Journal of Mathematical Modeling
Vol. 10, No. 2, 2022. pp. 349-365. Research Article

JMM

Numerical solutions of system of two-dimensional Volterra integral equations via operational matrices of hybrid functions

Khosrow Maleknejad*, **Maryam Shahabi**

School of Mathematics, Iran University of Science & Technology, Narmak, Tehran 16844, Iran
Email(s): maleknejad@iust.ac.ir, shahabi_te@yahoo.com

Abstract. The main target of this paper is to solve a system of two-dimensional Volterra integral equations (2-DVIEs). Operational Matrices of two-dimensional hybrid of block-pulse functions and Legendre polynomials are applied to reduce these systems of integral equations to a system of algebraic equations. The main benefit of these basic functions is their efficiency in dealing with non-sufficiently smooth functions. An error bound is provided and some examples are prepared to verify the applicability of the offered numerical technique.

Keywords: System of two-dimensional Volterra integral equations, two-dimensional hybrid functions, oroduct Operational matrix, operational matrices of integration.
AMS Subject Classification 2010: 45D05, 45F05, 65R20.

1 Introduction

Much attention has been devoted to Volterra integral equations considering the large number of applications in many natural phenomena including superfluidity, elasticity, plasticity, fluid mechanic, fracture mechanic theory, electromagnetic theory, molecular physics, heat transfer problems, etc [6, 8, 19, 27, 31]. System of integral equations also arise in various fields of science, engineering and physics such as describing the formation of liver zones and chemically reacting flowing systems and some problems in electromagnetic and elastic waves [1, 7, 16].

In the past decade, several numerical techniques and basic functions have been devoted to estimate the solution of two-dimensional integral equations of the second kind such as block-pulse functions method [4, 25], Franklin wavelet galerkin method [29], Legendre operational matrix method [28], hybrid functions method [22, 23], discrete Galerkin method [2], expansion method [9], expansion-iterative method [14], Bernoulli operational matrix method [5], Bernstein operational matrix method [30], Haar

*Corresponding author.

Received: 24 June 2021/ Revised: 8 October 2021 / Accepted: 12 October 2021
DOI: 10.22124/JMM.2021.19962.1727

wavelet [3] and rationalized haar functions method [11], shifted Jacobi operational matrix method [32] and Laguerre wavelet method [20]. There are fewer pieces of research focusing on computational methods for approximating the solution of system of two-dimensional integral equation. Maleknejad and Hoseingholipour have used two-dimensional Legendre wavelets, along with collocation method and numerical integration to solve system of two-dimensional integral equations numerically [21]. In [12] the modified Newton method has been implemented for the numerical solution of a general system of nonlinear Volterra type integral equations. Karimi et al. have considered Legendre wavelets for the numerical solution of a system of two-dimensional Volterra integral equations [17]. The authors of [26] have extended modified moving least squares methods for solving two-dimensional linear and nonlinear systems of integral equations. Adaptive iterative regularization schemes have been used to solve system of two-dimensional integral equations in [13]. Zaky et al. [33] have used a spectral collocation method for solving a general class of nonlinear systems of multi-dimensional integral equations.

Here, we plan to find the solution of linear system of 2-DVIEs of the form

$$\Lambda U(x,t) = G(x,t) + \int_0^t \int_0^x K(x,t,y,z)U(y,z)\,\mathrm{d}y\,\mathrm{d}z, \qquad (x,t) \in \Omega, \tag{1}$$

where $U(x,t) = (u_1(x,t), u_2(x,t))^T$, $G(x,t) = (g_1(x,t), g_2(x,t))^T$, and Λ and $K(x,t,y,z)$ are matrices of the form

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} \\ \lambda_{2,1} & \lambda_{2,2} \end{bmatrix}, \quad K(x,t,y,z) = \begin{bmatrix} k_{1,1}(x,t,y,z) & k_{1,2}(x,t,y,z) \\ k_{2,1}(x,t,y,z) & k_{2,2}(x,t,y,z) \end{bmatrix},$$

such that $u_1(x,t)$ and $u_2(x,t)$ are unknown functions defined on $\Omega = [0,1) \times [0,1)$, the functions $k_{l,s}(x,t,y,z)$ and $g_l(x,t)$ for $l,s = 1,2$ are given functions and $\lambda_{l,s}\,(l,s = 1,2)$ are constant values and $\det\Lambda \neq 0$. The existence and uniqueness results for the solution of systems of 2-DVIEs have been discussed in [33].

In this paper, we extend the method introduced in [24] to solve syatem of 2-DVIEs of the form (1). Two-dimensional hybrid of block-pulse functions and Legendre polynomials and their operational matrix of integration and product operational matrix and product operational vector are utilized to solve the considered problem. The most important aspect of the hybrid functions method is that the order of block-pulse functions and Legendre polynomials are adjustable, which helps us to change the number of subintervals and the degree of Legendre polynomials in each subinterval until getting the best possible result.

The remaining parts of this paper are structured as follows. In Section 2, two-dimensional hybrid of block-pulse functions and Legendre polynomials and their operational matrix of integration together with the product operational matrix and the product operational vector are introduced. Operational matrices are utilized to convert the system of integral equations into a system of algebraic equations in Section 3. Section 4 is devoted to the description of the error analysis of the proposed method. Some computed examples are prepared in Section 5 to test the accuracy of the hybrid functions method, followed by a conclusion in Section 6.

2 Basic concepts

2.1 Two-dimensional hybrid functions

Two-dimensional hybrid of block-pulse functions and Legendre polynomials on the interval Ω, introduced by $\phi_{i_1 j_1 i_2 j_2}$, are defined as

$$\phi_{i_1 j_1 i_2 j_2}(x,t) = \begin{cases} L_{j_1}(2Nx-2i_1+1)L_{j_2}(2Nt-2i_2+1), & (x,t)\in[\frac{i_1-1}{N},\frac{i_1}{N})\times[\frac{i_2-1}{N},\frac{i_2}{N}), \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where $i_1, i_2 = 1,2,\ldots,N$, $j_1, j_2 = 0,1,\ldots,M-1$, and N and M are positive integer numbers. Here L_{j_1} and L_{j_2} are the Legendre polynomials of order j_1 and j_2 respectively, which are defined on the interval $[-1,1]$ and can be determined recursively as

$$\begin{aligned} &L_0(x) = 1, \\ &L_1(x) = x, \\ &L_{m+1}(x) = \frac{2m+1}{m+1}xL_m(x) - \frac{m}{m+1}L_{m-1}(x), \quad m = 1,2,\ldots, -1 \leq x \leq 1. \end{aligned}$$

From Eq. (2) we have

$$\int_0^1\int_0^1 \phi_{i_1 j_1 i_2 j_2}(x,t)\phi_{n_1 m_1 n_2 m_2}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \begin{cases} \int_{\frac{i_2-1}{N}}^{\frac{i_2}{N}}\int_{\frac{i_1-1}{N}}^{\frac{i_1}{N}} L_{j_1}L_{j_2}L_{m_1}L_{m_2}\,\mathrm{d}x\,\mathrm{d}t, & i_1 = n_1, i_2 = n_2, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $L_{j_1} = L_{j_1}(2Nx-2i_1+1)$, $L_{j_2} = L_{j_2}(2Nt-2i_2+1)$, $L_{m_1} = L_{m_1}(2Nx-2i_1+1)$ and $L_{m_2} = L_{m_2}(2Nt-2i_2+1)$. Then, we transform the integrals over $[\frac{i_1-1}{N},\frac{i_1}{N}]$ and $[\frac{i_2-1}{N},\frac{i_2}{N}]$ to the integral over $[-1,1]$ by introducing the following transformations

$$\begin{aligned} \gamma &= 2Nx-2i_1+1, \qquad x\in[\frac{i_1-1}{N},\frac{i_1}{N}], \\ \eta &= 2Nt-2i_2+1, \qquad t\in[\frac{i_2-1}{N},\frac{i_2}{N}]. \end{aligned}$$

