neq 2\delta_i$, then S on $[x_i, x_{i+1}]$ is a quadratic spline with one breakpoint ξ_i, which can be freely chosen in the interval (x_i, x_{i+1})

$$S(x) \equiv S_i(x) = \begin{cases} f_i + m_i(x - x_i) + \dfrac{\mu_i - m_i}{2t_i h_i}(x - x_i)^2, & x_i \le x < \xi_i, \\ d_i + \mu_i(x - \xi_i) + \dfrac{m_{i+1} - \mu_i}{2(1 - t_i)h_i}(x - \xi_i)^2, & \xi_i \le x \le x_{i+1}, \end{cases} \tag{2}$$

  where

$$t_i = \frac{\xi_i - x_i}{h_i}, \quad \mu_i = 2\delta_i - t_i m_i - (1 - t_i)m_{i+1}, \tag{3}$$

$$d_i = (1 - t_i)f_i + t_i f_{i+1} + \frac{1}{2}t_i(1 - t_i)h_i(m_i - m_{i+1}). \tag{4}$$

  Here $0 < t_i < 1$ and the values d_i and μ_i are defined so that C^1 continuity is satisfied on $[x_i, x_{i+1}]$, that is, we have $d_i = S(\xi_i)$ and $\mu_i = S'(\xi_i)$.

Schumaker's quadratic spline, defined by equations (1) and (2), provides a solution to the HIP [12].

Now we state the positive Hermite interpolation problem (PHIP), which was studied by Lahtinen [8]. It should be noted here that, in this paper, the word "positive" is used as a synonym of non-negative, as used by Lahtinen in [8].

Problem 2. *(PHIP) For a positive set of data $\{f_i\}_{i=1}^n$ and known derivative values $\{m_i\}_{i=1}^n$, find a solution to the corresponding HIP which is positive on $[a,b]$.*

In [8], Lahtinen has given necessary and sufficient conditions, based on $\{m_i\}_{i=1}^n$ values, for the corresponding interpolant to be positive. Here we study the bounded Hermite interpolation problem (BHIP), which can be stated as follows.

Problem 3. *(BHIP) For a data set $\{f_i\}_{i=1}^n$, generated from a bounded function, find a quadratic Hermite interpolant spline which satisfies the same bounds.*

Here the derivative values are not given and this is a difference between our problem and the cases studied by Schumaker and Lahtinen. In both HIP and PHIP, the methodology is based on the given $\{m_i\}_{i=1}^n$ values. We assume that the $\{f_i\}_{i=1}^n$ values are a sample of a bounded function, say $f_i \in [m, M]$. Without loss of generality, we can assume that $m = 0$ and $M = 1$. Therefore, we wish to find a quadratic interpolant $S(x)$ on $[a,b]$ with $0 \le S(x) \le 1$.

Schumaker's quadratic spline, defined by Eqs. (1) and (2), provides with a solution to the BHIP. The basic idea can be summarized as follows: we impose conditions on the unknown derivative values m_i to make the resulting interpolant satisfy boundedness. Hence, the boundedness conditions as well as C^1-continuity conditions are put into a mixed integer linear programming (MILP) problem to obtain suitable m_i values.

The following theorem states the fundamental result.

Theorem 1. *The quadratic spline S, defined by Eqs. (1) and (2), provides with a solution to the BHIP when the following conditions are verified in each sub-interval* $[x_i, x_{i+1}]$*:*

- *If* $m_i + m_{i+1} = 2\delta_i$*, then the* m_i*-values must satisfy*

$$\frac{-2}{h_i}(f_i + \sqrt{f_i f_{i+1}}) \le m_i \le \frac{2}{h_i}(1 - f_i + \sqrt{(1-f_i)(1-f_{i+1})}), \tag{5}$$

- *If* $m_i + m_{i+1} \ne 2\delta_i$*, then*

$$\frac{2f_{i+1} - 2}{h_i} \le m_{i+1} - m_i \le \frac{2f_{i+1}}{h_i}, \tag{6}$$

$$\frac{-2f_i}{h_i} \le m_i \le \frac{2 - 2f_i}{h_i}, \tag{7}$$

$$\frac{2f_{i+1} - 2}{h_i} \le m_{i+1} \le \frac{2f_{i+1}}{h_i}. \tag{8}$$

The idea of the proof is based on the following lemma from [11].

