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Combination of Sinc and radial basis functions for time-space fractional diffusion equations

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Abstract. We study the combination of the Sinc and the Gaussian radial basis functions (GRBF) to develop the numerical methods for the time–space fractional diffusion equations with the Riesz fractional derivative. The GRBF is used to approximate the unknown function in spatial direction and the Sinc quadrature rule associated with double exponential transformation is applied to approximate the arising integrals. Three practical examples are considered for testing the ability of the proposed method.

Keywords: Fractional diffusion equations, Sinc method, double exponential transformation, Gaussian radial basis functions.
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1 Introduction

Fractional differentials interpret the mathematical models that arise in science and engineering more accurately and naturally. Theory of fractional calculus can be studied in [15,30,34]. Several textbooks have studied various models of fractional equations arising in many fields of science and engineering, such as chemistry [37], physics [13] and continuum mechanics [6]. Also several articles have studied various models of fractional equations in science and engineering, for example one can see chaotic behavior of fractional predator-prey dynamical system in [21], the behaviour of immune and tumor cells in immunogenetic tumor model with non-singular fractional derivative [12, 22] and the Boussinesq-Burgers equations, which arise in propagation of shallow-water waves [20]. The distributed order fractional Riccati differential equation and fractional Bagley–Torvik equation have been considered in [1, 2] by Aminikhah et al.

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Here we consider both the time and space fractional diffusion equations

$$ {}_0^C D_t^{\beta} u(x,t) - \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} = g(x,t), \quad x \in (a,b),\, t \in (0,T], \tag{1} $$

where $1 < \alpha \leq 2,\ 0 < \beta \leq 1$ and $\frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}}$ is the Riesz fractional derivative which is defined by [3]

$$ \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} = \frac{-1}{2\cos\left(\frac{\pi\alpha}{2}\right)} \left(\frac{\partial^{\alpha} u(x,t)}{\partial_{+} x^{\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial_{-} x^{\alpha}} \right), \qquad x \in (a,b), \tag{2} $$

with the boundary conditions

$$ u(a,t) = u(b,t) = 0, \quad t \in (0,T], \tag{3} $$

and the initial condition

$$ u(x,0) = 0, \quad x \in [a,b]. \tag{4} $$

In Eq. (2), when $\lceil \alpha \rceil = n$, the left and right fractional derivatives for given function $v(x)$ are defined using the Riemann-Liouville and Caputo fractional derivatives as shown in [34, 37, 47]

$$ \tfrac{\partial^{\alpha} v(x)}{\partial_{+} x^{\alpha}} = {}_a D_x^{\alpha} v(x) \quad = \sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{\Gamma(k-\alpha+1)} (x-a)^{k-\alpha} + {}_a^C D_x^{\alpha} v(x), \tag{5} $$

$$ \tfrac{\partial^{\alpha} v(x)}{\partial_{-} x^{\alpha}} = {}_x D_b^{\alpha} v(x) \quad = \sum_{k=0}^{n-1} \frac{v^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-x)^{k-\alpha} + {}_x^C D_b^{\alpha} v(x) \cdot \tag{6} $$

In Eqs. (1), (5) and (6), the left and right Caputo fractional derivatives of order $n-1 \leq \gamma < n$ for the function $v(\tau)$ are given by

$$ {}_a^C D_{\tau}^{\gamma} v(\tau) = {}_a I_{\tau}^{\gamma} \left(\frac{d^n}{d\tau^n} v(\tau) \right) = \frac{1}{\Gamma(n-\gamma)} \int_a^{\tau} \frac{\frac{d^n}{ds^n} v(s)}{(\tau - s)^{\gamma-n+1}} ds, \tag{7} $$

$$ {}_{\tau}^C D_b^{\gamma} v(\tau) = {}_{\tau} I_b^{\gamma} \left(\frac{d^n}{d\tau^n} v(\tau) \right) = \frac{1}{\Gamma(n-\gamma)} \int_{\tau}^{b} \frac{\frac{d^n}{ds^n} v(s)}{(s - \tau)^{\gamma-n+1}} ds. \tag{8} $$

