Journal of Mathematical Modeling
Vol. 10, No. 2, 2022. pp. 331-348. Research Article

JMM

Study on the stability for implicit second-order differential equation via integral boundary conditions

Ahmed Mohamed Ahmed El-Sayed[†], **Hind Hassan Gaber Hashem**[‡], **Shorouk Mahmoud Al-Issa**[§*]

[†]*Department of Mathematics, Alexandria University, Alexandria, Egypt*
[‡]*Department of Mathematics, College of Science, Qassim University, P.O. Box 6644 Buraidah 51452, Saudi Arabia*
[§]*Department of Mathematics, Lebanese International University, Saida, Lebanon*
[§]*Department of Mathematics, The International University of Beirut, Beirut, Lebanon*
Email(s): amasayed@alexu.edu.eg,3922@qu.edu.sa, shorouk.alissa@liu.edu.lb

Abstract. In this paper, the existence and the Ulam-Hyers stability of solutions for the implicit second-order differential equations are investigated via fractional-orders integral boundary conditions by direct application of the Banach contraction principle. Finally, we present some particular cases and two examples to illustrate our results.

Keywords: Caputo fractional derivative, second-order fractional-order differential equation, Green's function, boundary value problems, nonlocal boundary conditions.
AMS Subject Classification 2010: 26A33, 34K45, 47G10.

1 Introduction

We discuss a new class of implicit second-order nonlocal boundary value problems as follows

$$\frac{d^2}{dt^2}y(t) = \mathbf{f}\big(t,y(t),{}^cD^{\xi}y(t), \int_0^t \theta(t,s){}^cD^{\zeta}y(s)ds\big), \quad t \in I = (0,1), \tag{1}$$

equipped with the two sets of nonlocal boundary conditions

$$y(0) = 0 \text{ and } y(1) = \frac{\lambda}{\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}L_1(s,y(s))ds + \frac{\mu}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}L_2(s,y(s))ds, \tag{2}$$

$$y(0) = \frac{\lambda}{\Gamma(\gamma)}\int_0^{\tau}(\tau-s)^{\gamma-1}L_1(s,y(s))\,ds, \text{ and } y(1) = \frac{\mu}{\Gamma(\gamma)}\int_0^{\eta}(\eta-s)^{\gamma-1}L_2(s,y(s))\,ds, \tag{3}$$

*Corresponding author.

Received: 20 June 2021/ Revised: 23 August 2021 / Accepted: 29 October 2021
DOI: 10.22124/jmm.2021.19931.1725

where ${}^{c}D^{\delta}$ is Caputo fractional derivative of order $\delta \in \{\xi, \zeta\}$, with $1 < \xi < \zeta \leq 2$, $0 < \gamma < 1$, $\mathbf{f} : I \times \mathbb{R}^3 \to \mathbb{R}$, $\theta : I \times I \to \mathbb{R}$ are given functions satisfying some assumptions that will be specified later, and $\mathbf{L}_i : I \times \mathbb{R} \to \mathbb{R}$, $(i = 1, 2)$ are continuous. This type of boundary value problems appear in many applications and real problems such as chemical diffusion, thermoelasticity, heat conduction processes, population dynamics, vibration problems, nuclear reactor dynamics, inverse problems, control theory, medical science, biochemistry and certain biological processes [3, 9, 10].

The theory of fractional differential equations is a branch of differential equations with a great physical foundation. Many significant and physical applications whose states are subject to rapid change at particular points are modeled using impulsive differential equations (see, for example, the monographs [4, 8] and the references therein).

On the other hand, the stability analysis of integral and differential equations is critical in many applications. Ulam [18] was the first to raise the issue of functional equation stability, followed by Hyers [7]. Rassias [17] improved Ulam-Hyers stability in 1978, resulting in the so-called Ulam-Hyers-Rassias (UHR). For essential results on Ulam stability of integral, we suggest the references [1, 5, 14].

In [6], Hu and Wang investigated the existence of solution of the nonlinear fractional differential equation with integral boundary condition:

$$\begin{aligned} &D^{\alpha}u(t) = f\big(t, u(t), D^{\beta}u(t)\big), \quad t \in (0,1), 1 < \alpha \leq 2,\ 0 < \beta \leq 1, \\ &u(0) = u_{\circ}, \quad u(1) = \int_0^1 g(s)u(s))\, ds, \end{aligned}$$

where D^{α} is the Riemann-Liouville fractional derivative. In [13], Murad and Hadid considered the boundary value problem of the fractional differential equation:

$$\begin{aligned} &D^{\alpha}y(t) = f\big(t, y(t), D^{\beta}y(t)\big), \quad t \in (0,1), 1 < \alpha \leq 2,\ 0 < \beta \leq 1, \\ &y(0) = 0, \quad y(1) = I_0^{\gamma}y(s), \end{aligned}$$

where D^{α} is the Riemann-Liouville fractional derivative, In [3], Chasreechai and Tariboon gave some existence theorems for the positive solutions of problems of the following type:

$$\begin{aligned} &u''(t) + \lambda g(t)\ f(u(t)) = 0, \quad t \in (0,1), \\ &u(0) = \beta_1 \int_0^{\eta}\ u(s)\ ds, \quad u(1) = \beta_2 \int_0^{\eta}\ u(s)\ ds. \end{aligned}$$

Galvis [9] applied the Schauder's fixed point theorem to prove the existence of solutions of the nonlinear second-order problem

$$\begin{gathered} u'' + u(t)g(t,u) = 0, \quad 0 \leq t \leq r, \\ u(0) = 0,\ u(r) = \beta \int_0^a x(s)ds, \end{gathered}$$

Several papers on fractional differential equations with integral boundary conditions can be found in [12, 21–23].

In this paper, motivated by the above works, we consider implicit second-order nonlocal boundary value problem (SNBVP) with integral boundary conditions (1)-(2) and (1)-(3). Moreover, some results are obtained as special cases of the illustrated results in this paper (see Section 5).

The remaining part of the paper is set as follows: In Section 2, we recall some concepts and demonstrate a basic lemma that allows us to convert SNBVP's (1)-(2) and (1)-(3) into their equivalent integral equations. Section 3 establishes the main results, including the existence of a unique solution by applying Banach's contraction mapping principle. Also, we discuss the Ulam-Hyers-Rassias stability of our problem in Section 4. Section 5 contains particular cases of the results and two examples that demonstrate the application of our main results. Finally, in Section 6 some concluding results are given.

2 Preliminaries

Let us starting with some definitions relevant to our study.

Definition 1. *[15] The Riemann-Liouville fractional integral of the function $f \in L^1(I)$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_a^{\alpha} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

and when $a = 0$, we have $I^{\alpha} f(t) = I_0^{\alpha} f(t)$, where $\Gamma(.)$ is Eulers Gamma function.