Lemma 1. *A quadratic polynomial S is positive on* $[x_i, x_{i+1}]$ *if and only if the following conditions hold:*

$$S(x_i) \ge 0, \quad S(x_{i+1}) \ge 0, \quad 2S(x_{i+1}) - (x_{i+1} - x_i)S'(x_{i+1}) \ge -2\sqrt{S(x_i)S(x_{i+1})}.$$

This lemma is a straightforward corollary of *Proposition 3* in [11], and it can be deduced by a simple change of variable.

In order to prove Theorem 1, we distinguish two cases and impose the desired bounding conditions for each of the cases separately. In each case, Lemma 1 is employed to impose positivity conditions on the functions S and $1 - S$, simultaneously. A detailed proof is presented in the Appendix.

2.1 Algorithm Description

Here we explain the technique which is used to obtain suitable m_i values according to the boundedness conditions. For each sub-interval $[x_i, x_{i+1}]$ a variable $Z_i = |m_i + m_{i+1} - 2\delta_i|$ is defined. For *Case 1*, we have $Z_i = 0$ and for *Case 2*, Z_i would be a positive value. To be able to switch between these two cases (which impose different constraints on m_i), we define two binary variables p_i and q_i with the constraint

$p_i + q_i = 1$. With this notation, it is possible to put constraints of *Case 1* and *Case 2* in a single MILP problem. Whenever the objective implies *Case 1*, then the program must ignore the constraints of *Case 2*. To do so, we need to define some auxiliary constants. We assume M and K be sufficiently large values and ε be a relatively small positive constant. Now the constraints of the MILP problem can be summarized as follows.

$$\begin{cases} Z_i \geq m_i + m_{i+1} - 2\delta_i, \\ Z_i \geq -(m_i + m_{i+1} - 2\delta_i), \\ \varepsilon p_i \leq Z_i \leq M p_i, \end{cases}$$

$$-Kq_i - \frac{2}{h_i}(f_i + \sqrt{f_i f_{i+1}})p_i \leq m_i \leq \frac{2}{h_i}(1 - f_i + \sqrt{(1-f_i)(1-f_{i+1})})p_i + Kq_i,$$

$$\begin{cases} -Kp_i + \frac{2f_{i+1}-2}{h_i} q_i \leq m_{i+1} - m_i \leq \frac{2f_{i+1}}{h_i} q_i + Kp_i, \\ -Kp_i - \frac{2f_i}{h_i} q_i \leq m_i \leq \frac{2-2f_i}{h_i} q_i + Kp_i, \\ -Kp_i + \frac{2f_{i+1}-2}{h_i} q_i \leq m_{i+1} \leq \frac{2f_{i+1}}{h_i} q_i + Kp_i. \end{cases}$$

Remark 1. *In implementing the corresponding MILP problem, we have confined M and K to be greater than 50 and set $\varepsilon < 0.1$.*

3 Weighted quadratic spline method

The quadratic Hermite spline method, presented in the previous section, uses unknown derivative values to adjust the shape of the curve to attain boundedness. In this section we use a quadratic spline function equipped with control parameters to solve the bound-preserving interpolation problem. Once the shape parameters are suitably chosen, the method provides with a C^1 bounded interpolant. For each interval $[x_i, x_{i+1}]$, we define

$$\tau(x) \equiv \tau_i(x) := l_i(x) + \alpha_i(x - x_i)(x - x_{i+1}), \qquad i = 1, \ldots, n, \tag{9}$$

where $l_i(x) = \dfrac{f_{i+1} - f_i}{h_i}(x - x_i) + f_i$ is the linear interpolant and α_i is a control parameter (weight), hence the name "weighted quadratic spline".