So, Eq. (3) converts to

$$\begin{aligned} &\int_0^1\int_0^1 \phi_{i_1 j_1 i_2 j_2}(x,t)\phi_{n_1 m_1 n_2 m_2}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \begin{cases} \left(\frac{1}{2N}\int_{-1}^1 L_{j_1}(\gamma)L_{m_1}(\gamma)\mathrm{d}\gamma\right)\left(\frac{1}{2N}\int_{-1}^1 L_{j_2}(\eta)L_{m_2}(\eta)\mathrm{d}\eta\right), & i_1 = n_1, i_2 = n_2, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Now, using orthogonality properties of Legendre polynomials from [10], we arrive at

$$\begin{aligned} &\int_0^1\int_0^1 \phi_{i_1 j_1 i_2 j_2}(x,t)\phi_{n_1 m_1 n_2 m_2}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \begin{cases} \left(\frac{1}{2N}\times\frac{2}{2j_1+1}\right)\left(\frac{1}{2N}\times\frac{2}{2j_2+1}\right), & i_1 = n_1, i_2 = n_2, j_1 = m_1, j_2 = m_2, \\ 0, & \text{otherwise}. \end{cases} \end{aligned}$$

Thus, two-dimensional hybrid of block-pulse functions and Legendre polynomials are orthogonal to each other with the orthogonality clause

$$\int_0^1 \int_0^1 \phi_{i_1 j_1 i_2 j_2}(x,t)\phi_{n_1 m_1 n_2 m_2}(x,t)\,\mathrm{d}x\,\mathrm{d}t = \begin{cases} \frac{1}{N^2(2j_1+1)(2j_2+1)}, & i_1 = n_1, i_2 = n_2, j_1 = m_1, j_2 = m_2, \\ 0, & \text{otherwise.} \end{cases}$$

2.2 Function approximation

Now consider the space $X = L^2(\Omega)$ with the norm

$$\|g\|_2 =< g,g >^{\frac{1}{2}}= (\int_0^1 \int_0^1 |g(x,t)|^2\,\mathrm{d}t\mathrm{d}x)^{\frac{1}{2}},$$

where $\langle .,. \rangle$ denotes the inner product and let

$$X_{N,M} = span\{\phi_{1010}(x,t), \phi_{1011}(x,t), \dots, \phi_{101(M-1)}(x,t), \phi_{1020}(x,t), \dots, \phi_{102(M-1)}(x,t), \dots, \\ \phi_{N(M-1)N0}(x,t), \dots, \phi_{N(M-1)N(M-1)}(x,t)\}.$$

Since $X_{N,M}$ is a finite dimensional and closed subspace of the Hilbert Space X, for every $g \in X$ there exists a unique best approximation $g_{N,M} \in X_{N,M}$, such that

$$\| g - g_{N,M} \|_2 = \inf_{h \in X_{N,M}} \| g - h \|_2 .$$

A proof of the above result can be found in [18]. Moreover we have

$$g(x,t) \approx g_{N,M}(x,t) = \sum_{i_1=1}^{N} \sum_{j_1=0}^{M-1} \sum_{i_2=1}^{N} \sum_{j_2=0}^{M-1} g_{i_1 j_1 i_2 j_2} \phi_{i_1 j_1 i_2 j_2}(x,t) = G^T \Phi(x,t), \tag{4}$$

where

$$G = [g_{1010}, g_{1011}, \dots, g_{101(M-1)}, g_{1020}, \dots, g_{102(M-1)}, \dots, g_{N(M-1)N0}, \dots, g_{N(M-1)N(M-1)}]^T, \tag{5}$$

$$\Phi(x,t) = [\phi_{1010}(x,t), \phi_{1011}(x,t), \dots, \phi_{101(M-1)}(x,t), \phi_{1020}(x,t), \dots, \phi_{102(M-1)}(x,t), \dots, \\ \phi_{N(M-1)N0}(x,t), \dots, \phi_{N(M-1)N(M-1)}(x,t)]^T, \tag{6}$$

and the coefficients are uniquely obtained by

$$g_{i_1 j_1 i_2 j_2} = \frac{\langle g, \phi_{i_1 j_1 i_2 j_2} \rangle}{\langle \phi_{i_1 j_1 i_2 j_2}, \phi_{i_1 j_1 i_2 j_2} \rangle}.$$

In a similar way any function $k(x,t,y,z)$ in $L^2(\Omega \times \Omega)$ can be expanded in terms of two-dimensional hybrid of block-pulse functions and Legendre polynomials as

$$k(x,t,y,z) \approx k_{N,M}(x,t,y,z) = \Phi^T(x,t) K \Phi(y,z), \tag{7}$$

where K is an $N^2M^2 \times N^2M^2$ matrix with the entries

$$K_{i,j} = \frac{\langle \langle k(x,t,y,z), \Phi_{(j)}(y,z) \rangle, \Phi_{(i)}(x,t) \rangle}{\|\Phi_{(j)}(y,z)\|_2^2 \|\Phi_{(i)}(x,t)\|_2^2},$$

in which $\Phi_{(i)}(x,t)$ denotes the ith element of the column vector $\Phi(x,t)$.

2.3 Operational Matrices

The integration of the vector $\Phi(x,t)$ determined by (6) can be approximately obtained as [24]

$$\int_0^t \int_0^x \Phi(y,z)\,\mathrm{d}y\,\mathrm{d}z \approx Q\Phi(x,t) = (P\otimes P)\Phi(x,t), \tag{8}$$

where Q is an $N^2M^2 \times N^2M^2$ matrix called operational matrix of integration for two-dimensional hybrid of block-pulse functions and Legendre polynomials. Note that $\otimes$ denotes the Kronecker product and P is the operational matrix of integration for one-dimensional hybrid of block-pulse functions and Legendre polynomials given by [15]

$$P = \begin{bmatrix} S & E & E & \dots & E \\ 0 & S & E & \dots & E \\ 0 & 0 & S & \dots & E \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & S \end{bmatrix}_{NM\times NM}, \quad \text{with} \quad E = \frac{1}{N}\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{M\times M},$$

and S is the operational matrix of integration for one-dimensional shifted Legendre polynomials on the interval $[\frac{i-1}{N}, \frac{i}{N})$ and can be specified by

$$S = \frac{1}{2N}\begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{-1}{2M-3} & 0 & \frac{1}{2M-3} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{-1}{2M-1} & 0 \end{bmatrix}_{M\times M}.$$

The following property of the product of two vectors $\Phi(x,t)$ and $\Phi^T(x,t)$ defied by (6) will be usable

$$\Phi(x,t)\Phi^T(x,t)G \approx \tilde{G}\Phi(x,t), \tag{9}$$

where G is defined by (5) and $\tilde{G}$ is an $N^2M^2 \times N^2M^2$ matrix, called product operational matrix and can be determined by the next algorithm [24]:

1. D_{i_1,h_1,i_2}, for $i_1,i_2 = 1,\dots,N$ and $h_1 = 0,\dots,M-1$ are $M\times M$ matrices of the form

$$D_{i_1,h_1,i_2} = \left[D_{i_1,h_1,i_2}^{(j_2,m_2)}\right]_{j_2,m_2=0}^{M-1},$$

where

$$D_{i_1,h_1,i_2}^{(j_2,m_2)} = N(2m_2+1)\sum_{h_2=0}^{M-1} \varsigma_{i_2 j_2 h_2 m_2} g_{i_1 h_1 i_2 h_2}, \quad j_2,m_2 = 0,\dots,M-1,$$

in which

$$\varsigma_{ijhm} = \int_{\frac{i-1}{N}}^{\frac{i}{N}} L_j(2Nx-2i+1)L_h(2Nx-2i+1)L_m(2Nx-2i+1)\,\mathrm{d}x.$$

2. $A_{i_1h_1}$, for $i_1 = 1,\ldots,N$ and $h_1 = 0,\ldots,M-1$ are $NM \times NM$ block-diagonal matrices of the form

$$A_{i_1h_1} = \begin{bmatrix} D_{i_1,h_1,1} & & & \\ & D_{i_1,h_1,2} & & \\ & & \ddots & \\ & & & D_{i_1,h_1,N} \end{bmatrix}_{NM\times NM}.$$

3. $\tilde{G}_{i_1}$, for $i_1 = 1,\ldots,N$ are $NM^2 \times NM^2$ matrices determined by

$$\tilde{G}_{i_1} = \left[\tilde{G}_{i_1}^{(j_1,m_1)}\right]_{j_1,m_1=0,1}^{M-1},$$

such that

$$\tilde{G}_{i_1}^{(j_1,m_1)} = N(2m_1+1)\sum_{h_1=0}^{M-1} \varsigma_{i_1j_1h_1m_1}A_{i_1h_1}, \quad j_1,m_1 = 0,1,\ldots,M-1.$$