Definition 2. *[15] The Caputo fractional derivative of order $\alpha > 0$ of function $f \in C^{n-1}([a,b],\mathbb{R}^+)$, with $t \in [a,b]$ is given by*

$$({}^cD_{a+}^{\alpha} f)(t) = I_a^{n-\alpha} \frac{d^n}{dt^n} f(t) = \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s) ds,$$

where $n = [\alpha] + 1$. If $\alpha \in (0,1]$, then

$$({}^cD_{a+}^{\alpha} f)(t) = I_{a^+}^{1-\alpha} \frac{d}{dt} f(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f'(s) ds.$$

Lemma 1. *[24] Let $\beta > 0$. Then the differential equation $({}^cD_{a+}^{\beta} \mu)(t) = 0$ has the solution*

$$\mu(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1}, \quad \alpha_i \in \mathbb{R}, \quad i = 0,1,2,\ldots,n-1, \quad n = [\beta] + 1.$$

Lemma 2. *[24] Let $\alpha > 0$. Then*

$$I^{\alpha}({}^cD^{\beta} \mu(t)) = \mu(t) + \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{n-1} t^{n-1},$$

for arbitrary $\alpha_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\beta] + 1$.

For the existence of a solution for SNBVP's, we need the following auxiliary Lemma.

Lemma 3. *Let $1 < \alpha < \beta \le 2$ and $u \in C(I,R)$. Then, SNBVP (1)-(2) is equivalent to the integral equation*

$$y(t) = \mathbf{L}(t,y(t)) + \int_0^1 H(t,s) u(s) ds, \tag{4}$$

and the SNBVP (1)-(3) is equivalent to the integral equation

$$y(t) = \mathbf{g}(t,y(t)) + \int_0^1 H(t,s) u(s) ds, \tag{5}$$

where u is the solution of the functional integral equation

$$u(t) = \mathbf{f}\left(t, h(t) + \int_0^1 H(t,s)u(s)ds, I^{2-\xi}u(t), \int_0^t \theta(t,s)I^{2-\zeta}u(s)ds\right).$$

Here $H(t,s)$ is the Greens function defined by

$$H(t,s) = \begin{cases} (t-s) + t(1-s), & 0 \le s \le t \le 1, \\ \\ t(1-s), & 0 \le t \le s \le 1, \end{cases} \tag{6}$$

with $H_\circ := \max\{|H(t,s)|, (t,s) \in I \times I\}$, where

$$\mathbf{L}(t, y(t)) = \frac{\lambda\, t}{\Gamma(\gamma)} \int_0^\tau (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu\, t}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} L_2(s, y(s))ds, \tag{7}$$

and

$$\mathbf{g}(t, y(t)) = \frac{(1-t)\,\lambda}{\Gamma(\gamma)} \int_0^\tau (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu\, t}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} L_2(s, y(s))ds. \tag{8}$$

Proof. From the definition of Caputo fractional derivative, we have

$${}^cD^\xi y(t) = I^{2-\xi} \frac{d^2}{dt^2}\, y(t) \ \text{ and } \ {}^cD^\zeta y(t) = I^{2-\zeta} \frac{d^2}{dt^2}\, y(t) \ \text{ for } \ t \in I.$$

Hence, if y is a solution of equation (1), then

$$\frac{d^2}{dt^2} y(t) = \mathbf{f}\left(t, y(t), I^{2-\xi} \frac{d^2}{dt^2}\, y(t), \int_0^t \theta(t,s) I^{2-\zeta} \frac{d^2}{dt^2}\, y(s)ds\right).$$

Letting $\frac{d^2}{dt^2} y(t) = u(t)$, we get

$$u(t) = \mathbf{f}\left(t, y(t), I^{2-\xi}u(t), \int_0^t \theta(t,s)I^{2-\zeta}u(s)ds\right),$$

and Lemma 2 implies that

$$y(t) = \sigma_\circ + \sigma_1 t + \int_0^t (t-s)u(s)ds. \tag{9}$$

By virtue of (2), we get

$$y(0) = \sigma_\circ = 0, \tag{10}$$

$$\begin{aligned} y(1) &= \sigma_\circ + \sigma_1 + \int_0^1 (1-s)u(s)ds \\ &= \frac{\lambda}{\Gamma(\gamma)} \int_0^\tau (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu}{\Gamma(\gamma)} \int_0^\eta (\eta - s)^{\gamma-1} L_2(s, y(s))ds. \end{aligned} \tag{11}$$

By solving Eqs. (10) and (11), it is easily to obtain that

$$\sigma_1 = \frac{\lambda}{\Gamma(\gamma)}\int_0^{\tau} (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu}{\Gamma(\gamma)}\int_0^{\eta} (\eta - s)^{\gamma-1} L_2(s, y(s))ds - \int_0^1 (1-s)u(s)ds.$$

From Eqs. (10) and (11), the solution of the problem (1)-(2) is given by

$$\begin{aligned} y(t) = {} & \frac{\lambda\, t}{\Gamma(\gamma)}\int_0^{\tau} (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu\, t}{\Gamma(\gamma)}\int_0^{\eta} (\eta - s)^{\gamma-1} L_2(s, y(s))ds \\ & - t\int_0^1 (1-s)u(s)ds + \int_0^t (t-s)\, u(s)ds. \end{aligned}$$

Consequently, from the fact that the integral of a function on $[0,1]$ can be written as a sum of the integrals on $[0,t]$ and $t,1$, we get (4).

Conversely, if y satisfies the equation (4) then clearly the problem (1)-(2) holds true. This completes the proof of the equivalence between SNBVP (1)-(2) and the integral equation (4).

By a similar way, the solution of the problem (1)-(3) is given by

$$\begin{aligned} y(t) = {} & \frac{(1-t)\,\lambda}{\Gamma(\gamma)}\int_0^{\tau} (\tau - s)^{\gamma-1} L_1(s, y(s))ds + \frac{\mu\, t}{\Gamma(\gamma)}\int_0^{\eta} (\eta - s)^{\gamma-1} L_2(s, y(s))ds \\ & - t\int_0^1 (1-s)u(s)ds + \int_0^t (t-s)\, u(s)ds. \end{aligned}$$

This completes the proof. □

3 The main result

Consider the following assumptions, in aim of proving the existence of solutions for SNBVP's (1)-(2) and (1)-(3).

($\mathscr{H}_1$) $\mathbf{L}_i : I \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, are continuous and there exist constants $\ell_i \in [0,1)$ such that

$$|\mathbf{L}_i(t,u) - \mathbf{L}_i(t,v)| \le \ell_i |u - v|, \ \ \forall\, u, v \in \mathbb{R}.$$

($\mathscr{H}_2$) $\mathbf{f} : I \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exists $\psi \in C(I, \mathbb{R}_+)$, with norm $\|\psi\|$, such that:

$$|\mathbf{f}(t, \mu_1, \mu_2, \mu_3) - \mathbf{f}(t, \nu_1, \nu_2, \nu_3)| \le \psi(t)\big(|\mu_1 - \nu_1| + |\mu_2 - \nu_2| + |\mu_3 - \nu_3|\big),$$