The C^1 continuity condition, $\tau_i(x_{i+1}) = \tau_{i+1}(x_{i+1})$, implies the following linear system of equations

$$h_i \alpha_i + h_{i+1}\alpha_{i+1} = f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}], \qquad i = 1, \ldots, n-2, \tag{10}$$

where $f[x_i, x_{i+1}]$ is the Newton's divided difference. In order to have a bounded quadratic spline, i.e. $0 \leq \tau \leq 1$, we impose this condition to each sub-interval to have $0 \leq \tau_i \leq 1$. Now we can employ Lemma 1, which in turn forces the parameters α_i, $i = 1, \ldots, n$, to satisfy

$$\frac{-(f_i + f_{i+1} + 2\sqrt{f_i f_{i+1}})}{h_i^2} \leq \alpha_i \leq \frac{2 - f_i - f_{i+1} + 2\sqrt{(1-f_i)(1-f_{i+1})}}{h_i^2}. \tag{11}$$

Therefore, one should solve a constrained system of linear equations which results in a linear programming (LP) problem subject to conditions (10) and (11) above. One can state these constraints as sufficient conditions for having a solution to the BHIP.

Theorem 2. *The weighted quadratic spline τ, defined by Eq. (9), provides with a solution to the BHIP when conditions (10) and (11) are verified.*

4 Energy minimization technique

So far, we have proposed two methods to solve the bounded interpolation problem using quadratic splines. Both methods require solving an LP or an MILP problem. However, in both cases, when there is a solution it may not be unique. So it arises the matter of choosing the "*optimal solution*" in some sense. A possibility is proposed in Burmeister et al. [2], where they have presented an energy minimization method which results in an interpolating spline with minimum energy. It is generally based on minimizing the curvature of the spline which is represented by

$$E = \int_{x=x_1}^{x=x_n} \frac{S''^2(x)}{(1+[f'^2(x)])^{\frac{5}{2}}} dx. \tag{12}$$

Wolberg and Alfy [13] have used different, but closely related, quantities to express the energy of a spline. They introduce a discrete energy measure

$$E_D = \sum_{k=2}^{n-1} (S''(x_k^-) - S''(x_k^+))^2, \tag{13}$$

which is based on second derivative discontinuities. Then it has simplified to be linear with the first derivatives so that an LP procedure can be applied. The simplification is done by using the absolute values of the discontinuities:

$$\tilde{E}_D = \sum_{k=2}^{n-1} |S''(x_k^-) - S''(x_k^+)|. \tag{14}$$

Slack variables, s_k, are defined to be the absolute value of the discontinuity for each x_k. Using inequality constraints

$$S''(x_k^-) - S''(x_k^+) \leq s_k, \; -[S''(x_k^-) - S''(x_k^+] \leq s_k, \tag{15}$$

the discrete energy, $\tilde{E}_D$, can be written as

$$\tilde{E}_D = \sum_{k=2}^{n-1} s_k. \tag{16}$$

The same techniques is applied here and we try to reach bound-preserving interpolant splines with minimum energy. In both methods, introduced in Sections 2 and 3, we put $\tilde{E}_D$ as an objective function and minimize it subject to the following constraints:

- For the quadratic Hermite spline method (QHSM), proposed in Section 2:
  - Absolute value constraints: Eq. (15),
  - Boundedness constraints: Eqs. (6), (7) and (8),

  (of course we requires an MILP problem to handle this case).
- For the weighted quadratic spline method (WQSP), presented in Section 3:
  - Absolute value constraints: Eq. (15),
  - C^1 continuity constraints: Eq. (10),

– Boundedness constraints: Eq. (11).

The solution to these problems lead to a vector of values F^* in each case:

$$F^* = \begin{cases} (m_0, \dots, m_n, s_1, \dots, s_{n-1})^t, & \text{for QHSM,} \\ (\alpha_1, \dots, \alpha_{n-1}, s_1, \dots, s_{n-1})^t, & \text{for WQSM,} \end{cases}$$

which identifies a solution to the BHIP.

5 Numerical tests

In this section, we implement the proposed methods for numerical examples. To be able to compare the results, we consider data sets which are samples of known bounded functions. The error is calculated by maximum norm through a uniform partitioning with 10 points at each subinterval.