4. At the end, the block-diagonal matrix $\tilde{G}$ defined in (9) is obtained as

$$\tilde{G} = \begin{bmatrix} \tilde{G}_1 & & & \\ & \tilde{G}_2 & & \\ & & \ddots & \\ & & & \tilde{G}_N \end{bmatrix}_{N^2M^2\times N^2M^2}.$$

For an $N^2M^2 \times N^2M^2$ matrix Λ, it is useful to estimate an $N^2M^2 \times 1$ vector $\hat{\Lambda} = [\hat{\Lambda}_{(1)},\hat{\Lambda}_{(2)},\ldots,\hat{\Lambda}_{(N^2M^2)}]^T$, which satisfies the next formula:

$$\Phi^T(x,t)\Lambda\Phi(x,t) \approx \hat{\Lambda}^T\Phi(x,t). \tag{10}$$

$\hat{\Lambda}$ is called product operational vector for two-dimensional hybrid of block-pulse functions and Legendre polynomials and its entries are obtaiened with the aid of the following formulae [24]:

$$\hat{\Lambda}_{(m)} = \sum_{i=1}^{N^2M^2}\sum_{j=1}^{N^2M^2} \frac{\int_0^1\int_0^1 \Phi_{(i)}(x,t)\Phi_{(j)}(x,t)\Phi_{(m)}(x,t)\,\mathrm{d}x\,\mathrm{d}t}{\left\langle \Phi_{(m)},\Phi_{(m)}\right\rangle}\Lambda_{i,j}, \quad m = 1,2,\ldots,N^2M^2.$$

3 Implementation of the numerical method

In this section, we apply the operational matrices of two-dimensional hybrid of block-pulse functions and Legendre polynomials to discuss a numerical algorithm for the solution of the system of 2-DVIEs of the form (1).

First, let us rewrite Eq. (1) in the following form

$$\lambda_{l,1}u_1(x,t)+\lambda_{l,2}u_2(x,t)=g_l(x,t)+\int_0^t\int_0^x k_{l,1}(x,t,y,z)u_1(y,z)\,\mathrm{d}y\,\mathrm{d}z$$
$$+\int_0^t\int_0^x k_{l,2}(x,t,y,z)u_2(y,z)\,\mathrm{d}y\,\mathrm{d}z,\qquad l=1,2. \tag{11}$$

Expanding the functions in Eq. (11) with respect to the two-dimensional hybrid functions, we have

$$u_l(x,t)\approx U_l^T\Phi(x,t)=\bar{u}_{l,N,M}(x,t)\qquad l=1,2, \tag{12}$$
$$g_l(x,t)\approx G_l^T\Phi(x,t)=g_{l,N,M}(x,t)\qquad l=1,2, \tag{13}$$
$$k_{l,s}(x,t,y,z)\approx\Phi^T(x,t)K_{l,s}\Phi(y,z)=k_{l,s,N,M}(x,t,y,z)\qquad l,s=1,2, \tag{14}$$

in which U_1 and U_2 are $N^2M^2\times 1$ unknown vectors to be estimated. Substituting Eqs. (12)-(14) into Eq. (11), we obtain

$$\lambda_{l,1}U_1^T\Phi(x,t)+\lambda_{l,2}U_2^T\Phi(x,t)=G_l^T\Phi(x,t)+\Phi^T(x,t)K_{l,1}\int_0^t\int_0^x\Phi(y,z)\Phi^T(y,z)U_1\,\mathrm{d}y\,\mathrm{d}z$$
$$+\Phi^T(x,t)K_{l,2}\int_0^t\int_0^x\Phi(y,z)\Phi^T(y,z)U_2\,\mathrm{d}y\,\mathrm{d}z,\qquad l=1,2. \tag{15}$$

Making use of the operational matrix of integration (8) and the product operational matrix (9), Eq. (15) will be converted to

$$\lambda_{l,1}U_1^T\Phi(x,t)+\lambda_{l,2}U_2^T\Phi(x,t)=G_l^T\Phi(x,t)+\Phi^T(x,t)K_{l,1}\tilde{U}_1Q\Phi(x,t)+\Phi^T(x,t)K_{l,2}\tilde{U}_2Q\Phi(x,t),\quad l=1,2.$$

By setting $\Lambda_{l,s}=K_{l,s}\tilde{U}_sQ$ for $l,s=1,2$ and applying the operational vector introduced in (10), we obtain

$$\lambda_{l,1}U_1^T\Phi(x,t)+\lambda_{l,2}U_2^T\Phi(x,t)=G_l^T\Phi(x,t)+\hat{\Lambda}_{l,1}^T\Phi(x,t)+\hat{\Lambda}_{l,2}^T\Phi(x,t),\qquad l=1,2.$$

Thus we have

$$\lambda_{l,1}U_1+\lambda_{l,2}U_2-\hat{\Lambda}_{l,1}-\hat{\Lambda}_{l,2}=G_l,\qquad l=1,2,$$

which is a system of $2N^2M^2$ equations with the same number of unknowns that can be solved by any iterative mehod.

4 Error analysis

This section is devoted to the estimation of an error bound for the approximate solution of Eq. (1) based on two-dimensional hybrid of block-pulse functions and Legendre polynomials. Suppose that

$$\Omega=\bigcup_{1\le i_1,i_2\le N}\Omega_{i_1i_2},$$

where

$$\Omega_{i_1i_2}=[\frac{i_1-1}{N},\frac{i_1}{N})\times[\frac{i_2-1}{N},\frac{i_2}{N}).$$

We assume that the functions $k_{l,s}(x,t,y,z)$ and $g_l(x,t)$ for $l,s=1,2$ are sufficiently smooth on each subinterval $\Omega_{i_1i_2}$. Using the above assumptions, we have the next theorems.

Theorem 1. *[24] Let $g_{l,N,M}(x,t)$ be the two-dimensional hybrid expansion of real valued functions $g_l(x,t)\,(l=1,2)$ defined by (13) and suppose that $g_1(x,t)$ and $g_2(x,t)$ are sufficiently smooth on each subinterval $\Omega_{i_1 i_2}$ $i_1,i_2=1,2,\dots,N$, then there exist positive constants α_l such that*

$$\|g_l - g_{l,N,M}\|_2 \leq \frac{\alpha_l}{2^{2M-1}N^M M!}, \qquad l=1,2. \tag{16}$$

Theorem 2. *[24] Let $k_{l,s,N,M}(x,t,y,z)$ be the two-dimensional hybrid expansion of real valued function $k_{l,s}(x,t,y,z)$ defined by (14) and suppose that $k_{l,s}(x,t,y,z)$ for $l,s=1,2$ are sufficiently smooth on each subinterval $\Omega_{i_1,i_2}\times\Omega_{i_3,i_4}$, $i_1,i_2,i_3,i_4=1,2,\dots,N$, then there exist positive constants $\beta_{l,s}$ such that*

$$\|k_{l,s} - k_{l,s,N,M}\|_2 \leq \frac{\beta_{l,s}}{2^{2M-1}N^M M!}, \qquad l,s=1,2. \tag{17}$$

Now, consider Eq. (1) with the operator form

$$\Lambda U = G + \int_0^t \int_0^x KU\,\mathrm{d}y\,\mathrm{d}z.$$

Let us define the vectors $\bar{U}_{N,M}(x,t)$, $G_{N,M}(x,t)$ and the matrix $K_{N,M}(x,t,y,z)$ in the following form

$$\bar{U}_{N,M}(x,t) = \left(\bar{u}_{1,N,M}(x,t), \bar{u}_{2,N,M}(x,t)\right)^T, \qquad \text{approximate solution of system (1)}, \tag{18}$$

$$G_{N,M}(x,t) = \left(g_{1,N,M}(x,t), g_{2,N,M}(x,t)\right)^T, \qquad \text{hybrid expansion of G(x,t)}, \tag{19}$$

$$K_{N,M}(x,t,y,z) = \begin{bmatrix} k_{1,1,N,M}(x,t,y,z) & k_{1,2,N,M}(x,t,y,z) \\ k_{2,1,N,M}(x,t,y,z) & k_{2,2,N,M}(x,t,y,z) \end{bmatrix}, \qquad \text{hybrid expansion of K(x,t,y,z)}. \tag{20}$$