$\forall\, t \in I,\ \mu_i, \nu_i \in \mathbb{R},\ (i = 1,2,3).$

($\mathscr{H}_3$) $\theta(t,s)$ is continuous for all $(t,s) \in I \times I$, and there is a positive constant Θ such that

$$\max_{t,s \in [0,1]} |\theta(t,s)| = \Theta.$$

Remark 1. *From assumptions ($\mathscr{H}_1$) and ($\mathscr{H}_2$), we have*

$$|L_i(t,\mu)| - |L_i(t,0)| \le |L_i(t,\mu) - L_i(t,0)| \le \ell_i |\mu - 0|,$$

$$|\mathbf{f}(t,\mu_1,\mu_2,\mu_3)| - |\mathbf{f}(t,0,0,0)| \leq |\mathbf{f}(t,\mu_1,\mu_2,\mu_3) - \mathbf{f}(t,0,0,0)| \leq \psi(t)(|\mu_1|+|\mu_2|+|\mu_3|),$$

then

$$|L_i(t,\mu)| \leq \Omega_i + \ell_i|\mu|, \;\; where \;\; \Omega_i = \sup_{t\in I}|l_i(t,0)|, \; i=1,2,$$

and

$$|\mathbf{f}(t,\mu_1,\mu_2,\mu_3)| \leq \|\psi\|(|\mu_1|+|\mu_2|+|\mu_3|) + F, \;\; where \;\; F = \sup_{t\in I}|\mathbf{f}(t,0,0,0)|.$$

Lemma 4. *The function* $\mathbf{L}: I\times\mathbb{R}\to\mathbb{R}$ *satisfies the Lipschitz condition* $\|\mathbf{L}(t,\mu)-\mathbf{L}(t,\nu)\| \leq c\|\mu-\nu\|$.

Proof. For arbitrary $u,v\in\mathbb{R}$ and for each $t\in I$, we have

$$\begin{aligned}|\mathbf{L}(t,\mu(t))-\mathbf{L}(t,\nu(t))| &\leq \Big|\frac{\lambda\, t}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}L_1(s,\mu(s))ds + \frac{\mu\, t}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}L_2(s,\nu(s))ds \\ &\quad - \frac{\lambda\, t}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}L_1(s,\mu(s))ds - \frac{\mu\, t}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}L_2(s,\nu(s))ds\Big| \\ &\leq \frac{\lambda\, t}{\Gamma(\gamma)}\int_0^\tau (\tau-s)^{\gamma-1}\big|L_1(s,\mu(s))-L_1(s,\nu(s))\big|ds \\ &\quad + \frac{\mu\, t}{\Gamma(\gamma)}\int_0^\eta (\eta-s)^{\gamma-1}\big|L_2(s,\mu(s))-L_2(s,\nu(s))\big|ds \\ &\leq \frac{t(\lambda+\mu)}{\Gamma(\gamma+1)}\|\mu-\nu\|.\end{aligned}$$

Therefore, $\|\mathbf{L}(t,\mu)-\mathbf{L}(t,\nu)\| \leq c\|\mu-\nu\|$, where $c=\frac{t(\lambda+\mu)}{\Gamma(\gamma+1)}$. Thus $\mathbf{L}$ is Lipschitzian function with a Lipschitz constant c. In the same way, we can prove that $\mathbf{g}$ is a Lipschitzian function with a the Lipschitz constant c. □

Our result is based on the Banach's fixed point theorem to obtain the existence of a unique solution of SNBVPs (1)-(2) and (1)-(3).

Theorem 1. *Let the assumptions* $(\mathscr{H}_1)-(\mathscr{H}_3)$ *be satisfied. Then the SNBVP (1)-(2) has a unique solution on* $C(I,\mathbb{R})$, *provided that*

$$c + \frac{H_\circ\,\|\psi\|\,T}{\mathscr{M}} < 1, \;\; \mathscr{M} = 1-\|\psi\|\,\Big(\frac{1}{\Gamma(3-\xi)}+\frac{\Theta}{\Gamma(3-\zeta)}\Big). \tag{12}$$

Proof. The SNBVP (1)-(2) can be reduced to a fixed point problem. Define the operator $\mathscr{A}: C(I,\mathbb{R})\to C(I,\mathbb{R})$ by:

$$\mathscr{A}y(t) = \mathbf{L}(t,y(t)) + \int_0^1 H(t,s)v(s)ds, \tag{13}$$

where $v\in C(I,\mathbb{R})$ satisfies the implicit functional equation

$$v(t) = \mathbf{f}\Big(t,y(t),I^{2-\xi}v(t),\int_0^t \theta(t,s)I^{2-\zeta}v(s)ds\Big),$$

with H and $\mathbf{L}$ are the functions defined by (6) and (7), respectively.

We choose

$$\rho \geq \left(\frac{\lambda\ \Omega_1 + \mu\ \Omega_2}{\Gamma(\gamma+1)} + \frac{F\ H_\circ}{\mathscr{M}}\right)\left(1 - \left[\frac{\|\psi\|\ H_\circ}{\mathscr{M}} + \frac{\lambda \ell_1 + \mu \ell_2}{\Gamma(\gamma+1)}\right]\right)^{-1},$$

where $\mathscr{M} = 1 - \|\psi\|\ \left(\frac{1}{\Gamma(3-\xi)} + \frac{\Theta}{\Gamma(3-\zeta)}\right)$. Define the ball $\mathfrak{B}_\rho = \{y \in C(I,\mathbb{R}) : \|y\| \leq \rho\}$. The proof is divided into two steps:

Step 1: We first show that $\mathscr{A}(\mathfrak{B}_\rho) \subset \mathfrak{B}_\rho$. Let $y \in \mathfrak{B}_\rho$ and $t \in I$, we have

$$|\mathscr{A} y(t)| \leq |\mathbf{L}(t,y(t))| + \int_0^1 |H(t,s)||v(s)|ds, \tag{14}$$

where $v(t) = \mathbf{f}\big(t,y(t),I^{2-\xi}v(t),\int_0^t \theta(t,s)I^{2-\zeta}v(s)ds\big)$, and

$$\begin{aligned}
|v(t)| &= \left|\mathbf{f}\big(t,y(t),I^{2-\xi}v(t),\int_0^t \theta(t,s)I^{2-\zeta}v(s)ds\big)\right| \\
&\leq\ F + |\psi(t)|\ \Big(|y(t)| + \int_0^t \frac{(t-s)^{1-\xi}}{\Gamma(2-\xi)}|v(s)|\ ds + \int_0^t |\theta(t,s)| \int_0^s \frac{(s-\tau)^{1-\zeta}}{\Gamma(2-\xi)}|v(\tau)|\ ds\ d\tau\Big) \\
&\leq\ F + \|\psi\|\ \Big(\|y\| + \frac{s^{2-\xi}}{\Gamma(3-\xi)}\|v\| + \Theta\frac{s^{2-\zeta}}{\Gamma(3-\zeta)}\|v\|\Big).
\end{aligned}$$

Hence,

$$\|v\| \leq F + \|\psi\|\ \Big(\|y\| + \frac{\|v\|}{\Gamma(3-\xi)} + \frac{\Theta\ \|v\|}{\Gamma(3-\zeta)}\Big).$$