Example 1. Consider the data set in Table 1, which is a uniform sampling of the function

$$f(x) = \frac{11e^{\frac{-x}{4}}}{1+e^{1-\frac{x}{4}}} - \frac{11e^{\frac{-x}{4}}}{1+e^{1-\frac{x^2}{16}}} + \frac{1}{2}.$$

Table 1: Data set of a sampling.

x_i	1	2	3	4	5	6	7	8	9
f_i	0.8377	0.8784	0.7363	0.5000	0.2641	0.1200	0.1024	0.1702	0.2614

The QHSM is applied to this data set and then an energy minimization method (EMM) is used to get interpolants with minimum curvature. Figure 1 represents the original function (blue), the QHSM solution (green) and the EMM result (red). It is seen that the maximum error of QHSM solution is 0.0024, where the corresponding error for the EMM is 0.0020.

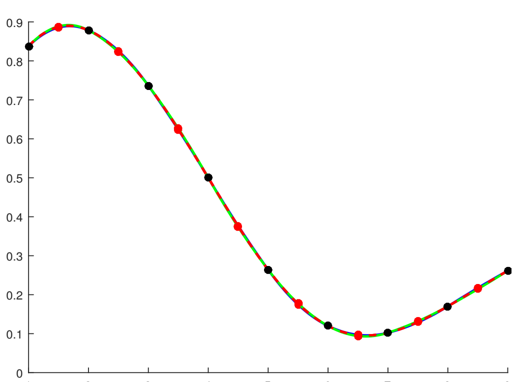

Figure 1: QHSM (green) and EMM (red) results of Example 1.

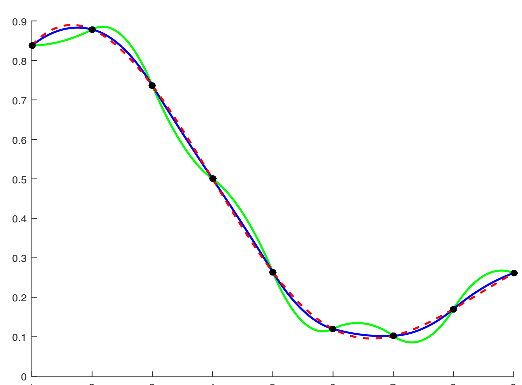

Figure 2: WQSM (green) and EMM (red) results of Example 1.

The WQSM is also applicable to Table 1 and the results are represented in Figure 2. Here the maximum error of the WQSM solution is 0.0387, while the corresponding EMM error is 0.0096. The QHSM ensures a better accuracy than the WQSM. However the latter has a simpler implementation.

Example 2. Table 2 presents data sampled from the function $f(t) = .0077t^3 - .1154t^2 + .4846t$.

Table 2: Data set from Example 2.

x_i	1	2	3	4	5	6	7	8	9
f_i	0.3769	0.5692	0.6231	0.5848	0.5005	0.4164	0.3787	0.4336	0.6273

The QHSM solution and the corresponding EMM result as well as the original curve are depicted in Figure 3. The maximum error in QHSM solution is 0.0860, while with an energy minimization technique it reduces to 0.0044. This example clearly shows the effect of EMM that results in a curve with minimum oscillations. Figure 4 illustrates the results of the WQSM.

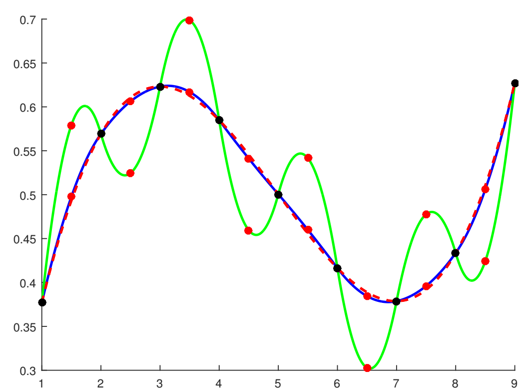

Figure 3: QHSM (green) and EMM (red) results of Example 2.