Using the above assumptions we have the following theorem:

Theorem 3. *Suppose that $U(x,t)$ is the exact solution of Eq. (1) and $\bar{U}_{N,M}(x,t)$ is its approximate solution obtained by the proposed method. Also, assume that the following conditions hold:*
(1) $\|U\|_2 = c_1 < \infty$, (2) $\|K\|_2 = c_2 < \infty$, (3) $\|\Lambda^{-1}\|_2 = c_3 < \infty$.
Then there exist positive constans ρ and ς such that

$$\|U - \bar{U}_{N,M}\|_2 \leq \frac{\dfrac{c_3\rho}{2^{2M-1}N^M M!}}{1 - c_2 c_3 - \dfrac{c_3\varsigma}{2^{2M-1}N^M M!}}. \tag{21}$$

Proof. From Eq. (1), we have

$$\Lambda\left(U(x,t) - \bar{U}_{N,M}(x,t)\right) = G(x,t) - G_{N,M}(x,t) + \int_0^t \int_0^x \left(K(x,t,y,z)U(y,z) - K_{N,M}(x,t,y,z)\bar{U}_{N,M}(y,z)\right)\mathrm{d}y\,\mathrm{d}z.$$

Since $0 \leq x,t \leq 1$ and $\det\Lambda \neq 0$, by tacking L^2-norm in the above equation we have

$$\|U - \bar{U}_{N,M}\|_2 \leq \|\Lambda^{-1}\|_2 \|G - G_{N,M}\|_2 + \|\Lambda^{-1}\|_2 \|KU - K_{N,M}\bar{U}_{N,M}\|_2. \tag{22}$$

It is clear that

$$\begin{aligned} K(x,t,y,z)U(y,z) - K_{N,M}(x,t,y,z)\bar{U}_{N,M}(y,z) &= K(x,t,y,z)\left(U(y,z) - \bar{U}_{N,M}(y,z)\right) \\ &+ \left(K(x,t,y,z) - K_{N,M}(x,t,y,z)\right)\left(U(y,z) - \bar{U}_{N,M}(y,z)\right) + \left(K(x,t,y,z) - K_{N,M}(x,t,y,z)\right)U(y,z), \end{aligned}$$

which leads to

$$\|KU - K_{N,M}\bar{U}_{N,M}\|_2 \leq \|K\|_2\|U - \bar{U}_{N,M}\|_2 + \|K - K_{N,M}\|_2\|U - \bar{U}_{N,M}\|_2 + \|K - K_{N,M}\|_2\|U\|_2. \quad (23)$$

From Eqs. (22) and (23), we obtain

$$\begin{aligned}\|U - \bar{U}_{N,M}\|_2 \leq & \|\Lambda^{-1}\|_2\|G - G_{N,M}\|_2 + \|\Lambda^{-1}\|_2\|K\|_2\|U - \bar{U}_{N,M}\|_2 \\ & + \|\Lambda^{-1}\|_2\|K - K_{N,M}\|_2\|U - \bar{U}_{N,M}\|_2 + \|\Lambda^{-1}\|_2\|K - K_{N,M}\|_2\|U\|_2.\end{aligned}$$

Using the assumptions 1–3 and the above inequality we have

$$\|U - \bar{U}_{N,M}\|_2 \leq \frac{c_3\|G - G_{N,M}\|_2}{1 - c_2c_3 - c_3\|K - K_{N,M}\|_2}. \quad (24)$$

Now consider definitions (19) and (20). It follows from Theorems 1 and 2 that

$$\|G - G_{N,M}\|_2 = \left(\sum_{l=1}^{2}\|g_l - g_{l,N,M}\|_2^2\right)^{\frac{1}{2}} \leq \left(\sum_{l=1}^{2}\left(\frac{\alpha_l}{2^{2M-1}N^M M!}\right)^2\right)^{\frac{1}{2}} = \frac{\rho}{2^{2M-1}N^M M!},$$

in which $\rho = \left(\alpha_l^2 + \alpha_2^2\right)^{\frac{1}{2}}$, and

$$\|K - K_{N,M}\|_2 = \left(\sum_{l=1}^{2}\sum_{s=1}^{2}\|k_{l,s} - k_{l,s,N,M}\|_2^2\right)^{\frac{1}{2}} \leq \left(\sum_{l=1}^{2}\sum_{s=1}^{2}\left(\frac{\beta_{l,s}}{2^{2M-1}N^M M!}\right)^2\right)^{\frac{1}{2}} = \frac{\varsigma}{2^{2M-1}N^M M!},$$

in which $\varsigma = \left(\beta_{1,1}^2 + \beta_{1,2}^2 + \beta_{2,1}^2 + \beta_{2,2}^2\right)^{\frac{1}{2}}$. From inequalities (24), (25) and (25), we obtain

$$\|U - \bar{U}_{N,M}\|_2 \leq \frac{\dfrac{c_3\rho}{2^{2M-1}N^M M!}}{1 - c_2c_3 - \dfrac{c_3\varsigma}{2^{2M-1}N^M M!}},$$

which completes the proof. □

Remark 1. *It is obvious in the case that $c_2c_3 \neq 1$ the right hand side of the inequality (21) tends to zero as $N, M \to \infty$, so $U - \bar{U}_{N,M} \to 0$and this proves convergence of the proposed method.*

5 Numerical results

In this section, some numerical examples are provided to indicate the accuracy and effectiveness of the hybrid functions method. In order to analyze the error of the method, we introduce the following functions:

$$e_{l,N,M}(x,t) = |u_l(x,t) - \bar{u}_{l,N,M}(x,t)|, \qquad (x,t) \in \Omega, \quad l = 1,2,$$

where $u_l(x,t)$ for $l = 1,2$ denote the exact solutions of Eq. (1) and $\bar{u}_{l,N,M}(x,t)$ for $l = 1,2$ are the approximate solutions obtained by the proposed method. All of the computations have been done using MATHEMATICA 11 on a personal computer.

Example 1. Cconsider the following system of 2-DVIEs [17]:

$$\begin{cases} u_1(x,t) - u_2(x,t) = g_1(x,t) + \int_0^t \int_0^x \left(zxu_1(y,z) + e^{-z}u_2(y,z)\right) \mathrm{d}y\,\mathrm{d}z, \\ \qquad\quad - u_2(x,t) = g_2(x,t) + \int_0^t \int_0^x \left(y^2z^3u_1(y,z) + ztu_2(y,z)\right) \mathrm{d}y\,\mathrm{d}z, \end{cases}$$

$$K(x,t,y,z) = \begin{bmatrix} k_{1,1}(x,t,y,z) & k_{1,2}(x,t,y,z) \\ k_{2,1}(x,t,y,z) & k_{2,2}(x,t,y,z) \end{bmatrix} = \begin{bmatrix} zx & e^{-z} \\ y^2z^3 & zt \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

where

$$g_1(x,t) = \frac{1}{12}x\left(4x^2e^{-t}\left(t^3+3t^2+6t+6\right) - 3\left(x^2-1\right)e^t - 3x\left(7x+4t^3\right)\right),$$
$$g_2(x,t) = -\frac{1}{16}x^4\left(e^t\left(t\left((t-3)t+6\right)-6\right)+6\right) - \frac{1}{15}x^3t^6 - x^2t^3.$$

The exact solutions of this system are $u_1(x,t) = \frac{1}{4}xe^t$ and $u_2(x,t) = x^2t^3$.