So

$$\|v\| \leq\ \frac{\rho\ \|\psi\| + F}{\mathscr{M}}.$$

And

$$\begin{aligned}
|\mathbf{L}(t,y(t))| &\leq \frac{\lambda\ t}{\Gamma(\gamma)}\int_0^\tau\ (\tau-s)^{\gamma-1}\ |L_1(s,y(s))|ds + \frac{\mu\ t}{\Gamma(\gamma)}\int_0^\eta\ (\eta-s)^{\gamma-1}\ |L_2(s,y(s))|ds \\
&\leq \frac{\lambda\ t}{\Gamma(\gamma)}\int_0^\tau\ (\tau-s)^{\gamma-1}[\Omega_1 + \ell_1|y(s)|]ds + \frac{\mu\ t}{\Gamma(\gamma)}\int_0^\eta\ (\eta-s)^{\gamma-1}[\Omega_2 + \ell_2|y(s)|]ds \\
&\leq \frac{\lambda\ t[\Omega_1 + \ell_1\|y\|]}{\Gamma(\gamma+1)} + \frac{\mu\ t[\Omega_2 + \ell_2\|y\|]}{\Gamma(\gamma+1)} \\
&\leq \frac{\lambda\ \Omega_1 + \mu\ \Omega_2 + \rho[\lambda\ell_1 + \mu\ell_2]}{\Gamma(\gamma+1)}.
\end{aligned}$$

Hence (14) implies that

$$|\mathscr{A} y(t)| \leq \frac{\lambda\ \Omega_1 + \mu\ \Omega_2 + \rho[\lambda\ell_1 + \mu\ell_2]}{\Gamma(\gamma+1)} + \frac{(\rho\ \|\psi\| + F)H_\circ}{\mathscr{M}} \leq\ \rho,$$

for each $t \in I$. Taking supremum over $t \in I$, we have $\|\mathscr{A} y\| \leq\ \rho$. This proves that $\mathscr{A} y \in \mathfrak{B}_\rho$ for every $y \in \mathfrak{B}_\rho$.

Step 2: We only need to show that the operator $\mathscr{A}$ which is defined by (15) is a contraction. Taking $x, y \in C(I, \mathbb{R})$, for $t \in I$, we have

$$\mathscr{A}x(t) - \mathscr{A}y(t) = \mathbf{L}(t, x(t)) + \int_0^1 H(t,s)u(s)ds - \mathbf{L}(t, y(t)) - \int_0^1 H(t,s)v(s)ds, \tag{15}$$

where $u, v \in C(I, \mathbb{R})$ such that

$$u(t) = \mathbf{f}\big(t, x(t), I^{2-\xi}u(t), \int_0^t \theta(t,s) I^{2-\zeta}\, u(s)ds\big),$$
$$v(t) = \mathbf{f}\big(t, y(t), I^{2-\xi}v(t), \int_0^t \theta(t,s) I^{2-\zeta}\, v(s)ds\big).$$

Then, for $t \in I$

$$|\mathscr{A}x(t) - \mathscr{A}y(t)| \leq\ |\mathbf{L}(t, x(t)) - \mathbf{L}(t, y(t))| + \int_0^1\ H(t,s)\ |u(s) - v(s)|ds, \tag{16}$$

but by assumption $(\mathscr{H}_2)$, we have

$$\begin{aligned}
&|v_n(t) - v(t)| \\
&= \big|f\big(t, x(t), I^{2-\xi}u(t), \int_0^t \theta(t,s) I^{2-\zeta}u(t)ds\big) - f\big(t, y(t), I^{2-\xi}v(t), \int_0^t \theta(t,s) I^{2-\zeta}v(s)ds\big)\big| \\
&\leq \psi(t)\Bigg(|x(t) - y(t)| + \int_0^t \frac{(t-s)^{1-\xi}}{\Gamma(2-\xi)}|u(s) - v(s)|ds + \int_0^t \theta(t,s) \int_0^\tau \frac{(s-\tau)^{1-\zeta}}{\Gamma(2-\zeta)}|u(\tau) - v(\tau)|dsd\tau\Bigg) \\
&\leq\ |\psi(t)|\ \big(\|x-y\| + \frac{s^{2-\xi}}{\Gamma(3-\xi)}\|u-v\| + \Theta\frac{s^{3-\zeta}}{\Gamma(3-\alpha)}\|u-v\|\ \big) \\
&\leq\ \|\psi\|\ \big(\|x-y\| + \frac{\|u-v\|}{\Gamma(3-\xi)} + \frac{\Theta\ \|u-v\|}{\Gamma(3-\zeta)}\ \big).
\end{aligned}$$

Thus

$$\|u-v\| \leq \frac{\|\psi\|}{\mathscr{M}}\|x-y\|.$$

From (16) and by Lemma 4, we have

$$|\mathscr{A}x(t) - \mathscr{A}y(t)| \leq\ c\|x-y\| + \frac{H_\circ\ |\psi\|}{\mathscr{M}}\|x-y\| \leq \big(c + \frac{H_\circ\ \|\psi\|}{\mathscr{M}}\big)\|x-y\|.$$

Taking supermum for $t \in I$, we have

$$\|\mathscr{A}x - \mathscr{A}y\| \leq \big(c + \frac{H_\circ\ \|\psi\|}{\mathscr{M}}\big)\|x-y\|.$$

Now if $\big(c + \frac{G_\circ\ \|\psi\|}{\mathscr{M}}\big) < 1$, then the operator $\mathscr{A}$ is contraction. Hence, by Banach's contraction principle, $\mathscr{A}$ has a unique fixed point which is a solution of the SNBVP (1)-(2) on I. □

By a similar way as done above we can prove the following theorem.

Theorem 2. *Let the assumptions $(\mathscr{H}_1) - (\mathscr{H}_3)$ be satisfied, if condition (12) holds. Then the SNBVP (1)-(3) has a unique continuous solution.*

4 Stability of solutions

Now, we study the Ulam stability for SNBVP's (1)-(2) and (1)-(3). Let $\varepsilon > 0$ and $\Phi : I \to \mathbb{R}_+$ be a continuous function. We consider the following inequalities:

$$\left| \frac{d^2}{dt^2} y(t) - \mathbf{f}\big(t, y(t), {}^c D^{\xi} y(t), \int_0^t \theta(t,s) {}^c D^{\zeta} y(s) ds\big) \right| \leq \varepsilon, \quad t \in I \tag{17}$$

$$\left| \frac{d^2}{dt^2} y(t) - \mathbf{f}\big(t, y(t), {}^c D^{\xi} y(t), \int_0^t \theta(t,s) {}^c D^{\zeta} y(s) ds\big) \right| \leq \Phi(t), \; t \in I \tag{18}$$

$$\left| \frac{d^2}{dt^2} y(t) - \mathbf{f}\big(t, y(t), {}^c D^{\xi} y(t), \int_0^t \theta(t,s) {}^c D^{\zeta} y(s) ds\big) \right| \leq \varepsilon \, \Phi(t), \quad t \in I. \tag{19}$$

Definition 3. *[7] The SNBVPs (1)-(2) and (1)-(3) are Ulam−Hyers stable if there exists a real number $c_f > 0$ such that there exists a solution $x \in C(I,\mathbb{R})$ of (1)-(2) and (1),-(3), respectively, stasfying*

$$|y(t) - x(t)| \leq \varepsilon \, c_f, \;\; t \in I,$$

for each solution $y \in C(I,\mathbb{R})$ of the inequality (17).