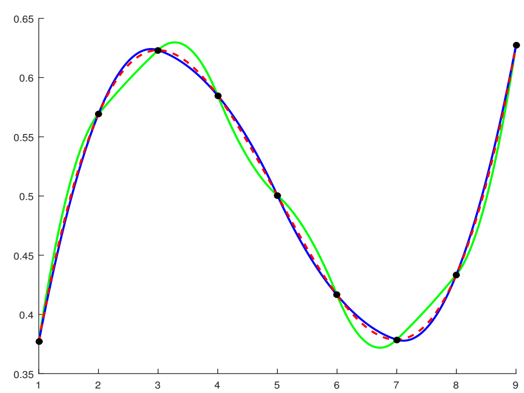

Figure 4: WQSM (green) and EMM (red) results of Example 2.

As it is seen in the solution curves, unlike Example 1, here the WQSM results in a curve with fewer oscillations compared to the curve obtained by QHSM. The maximum error is 0.0129, while with EMM it reduces to 0.0033. So in this example the WQSM is more promising than QHSM.

The two reported examples are evidences of the fact that it is difficult to establish the superiority of one method over another. These examples could be considered as a sample of similar numerical experiments. In some cases the QHSM proves to be superior, while in some other the WQSM gives more accurate approximations. However, in each method when the EMM is employed one observes notable improvement in the approximations.

6 Conclusions

The bounded interpolation problem may be handed by monotonicity preserving techniques piece-wisely. However, thereby the resulting spline would be bounded by the extreme values of the original data and not by the bounding values of the original function. Therefore, in order to gain a really bounding interpolating curve one may consider it as two simultaneous positivity-preserving problems. Employing the positivity preserving conditions and using quadratic splines, we have proposed two techniques which result in linear programming problems. Once we have the solution candidates we apply energy minimization techniques to gain a curve with minimum curvature.

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A Appendix: Proof of Theorem 1

The quadratic spline, S, defined in Eqs. (1) and (2), must satisfy the bounding constraints $0 \le S \le 1$. We get advantage of Lemma 1 to impose positivity conditions on S and $1-S$, in each case.

A.1 Case 1: $m_i + m_{i+1} = 2\delta_i$

According to Lemma 1, in order to have $S(x) \ge 0$ on $[x_i, x_{i+1}]$, the f_i-values must satisfy the following inequalities:

$$f_i \ge 0, \quad f_{i+1} \ge 0, \quad 2f_{i+1} - h_i m_{i+1} \ge -2\sqrt{f_i f_{i+1}}.$$

We assumed that $0 \le f_i \le 1$, so the first two inequalities are satisfied. The third one reads

$$m_{i+1} \le \frac{1}{h_i}(2f_{i+1} + 2\sqrt{f_i f_{i+1}}),$$

which, in turn, by substituting $m_{i+1} = 2\delta_i - m_i$ leads to

$$m_i \ge \frac{-2}{h_i}(f_i + \sqrt{f_i f_{i+1}}). \tag{17}$$

On the other hand, for condition $S(x) \le 1$ to hold, we can impose positivity conditions on $1-S(x)$. Employing Lemma 1, we come to the restriction

$$m_i \le \frac{2}{h_i}(1 - f_i + \sqrt{(1-f_i)(1-f_{i+1})}). \tag{18}$$

Inequalities (17) and (18) must hold simultaneously so this completes the proof for Case 1.

A.2 Case 2: $m_i + m_{i+1} \ne 2\delta_i$

In this case there is a breakpoint, ξ_i, in the definition of the quadratic spline in each sub-interval. We denote

$$\begin{aligned} P_1 = P_{i,1} :=& \quad f_i + m_i(x - x_i) + \frac{\mu_i - m_i}{2t_i h_i}(x - x_i)^2, && x_i \le x < \xi_i, \\ P_2 = P_{i,2} :=& \quad d_i + \mu_i(x - \xi_i) + \frac{m_{i+1} - \mu_i}{2(1-t_i)h_i}(x - \xi_i)^2, && \xi_i \le x \le x_{i+1}. \end{aligned}$$