Expanding the functions $g_l(x,t)$ and $k_{l,s}(x,t,y,z)$ for $l,s = 1,2$ with respect to two-dimensional hybrid of block-pulse functions and Legendre polynomials and substituting in the above system, we obtain

$$\begin{aligned} U_1^T\Phi(x,t) - U_2^T\Phi(x,t) =& G_1^T\Phi(x,t) + \Phi^T(x,t)K_{1,1}\int_0^t\int_0^x \Phi(y,z)\Phi^T(y,z)U_1\,\mathrm{d}y\,\mathrm{d}z \\ &+ \Phi^T(x,t)K_{1,2}\int_0^t\int_0^x \Phi(y,z)\Phi^T(y,z)U_2\,\mathrm{d}y\,\mathrm{d}z, \\ -U_2^T\Phi(x,t) =& G_2^T\Phi(x,t) + \Phi^T(x,t)K_{2,1}\int_0^t\int_0^x \Phi(y,z)\Phi^T(y,z)U_1\,\mathrm{d}y\,\mathrm{d}z \\ &+ \Phi^T(x,t)K_{2,2}\int_0^t\int_0^x \Phi(y,z)\Phi^T(y,z)U_2\,\mathrm{d}y\,\mathrm{d}z, \end{aligned}$$

in which $G_l(l = 1,2)$ are $N^2M^2 \times 1$ vectors introduced in (4), and $K_{l,s}(l,s = 1,2)$ are $N^2M^2 \times N^2M^2$ matrices introduced in (7). Making use of the operational matrix of integration (8) and the product operational matrix (9), the above equations will be reduced to

$$\begin{aligned} U_1^T\Phi(x,t) - U_2^T\Phi(x,t) &= G_1^T\Phi(x,t) + \Phi^T(x,t)K_{1,1}\tilde{U}_1Q\Phi(x,t) + \Phi^T(x,t)K_{1,2}\tilde{U}_2Q\Phi(x,t), \\ -U_2^T\Phi(x,t) &= G_2^T\Phi(x,t) + \Phi^T(x,t)K_{2,1}\tilde{U}_1Q\Phi(x,t) + \Phi^T(x,t)K_{2,2}\tilde{U}_2Q\Phi(x,t). \end{aligned}$$

By setting $\Lambda_{l,s} = K_{l,s}\tilde{U}_sQ$ for $l,s = 1,2$ and applying the operational vector introduced in (10), we have

$$\begin{aligned} U_1 - U_2 - \hat{\Lambda}_{1,1} - \hat{\Lambda}_{1,2} &= G_1, \\ -U_2 - \hat{\Lambda}_{2,1} - \hat{\Lambda}_{2,2} &= G_2, \end{aligned}$$

which is a system of $2(NM)^2$ equations in terms of unknown vectors U_1 and U_1. The Newton iteration method with zero vector as the initial guess was used to solve this nonlinear system. Absolute values of the errors are reported in Table 1 for $N = 2$ and $M = 2,3,5$. The results of our method and the Legendre Wavelet method [17] are compared in Table 2. The number of basic functions in our method is $2(NM)^2$ and for Legendre wavelet method is $2\left(2^kM\right)^2$. By comparing the results, we can conclude that the hybrid functions method gives better results by using fewer basic functions.

Table 1: Absolute errors for Example 1.

(x,t)	$(e_{1,2,2}(x,t), e_{2,2,2}(x,t))$	$(e_{1,2,3}(x,t), e_{2,2,3}(x,t))$	$(e_{1,2,5}(x,t), e_{2,2,5}(x,t))$
$(0,0)$	$(8.97\times10^{-6}, 1.03\times10^{-3})$	$(7.36\times10^{-8}, 1.92\times10^{-7})$	$(4.29\times10^{-10}, 1.67\times10^{-11})$
$(0.1,0.1)$	$(2.77\times10^{-5}, 3.13\times10^{-5})$	$(1.16\times10^{-5}, 2.25\times10^{-5})$	$(5.60\times10^{-9}, 3.82\times10^{-10})$
$(0.2,0.2)$	$(6.16\times10^{-4}, 8.46\times10^{-4})$	$(1.94\times10^{-5}, 7.00\times10^{-5})$	$(2.27\times10^{-8}, 1.69\times10^{-9})$
$(0.3,0.3)$	$(8.40\times10^{-4}, 2.16\times10^{-3})$	$(2.68\times10^{-5}, 1.57\times10^{-4})$	$(2.29\times10^{-8}, 5.61\times10^{-9})$
$(0.4,0.4)$	$(3.45\times10^{-4}, 3.86\times10^{-5})$	$(4.95\times10^{-5}, 3.59\times10^{-4})$	$(1.09\times10^{-8}, 2.30\times10^{-8})$
$(0.5,0.5)$	$(4.86\times10^{-3}, 2.36\times10^{-2})$	$(2.71\times10^{-5}, 1.56\times10^{-3})$	$(3.75\times10^{-7}, 9.44\times10^{-8})$
$(0.6,0.6)$	$(9.52\times10^{-5}, 2.42\times10^{-3})$	$(1.09\times10^{-4}, 8.05\times10^{-4})$	$(1.24\times10^{-7}, 6.35\times10^{-8})$
$(0.7,0.7)$	$(3.39\times10^{-3}, 2.63\times10^{-2})$	$(1.06\times10^{-4}, 8.53\times10^{-4})$	$(1.80\times10^{-7}, 5.48\times10^{-8})$
$(0.8,0.8)$	$(3.28\times10^{-3}, 3.71\times10^{-3})$	$(1.23\times10^{-4}, 1.12\times10^{-3})$	$(4.35\times10^{-8}, 7.70\times10^{-8})$
$(0.9,0.9)$	$(2.22\times10^{-3}, 3.92\times10^{-3})$	$(1.66\times10^{-4}, 1.80\times10^{-3})$	$(1.36\times10^{-7}, 1.84\times10^{-7})$

Table 2: Absolute errors for Example 1.

	Present method with $N=2, M=3$	Method of [17] with $k=2, M=4$
(x,t)	$(e_{1,2,3}(x,t), e_{2,2,3}(x,t))$	$(e_1(x,t), e_2(x,t))$
$(\frac{3}{8},\frac{5}{8})$	$(8.73\times10^{-5}, 3.83\times10^{-4})$	$(1.12\times10^{-2}, 1.05\times10^{-2})$
$(\frac{3}{8},\frac{7}{8})$	$(9.36\times10^{-5}, 3.85\times10^{-4})$	$(1.59\times10^{-2}, 1.41\times10^{-2})$
$(\frac{5}{8},\frac{5}{8})$	$(1.42\times10^{-4}, 1.06\times10^{-3})$	$(3.24\times10^{-2}, 2.80\times10^{-2})$
$(\frac{1}{8},\frac{7}{8})$	$(3.30\times10^{-5}, 4.49\times10^{-5})$	$(1.72\times10^{-3}, 1.67\times10^{-3})$
$(\frac{7}{8},\frac{1}{8})$	$(1.18\times10^{-4}, 2.08\times10^{-3})$	$(1.21\times10^{-2}, 1.17\times10^{-2})$
$(\frac{3}{8},\frac{3}{8})$	$(5.56\times10^{-5}, 3.84\times10^{-4})$	$(6.62\times10^{-3}, 6.44\times10^{-3})$
$(\frac{5}{8},\frac{3}{8})$	$(9.06\times10^{-5}, 1.06\times10^{-3})$	$(1.88\times10^{-2}, 1.76\times10^{-2})$
$(\frac{1}{8},\frac{1}{8})$	$(1.79\times10^{-5}, 4.27\times10^{-5})$	$(2.43\times10^{-4}, 2.43\times10^{-4})$

Example 2. Consider the following system of 2-DVIEs [33]:

$$\begin{cases} u_1(x,t) = g_1(x,t) + \displaystyle\int_0^t \int_0^x \frac{x+y-t-z}{4} u_2(y,z)\,\mathrm{d}y\,\mathrm{d}z, \\ u_2(x,t) = g_2(x,t) + \displaystyle\int_0^t \int_0^x \frac{x+y-t-z}{4} u_1(y,z)\,\mathrm{d}y\,\mathrm{d}z, \end{cases}$$

$$K(x,t,y,z) = \begin{bmatrix} 0 & \dfrac{x+y-t-z}{4} \\ \dfrac{x+y-t-z}{4} & 0 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$g_1(x,t) = xe^{1-t} - \frac{e}{24}t^2\left(3+3x-5t+e^{-x}(-3-6x+5t)\right),$$
$$g_2(x,t) = te^{1-x} + \frac{e}{24}x^2\left(3-5x+3t+e^{-t}(-3+5x-6t)\right).$$

The exact solutions of this system are $u_1(x,t) = xe^{1-t}$ and $u_2(x,t) = te^{1-x}$. Absolute values of errors for $N = 2$ and $M = 2,3,5$ are listed in Table 3. As expected from the theoretical achievements, we obtain better numerical results by increasing the values M.

Table 3: Absolute errors for Example 2.