Definition 4. *[7] The SNBVPs (1)-(2) and (1)-(3) are generalized Ulam−Hyers stable if there is $c_f \in C(\mathbb{R}_+,\mathbb{R}_+)$ with $c_f(0) = 0$ so that there is a solution $x \in C(I,\mathbb{R})$ of (1)-(2) and (1)-(3) respectively, satisfying $|y(t) - x(t)| \leq c_f(\varepsilon)$, $t \in I$, for each $\varepsilon > 0$ and for each solution $y \in C(I,\mathbb{R})$ of the inequality (17).*

Definition 5. *[7] The SNBVPs (1)-(2) and (1)-(3) are Ulam−Hyers−Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that there is a solution $x \in C(I,\mathbb{R})$ of (1)-(2) and (1)-(3) with*

$$|y(t) - x(t)| \leq \varepsilon \, c_{f,\Phi} \Phi(t), \;\; t \in I.$$

for each $\varepsilon > 0$ and for each solution $y \in C(I,\mathbb{R})$ of the inequality (19).

Definition 6. *[7] The SNBVPs (1)-(2) and (1)-(3) are generalized Ulam−Hyers−Rassias stable with respect to Φ if the actual number $c_{f,\Phi} > 0$ exists in such a way that for each solution $y \in C(I,\mathbb{R})$ of the inequality (18) there is a solution $x \in C(I,\mathbb{R})$ of (1)-(2) and (1)-(3) with the solution $x \in C(I,\mathbb{R})$ of the inequality $|y(t) - x(t)| \leq c_{f,\Phi}\Phi(t)$, $t \in I$.*

4.1 Ulam-Hyers Stability

Next, we present the following Ulam−Hyers stable result.

Theorem 3. *Let the assumptions of Theorem 1 be satisfied. Then SNBVP (1)-(2) is Ulam−Hyers stable.*

Proof. Let $\varepsilon > 0$ and let $z \in C(I, \mathbb{R})$ be a function which satisfies the inequality (17), i.e.,

$$\left| \frac{d^2}{dt^2} z(t) - \mathbf{f}\big(t, z(t), {}^cD^{\xi} z(t), \int_0^t \theta(t,s) {}^cD^{\zeta} z(s) ds\big) \right| \leq \varepsilon, \quad t \in I,$$

and let $y \in C(I, \mathbb{R})$ be the unique solution of SNBVP (1)-(2), which is by Lemma 3, the unique solution of integral equation

$$y(t) = \mathbf{L}(t, y(t)) + \int_0^1 H(t,s) u(s) ds,$$

where u is the solution of the functional integral equation

$$u(t) = \mathbf{f}\big(t, y(t), I^{2-\xi} u(t), \int_0^t \theta(t,s) I^{2-\zeta} u(s) ds\big).$$

Operating by I^2 on both sides of (17), and then integrating, we get

$$\left| z(t) - \mathbf{L}(t, z(t)) - \int_0^1 H(t,s) v(s) ds \right| \leq \frac{\varepsilon}{2}. \tag{20}$$

For each $t \in I$, we have

$$\begin{aligned}
|z(t) - y(t)| &= \big|z(t) - \mathbf{L}(t, y(t) - \int_0^1 H(t,s) u(s) ds\big| \\
&\leq \big|z(t) - \mathbf{L}(t, z(t) + \int_0^1 H(t,s) v(s) ds\big| \\
&\quad + \big|\mathbf{L}(t, z(t) + \int_0^1 H(t,s) v(s) ds - \mathbf{L}(t, y(t) - \int_0^1 H(t,s) u(s) ds\big| \\
&\leq \frac{\varepsilon}{2} + \big|\mathbf{L}(t, z(t)) - \mathbf{L}(t, y(t))\big| + \int_0^1 H(t,s) |v(s) - u(s)| ds \\
&\leq \frac{\varepsilon}{2} + c\, |z(t) - y(t)| + \int_0^1 H(t,s) |v(s) - u(s)| ds \\
&\leq \frac{\varepsilon}{2} + c\, \|z - y\| + H_{\circ} \|u - v\|.
\end{aligned}$$

Indeed, from proof of Theorem 1, we have

$$\|u - v\| \leq \frac{\|\psi\|}{1 - \|\psi\| \big(\Theta \frac{1}{\Gamma(3-\zeta)} + \frac{1}{\Gamma(3-\xi)}\big)} \|z - y\|.$$

Then, for each $t \in I$

$$\|z - y\| \leq \frac{\varepsilon}{2} + c\, \|z - y\| + \frac{H_{\circ}\, \|\psi\|}{1 - \|\psi\| \big(\Theta \frac{1}{\Gamma(3-\alpha)} + \frac{1}{\Gamma(3-\xi)}\big)} \|z - y\|.$$

Thus

$$\|z - y\| \quad \leq \quad \frac{\varepsilon}{2} \left[1 - \left(c + \frac{H_{\circ}\, \|\psi\|}{1 - \|\psi\| \big(\Theta \frac{1}{\Gamma(3-\zeta)} + \frac{1}{\Gamma(3-\xi)}\big)} \right) \right]^{-1} = \Lambda\, \varepsilon,$$

where

$$\Lambda = \frac{1}{2}\left[1-\left(c+\frac{H_\circ\,\|\psi\|}{1-\|\psi\|\big(\Theta\,\frac{1}{\Gamma(3-\zeta)}+\frac{1}{\Gamma(3-\xi)}\big)}\right)\right]^{-1}.$$

Then, SNBVP (1)-(2) is Ulam-Hyers stable. By putting $\Phi(\varepsilon)=\Lambda\,\varepsilon$, $\Phi(0)=0$, we deduce that SNBVP (1)-(2) is generalized Ulam-Hyers stable. □

By similar calculation as done before, we can prove the following theorem.

Theorem 4. *Let the assumptions of Theorem 1 be satisfied. Then SNBVP (1)-(3) is generalized Ulam-Hyers stable.*

4.2 Ulam-Hyers-Rassias Stability

In aim of proving some Ulam−Hyers−Rassias stability results, we consider the following assumption.

($\mathscr{H}_4$) The function $\Phi \in C(I,\mathbb{R}_+)$ is increasing and there exists $\lambda_\Phi > 0$ such that, for each $t \in J$, we have $I^2\,\Phi(t) \le \lambda_\Phi\,\Phi(t)$.