We observe that the d_i, which is defined in Eq. (4), has an equivalent representation

$$d_i = f_i + \frac{1}{2}(m_i + \mu_i)(\xi_i - x_i). \tag{19}$$

First, we need to verify the following results which will be needed in our forthcoming calculations.

$$m_{i+1}-m_i \le \frac{2f_{i+1}}{h_i} \Longrightarrow d_i \ge 0, \tag{20}$$

$$m_{i+1}-m_i \ge \frac{2(f_{i+1}-1)}{h_i} \Longrightarrow d_i \le 1. \tag{21}$$

For (20), we see that

$$m_i+\mu_i = m_i+2\delta_i-\frac{\xi_i-x_i}{h_i}m_i-\frac{x_{i+1}-\xi_i}{h_i}m_{i+1} = 2\delta_i+\frac{x_{i+1}-\xi_i}{h_i}(m_i-m_{i+1}),$$

which results in

$$\begin{aligned}\frac{1}{2}(m_i+\mu_i)(\xi_i-x_i) &= \{\delta_i+\frac{x_{i+1}-\xi_i}{2h_i}(m_i-m_{i+1})\}(\xi_i-x_i) = \{f_{i+1}-f_i+\frac{x_{i+1}-\xi_i}{2}(m_i-m_{i+1})\}\frac{\xi_i-x_i}{h_i}\\ &\ge \{f_{i+1}-f_i+\frac{x_{i+1}-\xi_i}{2}\frac{-2f_{i+1}}{h_i}\}\frac{\xi_i-x_i}{h_i} \ge -f_i\frac{\xi_i-x_i}{h_i},\end{aligned}$$

where we have used the assumption $m_{i+1}-m_i \le \frac{2f_{i+1}}{h_i}$, in the first inequality. Now we can conclude

$$d_i = f_i+\frac{1}{2}(m_i+\mu_i)(\xi_i-x_i) \ge f_i+\big(-f_i\frac{\xi_i-x_i}{h_i}\big) = f_i\frac{x_{i+1}-\xi_i}{h_i} \ge 0.$$

In order to prove (21), one observes that

$$m_i-m_{i+1} \le \frac{2(1-f_{i+1})}{h_i} \le \frac{2(1-f_{i+1})}{x_{i+1}-\xi_i},$$

which gives

$$(m_i-m_{i+1})(x_{i+1}-\xi_i) \le 2(1-f_{i+1}).$$

Therefore we can conclude

$$(m_i-m_{i+1})\frac{x_{i+1}-\xi_i}{h_i} \le \frac{2-2f_i+2f_i-2f_{i+1}}{h_i} = \frac{2-2f_i}{h_i}-2\delta_i \le \frac{2-2f_i}{\xi_i-x_i}-2\delta_i,$$

so it would lead to

$$2\delta_i+\frac{x_{i+1}-\xi_i}{h_i}(m_i-m_{i+1}) \le \frac{2-2f_i}{\xi_i-x_i},$$

which, in view of the definition of μ_i (Eq. (3)), results in

$$m_i+\mu_i \le \tfrac{2-2f_i}{\xi_i-x_i} \Longrightarrow (m_i+\mu_i)(\xi_i-x_i) \le 2(1-f_i) \Longrightarrow d_i = f_i+\tfrac{1}{2}(m_i+\mu_i)(\xi_i-x_i) \le 1.$$

This completes the reasoning for inequalities (20) and (21).

To have constraints $0 \le S \le 1$, one can employ Lemma 1 to impose the positivity conditions on P_1, $1-P_1$, P_2 and $1-P_2$, simultaneously.

We will study each case separately.

- $P_1 \geq 0$

  According to Lemma 1, a sufficient condition reads

P1-1: $f_i \geq 0$,

P1-2: $d_i \geq 0$,

P1-3: $2d_i - (\xi_i - x_i)\mu_i \geq -2\sqrt{f_i d_i}$.