(x,t)	$(e_{1,2,2}(x,t), e_{2,2,2}(x,t))$	$(e_{1,2,3}(x,t), e_{2,2,3}(x,t))$	$(e_{1,2,5}(x,t), e_{2,2,5}(x,t))$
$(0,0)$	$(8.74\times10^{-6}, 8.77\times10^{-6})$	$(1.02\times10^{-7}, 1.02\times10^{-7})$	$(1.11\times10^{-7}, 5.02\times10^{-11})$
$(0.1,0.1)$	$(9.50\times10^{-5}, 9.34\times10^{-5})$	$(8.29\times10^{-5}, 8.29\times10^{-5})$	$(4.90\times10^{-8}, 3.25\times10^{-8})$
$(0.2,0.2)$	$(4.02\times10^{-3}, 4.01\times10^{-3})$	$(1.20\times10^{-4}, 1.20\times10^{-4})$	$(1.41\times10^{-7}, 1.34\times10^{-7})$
$(0.3,0.3)$	$(5.66\times10^{-3}, 5.65\times10^{-3})$	$(1.91\times10^{-4}, 1.91\times10^{-4})$	$(2.10\times10^{-7}, 2.03\times10^{-7})$
$(0.4,0.4)$	$(1.00\times10^{-3}, 1.02\times10^{-3})$	$(3.06\times10^{-4}, 3.05\times10^{-4})$	$(1.53\times10^{-7}, 1.37\times10^{-7})$
$(0.5,0.5)$	$(1.41\times10^{-2}, 1.40\times10^{-2})$	$(6.96\times10^{-4}, 6.95\times10^{-4})$	$(6.80\times10^{-7}, 6.80\times10^{-7})$
$(0.6,0.6)$	$(3.46\times10^{-4}, 3.39\times10^{-4})$	$(3.02\times10^{-4}, 3.01\times10^{-4})$	$(1.18\times10^{-7}, 1.18\times10^{-7})$
$(0.7,0.7)$	$(8.54\times10^{-3}, 8.51\times10^{-3})$	$(2.55\times10^{-4}, 2.54\times10^{-4})$	$(2.84\times10^{-7}, 2.84\times10^{-7})$
$(0.8,0.8)$	$(9.17\times10^{-3}, 9.11\times10^{-3})$	$(3.09\times10^{-4}, 3.09\times10^{-4})$	$(3.29\times10^{-7}, 3.28\times10^{-7})$
$(0.9,0.9)$	$(1.35\times10^{-3}, 1.41\times10^{-3})$	$(4.18\times10^{-4}, 4.16\times10^{-4})$	$(1.86\times10^{-7}, 1.87\times10^{-7})$

Example 3. Consider the following system of 2-DVIEs:

$$\begin{cases} u_1(x,t) = g_1(x,t) + \int_0^t \int_0^x (u_1(y,z) + u_2(y,z))\,\mathrm{d}y\,\mathrm{d}z, \\ u_2(x,t) = g_2(x,t) + \int_0^t \int_0^x (u_1(y,z) - u_2(y,z))\,\mathrm{d}y\,\mathrm{d}z, \end{cases}$$

where

$$g_1(x,t) = -\frac{t^2}{2} + \frac{1}{2}t^2\cos x + t\sin x - \frac{1}{2}t^2\sin x,$$
$$g_2(x,t) = -\frac{t^2}{2} + \frac{1}{2}t^2\cos x + t\cos x + \frac{1}{2}t^2\sin x.$$

The exact solutions of this system are $u_1(x,t) = t\sin x$ and $u_2(x,t) = t\cos x$. The absolute errors of this example for $N = 2$ and $M = 2,3,5$ are shown in Table 4.

Example 4. Consider the following system of 2-DVIEs:

$$\begin{cases} u_1(x,t) = g_1(x,t) + \int_0^t \int_0^x xz\,(u_1(y,z) + u_2(y,z))\,\mathrm{d}y\,\mathrm{d}z, \\ u_2(x,t) = g_2(x,t) + \int_0^t \int_0^x (xt + yz)\,u_2(y,z)\,\mathrm{d}y\,\mathrm{d}z, \end{cases}$$

$$K(x,t,y,z) = \begin{bmatrix} xz & xz \\ 0 & xt+yz \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where $g_1(x,t)$ and $g_2(x,t)$ are defined in such a way that

$$u_1(x,t) = \begin{cases} xt + x, & 0 \le x < \frac{1}{2}, 0 \le t < 1, \\ x + t, & \frac{1}{2} \le x < 1, 0 \le t < 1, \end{cases}$$

Table 4: Absolute errors for Example 3.

(x,t)	$(e_{1,2,2}(x,t),e_{2,2,2}(x,t))$	$(e_{1,2,3}(x,t),e_{2,2,3}(x,t))$	$(e_{1,2,5}(x,t),e_{2,2,5}(x,t))$
$(0,0)$	$(2.65\times10^{-5},1.47\times10^{-5})$	$(3.74\times10^{-9},1.00\times10^{-9})$	$(2.85\times10^{-12},5.38\times10^{-12})$
$(0.1,0.1)$	$(6.01\times10^{-5},7.05\times10^{-5})$	$(3.65\times10^{-5},7.78\times10^{-6})$	$(1.51\times10^{-8},4.23\times10^{-9})$
$(0.2,0.2)$	$(3.88\times10^{-4},1.78\times10^{-3})$	$(5.61\times10^{-5},1.56\times10^{-5})$	$(6.14\times10^{-8},1.59\times10^{-8})$
$(0.3,0.3)$	$(7.75\times10^{-4},2.62\times10^{-3})$	$(8.46\times10^{-5},1.99\times10^{-5})$	$(9.21\times10^{-8},2.32\times10^{-8})$
$(0.4,0.4)$	$(1.20\times10^{-4},4.01\times10^{-4})$	$(1.43\times10^{-4},4.26\times10^{-5})$	$(6.08\times10^{-8},1.49\times10^{-8})$
$(0.5,0.5)$	$(6.82\times10^{-3},7.92\times10^{-3})$	$(3.92\times10^{-4},3.33\times10^{-4})$	$(3.85\times10^{-7},3.38\times10^{-7})$
$(0.6,0.6)$	$(6.71\times10^{-4},2.16\times10^{-4})$	$(1.70\times10^{-4},1.46\times10^{-4})$	$(6.76\times10^{-8},6.37\times10^{-8})$
$(0.7,0.7)$	$(4.12\times10^{-3},4.81\times10^{-3})$	$(1.45\times10^{-4},1.37\times10^{-4})$	$(1.61\times10^{-7},1.50\times10^{-7})$
$(0.8,0.8)$	$(5.21\times10^{-3},5.17\times10^{-3})$	$(1.74\times10^{-4},1.60\times10^{-4})$	$(1.86\times10^{-7},1.74\times10^{-7})$
$(0.9,0.9)$	$(6.29\times10^{-5},8.12\times10^{-4})$	$(2.35\times10^{-4},2.38\times10^{-4})$	$(1.06\times10^{-7},9.94\times10^{-8})$

and

$$u_2(x,t)=\begin{cases} xt+t, & 0\le x<\frac{1}{2}, 0\le t<1,\\ x-t, & \frac{1}{2}\le x<1, 0\le t<1,\end{cases}$$

to be the exact solutions. Absolute values of the errors are reported in Table 5 for different values of N And M. The results Confirm proficiency of the hybrid functions method in dealing with non-smooth solutions. Also, the graphical representation of the error functions $e_{l,N,M}(x,t)$ for $l=1,2$, $N=2$ and $M=2,3$ are illustrated in Figures 1 and 2.

Table 5: Absolute errors for Example 4.