Theorem 5. *Let the assumptions $(\mathscr{H}_1)-(\mathscr{H}_3)$ and $(\mathscr{H}_4)$ be satisfied. Then SBVP (1)-(2) is Ulam-Hyers-Rassias stable with respect to Φ.*

Proof. Let $z \in C(I,\mathbb{R})$ be a solution of the inequality (19), i.e.,

$$\left|\frac{d^2}{dt^2}z(t)-\mathbf{f}\big(t,z(t),{}^cD^{\xi}z(t),\int_0^t\theta(t,s){}^cD^{\zeta}z(s)ds\big)\right| \le \varepsilon\,\Phi,\quad t\in I$$

and let us assume that y is a solution of the problem (1)−(2). Thus, we have

$$y(t)=\mathbf{L}(t,y(t))+\int_0^1 H(t,s)u(s)ds,$$

where $u \in C(I,\mathbb{R})$ such that

$$u(t)=\mathbf{f}\big(t,y(t),I^{2-\xi}u(t),\int_0^t\theta(t,s)I^{2-\zeta}u(s)ds\big).$$

Operating by I^2 on both sides of the inequality (19) and then integrating, we get

$$\left|z(t)-\mathbf{L}(t,z(t))-\int_0^1 H(t,s)v(s)ds\right| \le \varepsilon\int_0^t (t-s)\,\Phi(s)ds \le \varepsilon\lambda_\Phi\Phi(t),$$

where $v \in C(I,\mathbb{R})$ such that

$$v(t)=\mathbf{f}\big(t,z(t),I^{2-\xi}v(t),\int_0^t\theta(t,s)I^{2-\zeta}v(s)ds\big).$$

For each $t \in I$, we have

$$\begin{aligned}
|z(t)-y(t)| &= \left|z(t)-\left|\mathbf{f}(t,y(t))+\int_0^1 H(t,s)u(s)ds\right|\right. \\
&\leq \left|z(t)-\mathbf{L}(t,z(t))+\int_0^1 H(t,s)v(s)ds\right| \\
&\quad +\left|\mathbf{L}(t,z(t))+\int_0^1 H(t,s)v(s)ds-\mathbf{L}(t,y(t))-\int_0^1 H(t,s)u(s)ds\right| \\
&\leq \varepsilon\, \lambda_\Phi\, \Phi(t)+\left|\mathbf{L}(t,z(t))-\mathbf{L}(t,y(t))\right|+\int_0^1 H(t,s)|v(s)-u(s)|ds \\
&\leq \varepsilon\, \lambda_\Phi\, \Phi(t)+c\,|z(t))-y(t)|+\int_0^1 H(t,s)|v(s)-u(s)|ds \\
&\leq \varepsilon\, \lambda_\Phi\, \Phi(t)+c\,\|z-y\|+H_\circ\|v-u\|\,T.
\end{aligned}$$

Indeed, from proof of Theorem 1, we have

$$\|u-v\| \leq \frac{\|\psi\|}{1-\|\psi\|\left(\Theta\,\frac{1}{\Gamma(3-\zeta)}+\frac{1}{\Gamma(3-\xi)}\right)}\|z-y\|.$$

Then, for each $t \in I$

$$\|z-y\| \leq \varepsilon\, \lambda_\Phi\, \Phi(t)+c\,\|z-y\|+\frac{H_\circ\,\|\psi\|}{1-\|\psi\|\left(\Theta\,\frac{1}{\Gamma(3-\zeta)}+\frac{1}{\Gamma(3-\xi)}\right)}\|z-y\|.$$

Thus

$$\|z-y\| \leq \left[1-\left(c+\frac{H_\circ\,\|\psi\|}{1-\|\psi\|\left(\Theta\,\frac{1}{\Gamma(3-\zeta)}+\frac{1}{\Gamma(3-\xi)}\right)}\right)\right]\varepsilon\, \lambda_\Phi\, \Phi(t)=c_\Phi\varepsilon\,\Phi(t),$$

where

$$c_\Phi=\left[1-\left(c+\frac{H_\circ\,\|\psi\|}{1-\|\psi\|\left(\Theta\,\frac{1}{\Gamma(3-\zeta)}+\frac{1}{\Gamma(3-\xi)}\right)}\right)\right]\lambda_\Phi.$$

Then, the problem SNBVP (1)-(2) is Ulam-Hyers-Rassias stable with respect to Φ. □

In a similar way, we can prove the following theorem.

Theorem 6. *Assume that the assumptions* $(\mathscr{H}_1)-(\mathscr{H}_3)$ *and the condition* $(\mathscr{H}_4)$ *are satisfied. Then SNBVP (1)-(3) is Ulam-Hyers-Rassias stable with respect to* Φ.

5 Particular cases and examples

In this section, we introduce some existence results for some boundary value problems as particular cases of our main result.

- By taking $B_1(s)=\frac{(\tau-s)^\gamma}{\Gamma(\gamma+1)}$ and $B_2(s)=\frac{(\eta-s)^\gamma}{\Gamma(\gamma+1)}$, we obtain second order nonlocal boundary value problems with nonlocal boundary conditions involving Riemann-Stieltjes integrals

$$\frac{d^2}{dt^2}y(t)=\mathbf{f}\big(t,y(t),{}^cD^\xi y(t),\int_0^t \theta(t,s)\,{}^cD^\zeta y(s)ds\big),\quad t\in(0,1),$$

equipped with the two sets of boundary conditions

$$y(0) = 0 \text{ and } y(1) = \lambda \int_0^\tau L_1(s,y(s))dB_1(s) + \mu \int_0^\eta L_2(s,y(s))dB_2(s),$$
$$y(0) = \lambda \int_0^\tau L_1(s,y(s))\, dB_1(s), \text{ and } y(1) = \mu \int_0^\eta L_2(s,y(s))\, dB_2(s).$$

This type of boundary conditions are studied in many papers, for example [19].

- Letting $\zeta \to 2$, $\theta(t,s) = 1$, $\lambda = 0$, $\mu = 1$, $L_2(s,y) = y$, $\gamma \to 0$ and

$$f(t,y(t),u(t),v(t)) = f(t,y(t),y'(t)) - e(t),$$

we obtain the following three points boundary value problem

$$\frac{d^2}{dt^2}y(t) = \mathbf{f}\big(t,y(t),y'(t)\big) - e(t), \quad t \in (0,1)$$
$$y(0) = 0 \text{ and } y(1) = y(\eta),$$

which is studied in [10]. In the case of $f(t,y(t),u(t)) = p_0(t) + p_1(t)y + p_2(t)y'$, where $p_k : (0,1) \to R$, $k = 0,1,2$ are locally integrable, see [11].

- Taking $\gamma \to 0$, $\lambda = 0$, $L_1(s,y(s)) = L_2(s,y(s)) = y(s)$, $f(t,y(t),u(t),v(t)) = -a(t)f(y(t))$, we obtain

$$\frac{d^2}{dt^2}y(t) + a(t)f(y(t)) = 0, \quad t \in (0,1)$$
$$y(0) = 0 \text{ and } y(1) = \mu y(\eta),$$

which is studied in [16].

- Letting $\lambda = 0$, $\mu = \eta = 1$, $L_2(s,y) = g(s)y(s)$, $\gamma \to 1$ and $f(t,y(t),u(t),v(t)) = -f(t,y(t))$, we get the following two points boundary value problem

$$\frac{d^2}{dt^2}y(t) = -\mathbf{f}(t,y(t)), \quad t \in (0,1)$$
$$y(0) = 0 \text{ and } y(1) = \int_0^1 g(s)y(s)ds,$$

which is studied in [2].