The first condition is satisfied according to assumptions of the problem. For the second one ($d_i \geq 0$), we stated a sufficient condition in (20). For the third one we assume that $m_i \geq \frac{-2f_i}{h_i}$, in this way we have

$$m_i \geq \frac{-2f_i}{h_i} \geq \frac{-2f_i}{\xi_i - x_i},$$

leading to

$$2(f_i + \frac{1}{2}m_i(\xi_i - x_i) + \frac{1}{2}\mu_i(\xi_i - x_i)) - \mu_i(\xi_i - x_i) \geq 0,$$

which, from the definition of d_i, results in

$$2d_i - \mu_i(\xi_i - x_i) \geq 0 \geq -2\sqrt{f_i d_i},$$

and this is the desired condition *P1-3*.

So far we have verified that a sufficient condition for P_1 to be positive is to have

$$m_i \geq \frac{-2f_i}{h_i} \quad \text{and} \quad m_{i+1} - m_i \leq \frac{2f_{i+1}}{h_i},$$

which the latter appears in one side of Eq. (6) and the former in Eq. (7).

- $P_1 \leq 1$

  Again, employing Lemma 1 we come to the following sufficient conditions to have $1 - P_1 \geq 0$:

P1-4: $1 - f_i \geq 0 \Longrightarrow f_i \leq 1$,

P1-5: $1 - P_1(\xi_i) \geq 0 \Longrightarrow d_i \leq 1$,

P1-6: $2(1 - d_i) - (\xi_i - x_i)(-\mu_i) \geq -2\sqrt{(1 - f_i)(1 - d_i)}$.

Here, the first condition is a part of our general assumptions. For the second one we have already given a sufficient condition in (21). A sufficient condition for *P1-6* to hold would be $m_i \leq \frac{2(1-f_i)}{h_i}$, in this way we have

$$m_i \leq \frac{2(1 - f_i)}{h_i} \leq \frac{2(1 - f_i)}{\xi_i - x_i} \Longrightarrow m_i(\xi_i - x_i) \leq 2 - 2f_i,$$

which gives

$$2 - 2(f_i + \frac{1}{2}(m_i + \mu_i)(\xi_i - x_i)) + (\xi_i - x_i)\mu_i \geq 0.$$

Now, from the definition of d_i, one concludes that

$$2-2d_i+(\xi_i-x_i)\mu_i \geq 0 \geq -2\sqrt{(1-f_i)(1-d_i)}.$$

This completes our task.

We see that the two constraints $m_{i+1}-m_i \geq \frac{2(f_{i+1}-1)}{h_i}$ and $m_i \leq \frac{2(1-f_i)}{h_i}$ appear in Eqs. (6) and (7).

In this way, it is verified that the conditions in Eqs. (6) and (7) are sufficient conditions for $0 \leq P_1 \leq 1$ to hold.

- $P_2 \geq 0$

  Sufficient conditions for positivity of P_2 would be

P2-1: $d_i \geq 0$,

P2-2: $f_{i+1} \geq 0$,

P2-3: $2f_{i+1}-(x_{i+1}-\xi_i)m_{i+1} \geq -2\sqrt{f_{i+1}d_i}$.

  The first two constraints are satisfied according to the previous cases. For the third one to hold, we impose the condition $m_{i+1} \leq \frac{2f_{i+1}}{h_i}$, which provides a sufficient condition for *P2-3* to be valid.

  This constraint apears in the right hand side of Eq. (8).

- $P_2 \leq 1$

  Employing Lemma 1 on $1-P_2$, gives us the following set of sufficient conditions

P2-4: $d_i \leq 1$,

P2-5: $f_{i+1} \leq 1$,

P2-6: $2(1-f_{i+1})+(x_{i+1}-\xi_i)m_{i+1} \geq -2\sqrt{(1-f_{i+1})(1-d_i)}$.

  Here we just need to satisfy *P2-6*, where we see that $m_{i+1} \geq \frac{2f_{i+1}-2}{h_i}$ would provide a sufficient condition. This constraint appears in Eq. (8).

In view of the four mentioned cases, we see that the expressions Eqs. (6), (7) and (8) provide with sufficient conditions for the quadratic spline of the form (2) to satisfy $0 \leq S_i \leq 1$. This completes the proof.