(x,t)	$(e_{1,1,3}(x,t),e_{2,1,3}(x,t))$	$(e_{1,2,2}(x,t),e_{2,2,2}(x,t))$	$(e_{1,2,3}(x,t),e_{2,2,3}(x,t))$
$(0,0)$	$(4.60\times10^{-4},9.28\times10^{-2})$	$(5.22\times10^{-5},5.22\times10^{-5})$	$(1.16\times10^{-10},2.28\times10^{-10})$
$(0.1,0.1)$	$(4.68\times10^{-3},2.78\times10^{-2})$	$(7.30\times10^{-6},7.26\times10^{-6})$	$(4.36\times10^{-11},8.28\times10^{-11})$
$(0.2,0.2)$	$(8.80\times10^{-3},1.48\times10^{-2})$	$(1.52\times10^{-5},1.53\times10^{-5})$	$(5.69\times10^{-12},1.02\times10^{-11})$
$(0.3,0.3)$	$(3.48\times10^{-2},6.04\times10^{-2})$	$(2.83\times10^{-5},2.79\times10^{-5})$	$(4.87\times10^{-12},9.99\times10^{-12})$
$(0.4,0.4)$	$(6.76\times10^{-2},1.70\times10^{-1})$	$(1.23\times10^{-4},1.22\times10^{-4})$	$(4.37\times10^{-11},8.30\times10^{-11})$
$(0.5,0.5)$	$(1.48\times10^{-1},3.98\times10^{-1})$	$(3.49\times10^{-4},5.62\times10^{-4})$	$(1.02\times10^{-12},7.09\times10^{-12})$
$(0.6,0.6)$	$(1.09\times10^{-1},3.51\times10^{-1})$	$(1.11\times10^{-4},54.8\times10^{-4})$	$(7.61\times10^{-14},5.22\times10^{-13})$
$(0.7,0.7)$	$(5.93\times10^{-2},2.45\times10^{-1})$	$(6.60\times10^{-6},2.22\times10^{-4})$	$(1.06\times10^{-13},1.87\times10^{-13})$
$(0.8,0.8)$	$(4.31\times10^{-3},7.42\times10^{-2})$	$(3.77\times10^{-6},2.26\times10^{-4})$	$(5.10\times10^{-15},1.08\times10^{-12})$
$(0.9,0.9)$	$(5.10\times10^{-2},1.64\times10^{-1})$	$(1.19\times10^{-4},8.61\times10^{-4})$	$(1.92\times10^{-13},4.73\times10^{-13})$

Example 5. Consider the following system of 2-DVIEs:

$$\begin{cases} u_1(x,t)=g_1(x,t)+\int_0^t\int_0^x xz\,(u_1(y,z)+u_2(y,z))\,\mathrm{d}y\,\mathrm{d}z,\\ u_2(x,t)=g_2(x,t)+\int_0^t\int_0^x xz\,(u_1(y,z)-u_2(y,z))\,\mathrm{d}y\,\mathrm{d}z,\end{cases}$$

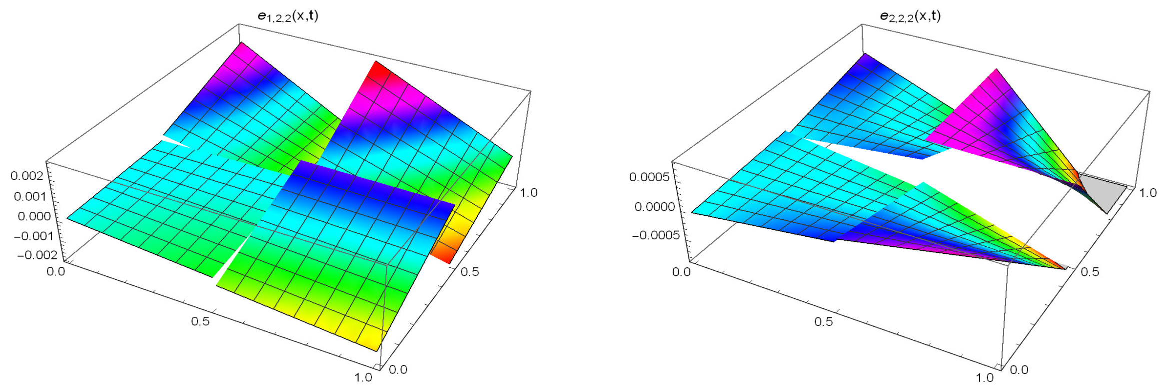

Figure 1: Plote of functions $e_{1,2,2}(x,t)$ and $e_{2,2,2}(x,t)$ for Example 4.

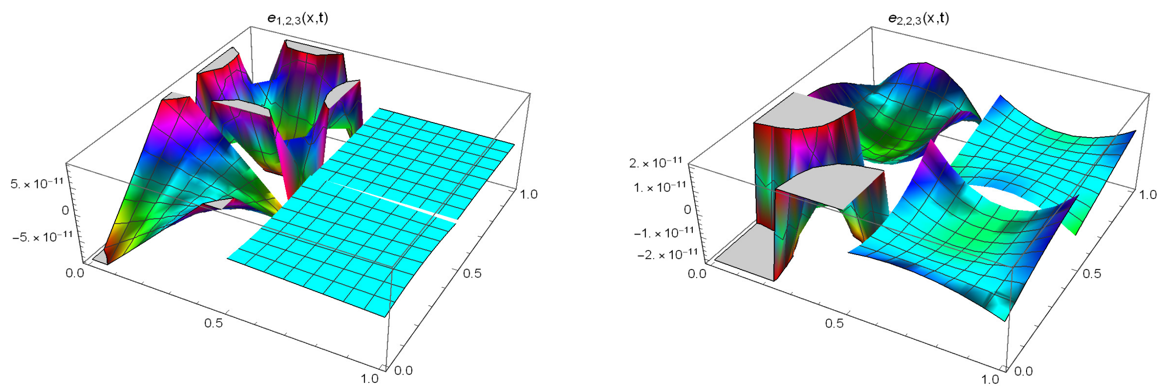

Figure 2: Plote of functions $e_{1,2,3}(x,t)$ and $e_{2,2,3}(x,t)$ for Example 4.

where $g_1(x,t)$ and $g_2(x,t)$ are defined in such a way that

$$u_1(x,t) = \begin{cases} x+xt, & 0 \le x < \frac{1}{3}, 0 \le t < 1, \\ x-xt, & \frac{1}{3} \le x < \frac{2}{3}, 0 \le t < 1, \\ x+t, & \frac{2}{3} \le x < 1, 0 \le t < 1, \end{cases}$$

and

$$u_2(x,t) = \begin{cases} 1+x\sin t, & 0 \le x < \frac{1}{3}, 0 \le t < 1, \\ 1-x\sin t, & \frac{1}{3} \le x < \frac{2}{3}, 0 \le t < 1, \\ x-t, & \frac{2}{3} \le x < 1, 0 \le t < 1, \end{cases}$$

to be the exact solutions. Absolute values of the errors are reported in Table 6 for $N = 2,3$ and $M = 2,4$. We observe that the values of errors decay as N and M increase. The results indicate that our algorithm also provides accurate results for discontinuous functions.

Table 6: Absolute errors for Example 5.

(x,t)	$(e_{1,2,2}(x,t), e_{2,2,2}(x,t))$	$(e_{1,3,2}(x,t), e_{2,3,2}(x,t))$	$(e_{1,3,4}(x,t), e_{2,3,4}(x,t))$
$(0,0)$	$(5.11 \times 10^{-5}, 1.17 \times 10^{-3})$	$(4.97 \times 10^{-6}, 5.68 \times 10^{-6})$	$(3.41 \times 10^{-15}, 4.86 \times 10^{-13})$
$(0.1,0.1)$	$(6.66 \times 10^{-3}, 6.75 \times 10^{-3})$	$(1.49 \times 10^{-6}, 2.46 \times 10^{-5})$	$(1.17 \times 10^{-10}, 1.75 \times 10^{-8})$
$(0.2,0.2)$	$(3.25 \times 10^{-2}, 3.24 \times 10^{-2})$	$(2.53 \times 10^{-6}, 1.54 \times 10^{-4})$	$(5.92 \times 10^{-10}, 6.53 \times 10^{-8})$
$(0.3,0.3)$	$(1.17 \times 10^{-1}, 1.15 \times 10^{-1})$	$(1.70 \times 10^{-5}, 1.99 \times 10^{-4})$	$(5.61 \times 10^{-10}, 1.01 \times 10^{-7})$
$(0.4,0.4)$	$(7.09 \times 10^{-2}, 6.91 \times 10^{-2})$	$(5.64 \times 10^{-6}, 8.19 \times 10^{-5})$	$(1.98 \times 10^{-9}, 5.65 \times 10^{-7})$
$(0.5,0.5)$	$(3.71 \times 10^{-2}, 1.15 \times 10^{-1})$	$(4.59 \times 10^{-7}, 1.10 \times 10^{-3})$	$(2.15 \times 10^{-11}, 6.59 \times 10^{-7})$
$(0.6,0.6)$	$(2.67 \times 10^{-1}, 1.46 \times 10^{-1})$	$(3.96 \times 10^{-5}, 1.18 \times 10^{-4})$	$(6.62 \times 10^{-9}, 8.65 \times 10^{-7})$
$(0.7,0.7)$	$(5.15 \times 10^{-1}, 3.51 \times 10^{-1})$	$(2.93 \times 10^{-4}, 9.04 \times 10^{-6})$	$(2.09 \times 10^{-9}, 2.07 \times 10^{-9})$
$(0.8,0.8)$	$(2.55 \times 10^{-1}, 1.54 \times 10^{-1})$	$(1.00 \times 10^{-4}, 2.06 \times 10^{-5})$	$(7.38 \times 10^{-9}, 7.31 \times 10^{-9})$
$(0.9,0.9)$	$(8.81 \times 10^{-2}, 7.62 \times 10^{-2})$	$(2.26 \times 10^{-4}, 9.29 \times 10^{-5})$	$(1.21 \times 10^{-8}, 1.21 \times 10^{-8})$