- Letting $\gamma \to 1$, $\tau = \eta$, $f(t,y(t),u(t),v(t)) = -\lambda_1 g(t)f(y(t))$, $L_1(s,y(s)) = L_2(s,y(s)) = y(s)$, we obtain a nonlocal value problem with integral condition

$$\frac{d^2}{dt^2}y(t) + \lambda_1 g(t)f(y(t)) = 0, \quad t \in (0,1)$$
$$y(0) = \lambda \int_0^\eta y(s)ds \text{ and } y(1) = \mu \int_0^\eta y(s)ds,$$

which is studied in [3].

- Letting $\gamma \to 1$, $\tau = \eta = 1$, $f(t, y(t), u(t), v(t)) = -f(t, y) - \omega^2 y(t)$, $L_1(s, y(s)) = L_2(s, y(s)) = sy(s)$, then we obtain the following boundary value problem [20]

$$\frac{d^2}{dt^2}y(t) + \omega^2 y(t) = -f(t, y(t)), \quad t \in (0,1)$$
$$y(0) = \lambda \int_0^1 s\, y(s)ds \text{ and } y(1) = \mu \int_0^1 s\, y(s)ds,$$

Finally, we give some examples to support the main results.

Example 1. Consider the following SNBVP:

$$\frac{d^2}{dt^2}y(t) = \frac{e^{-t}}{e^t+8}\left(\frac{|y(t)|}{1+|y(t)|} - \frac{|{}^cD^{\frac{4}{3}}y(t)|}{1+|{}^cD^{\frac{4}{3}}y(t)|} - \frac{|\int_0^1 \ln(t+s){}^cD^{\frac{5}{3}}y(t)|}{1+|\int_0^1 \ln(t+s){}^cD^{\frac{5}{3}}y(t)|}\right), \tag{21}$$

$$y(0) = 0,\ y(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{3}{4}} (\frac{3}{4}-s)^{\frac{1}{2}-1}\frac{\cos y(s)}{20}ds + \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{4}} (\frac{1}{4}-s)^{\frac{1}{2}-1}\frac{e^{-y(s)}}{30}ds. \tag{22}$$

Set

$$\mathbf{f}(t, \mu, \nu, w) = \frac{e^{-t}}{e^t+8}\left(\frac{|\mu(t)|}{1+|\mu(t)|} - \frac{|\nu(t)|}{1+|\nu(t)|} - \frac{|w(t)|}{1+|w(t)|}\right),$$

Clearly, the function $\mathbf{f}$ is continuous. In fact, for any $\mu_i.\nu_i, \omega_i, \in \mathbb{R}$ $(i = 1,2)$ and $t \in (0,1)$

$$\begin{aligned} |\mathbf{f}(t,\mu_1,\nu_1,w_1) - \mathbf{f}(t,\mu_2,\nu_2,w_2)| &\leq \frac{e^{-t}}{e^t+8}\big(|\mu_1-\mu_2| + |\nu_1-\nu_2| + |w_1-w_2|\big) \\ &\leq \frac{1}{9}\big(|\mu_1-\mu_2 + |\nu_1-\nu_2| + |w_1-w_2|\big). \end{aligned}$$

Hence the condition $(\mathscr{H}_2)$ holds with $\|\psi\| = \frac{1}{9}$.

Set $L_1(t,x(t)) = \frac{\cos x(t)}{20}$ and $L_2(t,x(t)) = \frac{e^{-x(t)}}{30}$. We can easily verify the condition $(\mathscr{H}_1)$ with $\ell_1 = \frac{1}{20}$ and $\ell_2 = \frac{1}{30}$. On the other hand, we have

$$\begin{aligned} |\mathbf{L}(t,x(t)) - \mathbf{L}(t,y(t))| \leq{}& \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{3}{4}}(\frac{3}{4}-s)^{\frac{1}{2}}\frac{|\cos x(s) - \cos y(s)|}{20}\,ds \\ &+ \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{4}}(\frac{1}{4}-s)^{\frac{1}{2}}\frac{|e^{-x(s)} - e^{-y(s)}|}{30}\,ds \\ \leq{}& 0.0258587\,|x(s) - y(s)|. \end{aligned}$$

This inequality shows that Lemma 4 holds, which means that $\mathbf{L}$ is Lipschitz with constant $c = 0.0258587$.

Clearly $H_0 < 2$. We shall check condition (12)

$$\big(c + \frac{H_\circ\,\|\psi\|}{\mathscr{M}}\big) \approx 0.3069135234 < 1,$$

where

$$\mathscr{M} = 1 - \|\psi\|\,\big(\frac{1}{\Gamma(\frac{5}{3})} + \frac{\Theta}{\Gamma(\frac{4}{3})}\big) = 0.7906721526,$$

which satisfied with $\xi = \frac{4}{3}$, $\zeta = \frac{5}{3}$, $c = 0.0258587$, $\|\psi\| = \frac{1}{9}$, and $\Theta = \ln(2)$. It follows from Theorem 1 that the SNBVP (21)-(22) has a unique solution on I.

Also, assumption $(\mathscr{H}_4)$ is satisfied with $\Phi(t) = e^2$, and $\lambda_\Phi = \frac{2}{\sqrt{\pi}}$. Indeed, for each $t \in (0,1)$, we get

$$I^2\,\Phi(t) \le \frac{e^2\,2}{\sqrt{\pi}} = \lambda_\Phi\,\Phi(t).$$

As a result, Theorem 5 implies that the SNBVP (21)-(22) is generalized Ulam-Hyers-Rassias stable.

Example 2. Consider the following SNBVP:

$$\frac{d^2}{dt^2}y(t) = \frac{2 + y(t) + {}^cD^{\frac{4}{3}}y(t) + \int_0^1 e^{t-s}\,{}^cD^{\frac{3}{2}}y(s)ds}{2e^{t+1}\big(1 + y(t) + {}^cD^{\frac{4}{3}}y(t) + \int_0^1 e^{t-s}\,{}^cD^{\frac{3}{2}}y(s)ds\big)}, \tag{23}$$

with

$$y(0) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{3}} (\frac{1}{3} - s)^{\frac{1}{2}} \frac{e^{s-2}\,|y|}{20(1+|y|)}\,ds, \text{ and } y(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}} \frac{\sin y}{40(s+3)}\,ds. \tag{24}$$

Set

$$\mathbf{f}(t,\mu,\nu,\omega) = \frac{2 + |\mu| + |\nu| + |\omega|}{2e^{t+1}\big(1 + |\mu| + |\nu| + |\omega|\big)}.$$

Clearly, the function $\mathbf{f}$ is continuous. In fact, for any $\mu_1.\nu_1,\omega_1,u_2,\nu_2,\omega_2 \in \mathbb{R}$ and $t \in (0,1)$

$$|\mathbf{f}(t,\mu_1,\nu_1,\omega_1) - \mathbf{f}(t,\mu_2,\nu_2,\omega_2)| \le \frac{1}{2e^2}\big(|\mu_1 - \mu_2| + |\nu_1 - \nu_2| + |\omega_1 - \omega_2|\big).$$

Hence the condition $(\mathscr{H}_2)$ holds with with $\psi(t) = \frac{1}{2e^{t+1}}$. Also, we have,

$$|\mathbf{f}(t,\mu,\nu,\omega)| = \frac{1}{2e^{t+1}}\big(2 + |\mu| + |\nu| + |\omega|\big),$$

where $\mathbf{f}(t,0,0,0) = \frac{1}{e^{t+1}}$, and $\|\psi\| = \frac{1}{2e^2}$.