6 Concluding remarks

System of two-dimensional integral equations have applications in many branches of science and engineering. This article applied the operational matrices of the two-dimensional hybrid of block-pulse functions and Legendre polynomials to solve a system of two-dimensional Volterra integral equations of the second kind. These operational matrices transform the system of integral equations into a system of algebraic equations. The convergence analysis of the numerical method is considered and an error bound is obtained. Some numerical examples are provided to show the reliability of the proposed method. Numerical results indicate that the proposed method is effective to deal with smooth and non-smooth solutions.

References

[1] E. Adams, H. Spreuer, *Uniqueness and stability for boundary value problems with weakly coupled systems of nonlinear integro-differential equations and application to chemical reactions*, J. Math. Anal. Appl. **49** (1975) 393–410.

[2] P. Assari, M. Dehghan, *A meshless local discrete galerkin (mldg) scheme for numerically solving two-dimensional nonlinear volterra integral equations*, Appl. Math. Comput. **350** (2019) 249–265.

[3] I. Aziz, Siraj-Ul-Islam, F. Khan, *A new method based on Haar wavelet for the numerical solution of two-dimensional nonlinear integral equations*, J. Comput. Appl. Math. **272** (2014) 70-80.

[4] E. Babolian, K. Maleknejad, M. Mordad, B. Rahimi, *A numerical method for solving Fredholm–Volterra integral equations in two-dimensional spaces using block pulse functions and an operational matrix*, J. Comput. Appl. Math. **235** (2011) 3965-3971.

[5] S. Bazm, A. Hosseini, *Bernoulli operational matrix method for the numerical solution of nonlinear two-dimensional volterra-fredholm integral equations of hammerstein type*, Computat. Appl. Math. **39** (2020) 1–20.

[6] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Differential Equations*, Cambridge University Press, Cambridge, 2004.

[7] W.C. Chew, M.S. Tong, B. Hu, *Integral equation methods for electromagnetic and elastic waves*, Synth. Lect. Comput. Electromagn. **3** (2008) 1-241.

[8] C. Corduneanu, *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.

[9] N. M. Darani, K. Maleknejad , H. Mesgarani, *A new approach for two-dimensional nonlinear mixed volterra-fredholm integral equations and its convergence analysis*, TWMS J. Pure Appl. Math. **10** (2019) 132–139.

[10] K. B. Datta, B. M. Mohan, *Orthogonal Functions in Systems and Control*, World Scientific, 1995.

[11] M. Erfanian, H. Zeidabadi, R. Mehri, *Solving two-dimensional nonlinear volterra integral equations using rationalized haar functions in the complex plane*, Advanced Science, Engineering and Medicine. **12** (2020) 409–415.

[12] Z. K. Eshkuvatov, H. H. Hameed, B. M. Taib, N. M. A. Nik Long, *General 2 2 system of nonlinear integral equations and its approximate solution*, J. Comput. Appl. Math. **361** (2019) 528–546.

[13] M. S. Farahani, M. Hadizadeh, M. V. Bulatov, E. Chistyakova, *Adaptive iterative regularization schemes for twodimensional integralalgebraic systems*, Math. Methods Appl. Sci. **42** (2019) 6635–6647.

[14] S. Hatamzadeh-Varmazyar, Z. Masouri, *Numerical expansion-iterative method for analysis of integral equation models arising in one-and two-dimensional electromagnetic scattering*, Engineering Analysis with Boundary Elements. **36** (2012) 416–422.

[15] C. -H. Hsiao, *Hybrid function method for solving Fredholm and Volterra integral equations of the second kind*, J. Comput. Appl. Math. **230** (2009) 59–68.

[16] K. Holmåker, *Global asymptotic stability for a stationary solution of a system of integro-differential equations describing the formation of liver zones*, SIAM J. Math. Anal. **24** (1993) 116–128.

[17] A. Karimi, K. Maleknejad,R. Ezzati, *Numerical solutions of system of two-dimensional Volterra integral equations via Legendre wavelets and convergence*, Appl. Numer. Math. **156** (2020) 228–241.

[18] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons. Inc., New York, 1989.

[19] E. Ladopoulos, *Singular Integral Equations: Linear and Non-Linear Theory and Its Applications in Science and Engineering*, Springer-Verlag, Berlin, Heidelberg, 2000.

[20] K. Maleknejad, M. S. Dehkordi, *Numerical solutions of two-dimensional nonlinear integral equations via laguerre wavelet method with convergence analysis*, Appl. Math. J. Chinese Univ. Ser. B **36** (2021) 83–98.

[21] K. Maleknejad, A. Hoseingholipour, *The impact of legendre wavelet collocation method on the solutions of nonlinear system of two-dimensional integral equations*, Int. J. Comput. Math. **97** (2020) 2287–2302.

[22] K. Maleknejad, E. Saeedipoor, *Hybrid function method and convergence analysis for two-dimensional nonlinear integral equations*, J. Comput. Appl. Math. **322** (2017) 96-108.

[23] K. Maleknejad , E. Saeedipoor, *Convergence analysis of hybrid functions method for two-dimensional nonlinear volterra-fredholm integral equations*, J. Comput. Appl. Math. **368** (2020) 112533.

[24] K. Maleknejad, M. Shahabi, *Application of hybrid functions operational matrices in the numerical solution of two-dimensional nonlinear integral equations*, Appl. Numer. Math. **136** (2019) 46–65.

[25] K. Maleknejad, S. Sohrabi, B. Baranji, *Application of 2d-bpfs to nonlinear integral equations*, Commun. Nonlinear Sci. Numer. Simul. **15** (2010) 527–535.

[26] M. Matinfar, M. Pourabd, *Modified moving least squares method for two-dimensional linear and nonlinear systems of integral equations*, Comput. Appl. Math. **37** (2018) 5857–5875.

[27] F. Mirzaee, E. Hadadiyan, *Application of two-dimensional hat functions for solving spacetime integral equations*, J. Appl. Math. Comput. **51** (2016) 453-486.

[28] S. Nemati, P. M. Lima, and Y. Ordokhani, *Numerical solution of a class of two-dimensional nonlinear volterra integral equations using legendre polynomials*, J. Comput. Appl. Math. **242** (2013) 53–69.

[29] Y. Rostami , K. Maleknejad, *Franklin wavelet galerkin method (fwgm) for numerical solution of two-dimensional fredholm integral equations*, Mediterr. J. Math. **13** (2016) 4819–4828.

[30] F.H. Shekarabi, K. Maleknejad, R. Ezzati, *Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations*, Afr. Math. **26** (2015) 1237-1251.

[31] G. Vainikko, *Multidimensional Weakly Singular Integral Equations*, Springer-Verlag, Berlin, Heidelberg, 1993.

[32] Y. Wang, S.S. Ezz-Eldien, A.A. Aldraiweesh, *A new algorithm for the solution of nonlinear two-dimensional volterra integro-differential equations of high-order*, J. Comput. Appl. Math. **364** (2020) 112301.

[33] M.A. Zaky, I.G. Ameen, N.A. Elkot, E.H. Doha, *A unified spectral collocation method for nonlinear systems of multi-dimensional integral equations with convergence analysis*, Appl. Numer. Math. **161** (2021) 27–45.