Set $L_1(t,x(t)) = \frac{e^{t-2}|x|}{20(1+|x|)}$, and $L_2(t,x(t)) = \frac{\sin x}{40(t+3)}$. We can easily verify the condition $(\mathscr{H}_1)$ with $\ell_1 = \frac{1}{20e}$, and $\ell_2 = \frac{1}{40}$. On the other hand, we have

$$\begin{aligned} |\mathbf{g}(t,x(t)) - \mathbf{g}(t,y(t))| \le\ & \frac{1}{2\,\Gamma(\frac{1}{2})}\int_0^{\frac{1}{3}} (\frac{1}{3} - s)^{\frac{1}{2}}\,\Big|\frac{|x|}{20e^{-t+2}(1+|x|)} - \frac{|y|}{20e^{-t+2}(1+|y|)}\Big|\,ds \\ & + \frac{1}{2\,\Gamma(\frac{1}{2})}\int_0^{\frac{1}{4}} (\frac{1}{4} - s)^{\frac{1}{2}}\,\Big|\frac{\sin x}{40(t+3)} - \frac{\sin y}{40(t+3)}\Big|\,ds \\ \le\ & 0.01292934462\,|x(t) - y(t)|. \end{aligned}$$

It follows from the above inequality that Lemma 4 holds, which means that $\mathbf{g}$ is Lipschitz with constant $c = 0.01292934462$.

Clearly $H_0 < 2$. We shall check condition (12)

$$\big(c + \frac{H_\circ \, \|\psi\|}{\mathscr{M}}\big) \approx 0.1069266327 < 1$$

where

$$\mathscr{M} = 1 - \|\psi\| \, \big(\frac{1}{\Gamma(\frac{5}{3})} + \frac{\Theta}{\Gamma(\frac{3}{2})}\big) = 0.7198892968$$

which is satisfied with $\xi = \frac{4}{3}$, $\zeta = \frac{3}{2}$, $c = 0.01292934462$, $\|\psi\| = \frac{1}{2e^2}$, and $\Theta = e$. It follows from Theorem 2 that the the SNBVP (21)-(22) has a unique solution on I.

6 Conclusion

We have investigated existence and uniqueness of the solutions as well as the HyersUlam stability for some boundary value problems with integral boundary conditions. The given problems were converted into an analogous fixed point problem, which was solved using typical functional analysis tools to get uniqueness results for the original problem. Finally, we have provided some examples to demonstrate the validity of our theoretical conclusions and by setting the parameters involved in the integral boundary conditions, our results lead to some additional results as special cases which were studied before. We believe that the proposed boundary value problems are general, and that it may be applied to a wide range of fractional dynamical equations as particular cases in physics and other applied sciences.

Acknowledgements

The authors thank the editors and the reviewers for their useful comments.

References

[1] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967) 197–228.

[2] M.K. Benchohra, J. Nieto, A. Ouahab, *Second-Order Boundary Value Problem with Integral Boundary Conditions*, Bound. Value Probl. **2011** (2011) 260309.

[3] S. Chasreechai, J. Tariboon, *Positive solutions to generalized second-order three-point integral boundary-value problems*, Electron. J. Diff. Equ. **2011** (2011) 1–14.

[4] A.M.A. El-Sayed, H.M.H. Hashem, Sh.M. Al-Issa, *Existence of solutions for an ordinary second-order hybrid functional differential equation*, Adv Differ Equ. **2021** (2020) 296.

[5] A.M.A. El-Sayed, Sh.M. Al-Issa, M. Elmiari, *Ulam-type Stability for a Boundary Value Problem of Implicit Fractional-orders Differential Equation,* Adv. Dyn. Syst. Appl. **16** (2021) 75–89.

[6] M. Hu, L.Wang, *Existence of solutions for a nonlinear fractional differential equation with integral boundary condition*, Int. J. Math. Comput. Sci. **5** (2011) 55–58.

[7] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. **27** (1941) 222–224.

[8] F. Gaafar, *The existence of solutions for a nonlocal problem of an implicit fractional-order differential equation*, J. Egyptian Math. Soc. **26** (2018) 184–195.

[9] J. Galvis, E.M. Rojas, A.V. Sinitsyn, *Existence of positive solutions of a nonlinear second-order boundary-value problem with integral boundary conditions,* Electron. J. Differential Equations **2015** (2015) 1–7.

[10] C.P. Gupta, *Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation*, J. Math. Anal. Appl. **168** (1992) 540–551.

[11] I.T. Kiguradze, A.G. Lomtatidze, *On certain boundary value problems for secondorder linear ordinary differential equations with singularities*, J. Math. Anal. Appl. **101** (1984) 325–347.

[12] O. KyuNam, K. Jong, S. Pak, H. Choi, *A new approach to approximate solutions for a class of nonlinear multi-term fractional differential equations with integral boundary conditions*, Adv. Difference Equ. **2020** (2020) 271.

[13] S.A. Murad, S. Hadid, *An existence and uniqueness theorem for fractional differential equation with integral boundary condition*, J. Frac. Calc. Appl. **3** (2012) 1–9.

[14] M. Obloza, *Hyers stability of the linear differential equation*, Rocznik Nauk-Dydakt Prace Mat. **13** (1993) 259–270.

[15] I. Podlubny, A.M.A. EL-Sayed, *On two definitions of fractional calculus*, Preprint UEF 03-69 (ISBN 80-7099-252-2), Solvak Academy of science-Institute of Experimental Physics, 1996.

[16] R. Ma, *Positive solutions of a nonlinear three-point boundary-value problem*, Electron. J. Differential Equations. **1999** (1999) 1-8.

[17] T.M. Rassias, *On the stability of the linear mapping in Banach spaces*. Proc. Amer. Math. Soc. **72** (1978) 297–300.

[18] S.M. Ulam, *A Collection of Mathematical Problems*, New York: Interscience, 1968.

[19] J.R.L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc. **74** (2006) 673–693.

[20] J.R.L. Webb, *Nonexistence of positive solutions of nonlinear boundary value problems*, Electron. J. Qual. Theory Differ. Equ. **2012** (2012) 1–21.

[21] H.M. Srivastava, A.M.A. El-Sayed, H.H.G. Hashem, Sh.M. Al-Issa, *Analytical investigation of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders*, Rev. Real Acad. Cienc. Exactas Fis. Nat. -A: Mat. **2022** (2022) 116.

[22] Y. Sun, M. Zhao, *Positive solutions for a class of fractional differential equations with integral boundary conditions,* Appl. Math. Lett. **34** (2014) 1721.

[23] X.G. Zhang, L.S. Liu, B. Wiwatanapataphee, Y.H. Wu, *The eigenvalue for a class of singular p -Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary condition,* Appl. Comput. Math. **235** (2014) 412422.

[24] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional differential equations*, Electron. J. Differential Equations, **36** (2006) 1-12.