The fractional diffusion equations with left and right side fractional derivatives have been used in random motion of the particles, such as transport processes in physics [5, 7, 27], Brownian motion [8] and random walk [35]. In [47], the authors developed a fast direct solution of fractional diffusion equations by using a shifted Grnwald finite difference discretization that reduce the memory requirement of the finite difference method to $\mathscr{O}(\mathscr{N})$. The numerical contour integral method for fractional diffusion equations with variable coefficients were studied in [45]. The finite difference combining with the trapezoidal scheme was used to solve the time–space fractional diffusion equation in [3]. The coupling of the homotopy perturbation and the Laplace transform method to find the analytical solution of the new Yang-Abdel-Aty-Cattani fractional diffusion equation which converges to the exact solution in terms of Prabhaker function has been implemented by Kumar et al. in [23]. Kumar et al. in [17] applied the homotopy perturbation transform method and residual power series method to solve the multidimensional heat equations of arbitrary order, where the new fractional operator has been taken in the new Yang-Abdel-Aty-Cattani (YAC) sense. The Haar wavelet operational matrix of the fractional order integration and Adams-Bashforth-Moulton methods are derived and used to solve the fractional LV model in the Caputo sense in [19]. The fractional homotopy analysis transform method, which is an innovative adjustment in Laplace transform algorithm (LTA), is used for space-fractional telegraph equation by Kumar in [16]. The fractional homotopy analysis transform method (FHATM) has been applied to solve nonlinear homogeneous and nonhomogeneous time-fractional gas dynamics equations in [24]. The coupling of Adomian decomposition method (ADM) and Laplace transform method (LTM) is used to obtain the solution of time-fractional NavierStokes equation in [18] by Kumar et al.

In addition, there are several numerical methods for solving the fractional diffusion equations, such as the finite difference method [28, 29, 39, 46], the finite element method [11, 14, 40], the preconditioned iterative method [25], the RBF meshless method [36], the Chebyshev tau method [38] and the shifted Legendre tau method [41].

In our approach we use the double exponential transformation (DE) Sinc quadrature rule to approximate the integrand of fractional integral. The approximate formula based on the DE-Sinc method for Caputo fractional derivatives was developed in [32, 33]. A survey of application of Sinc methods in fractional calculus was provided in [4]. Also we use the GRBF method to approximate the solution of the fractional diffusion problems.

The paper is organized as follows. In Section 2, we introduce Sinc function and review some definitions, theorems and notations. In Section 3 we describe the Gaussian RBFs approximation method. In Section 4, we develop the RBF-Sinc approximation method to apply to both time and space fractional diffusion equations. In Section 5, we develop the RBF-Sinc approximation method to apply to the space fractional diffusion equations. Finally, in Section 6, we test our methods on three examples. The obtained results were are compared with those of the existing methods.

2 Sinc method

We first review some properties of the Sinc quadrature rule [26, 42]. The Sinc function is defined by

$$Sinc(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

Let j be an integer and h be a positive number. The jth shifted Sinc function is defined by

$$S(j,h)(t) = Sinc(\frac{t - jh}{h}).$$

Since most of the problems are defined over a finite interval (a,b), we need the transformation that maps the interval $(-\infty,\infty)$ to a finite interval (a,b). Here we use the double exponential transformation [31, 43, 44], as

$$t = \psi_{a,b}(z) = \frac{b-a}{2}\tanh(\frac{\pi}{2}\sinh(z)) + \frac{b+a}{2},$$

and its inverse function defined by

$$z = (\psi_{a,b})^{-1}(t) = \varphi_{a,b}(t) = \log\left[\frac{1}{\pi}\log(\frac{t-a}{b-t}) + \sqrt{1 + \left(\frac{1}{\pi}\log(\frac{t-a}{b-t})\right)^2}\right].$$

We define the Sinc points as $t_k = \psi_{a,b}(kh)$.

Let $f(t)$ be the analytic function on a strip domain $\mathscr{D}_d = \{z \in \mathbb{C} :| Im(z) |< d\}$ for some $d > 0$, and bounded in some sense. For the double exponential (DE) transformations, the condition should be considered on the translated domain

$$\psi_{a,b}(\mathscr{D}_d) = \left\{ z \in \mathbb{C} : \left| arg\left(\frac{1}{\pi}\log(\frac{t-a}{b-t}) + \sqrt{1+\left(\frac{1}{\pi}\log(\frac{t-a}{b-t})\right)^2} \right) \right| < d \right\}.$$

The truncated Sinc quadrature rule can be defined by

$$\int_a^b f(x)dx = \int_{-\infty}^{\infty} f(\psi_{a,b}(s))\psi'_{a,b}(s)ds = h\sum_{j\in\mathbb{Z}} f(\psi(jh))\psi'(jh). \tag{9}$$

Following [32], if $f \in L_\beta(\psi_{a,b}(\mathscr{D}_d))$ for $0 < d < \frac{\pi}{2}$. Let M be a positive integer, and h be selected by $h = \log(\frac{dM}{\beta})/M$, $\beta = 2-\alpha$, then there exist constants $C > 0$ independent of M, such that

$$\left| \int_a^b f(x)dx - h\sum_{k=-M}^{M} f(\psi_{a,b}(kh))(\psi_{a,b})'(kh) \right| \leq C\exp\left(\frac{-\pi dM}{\log(\frac{dM}{\beta})} \right). \tag{10}$$

3 The GRBF-Sinc approximation method for left and right fractional derivatives

The radial basis functions (RBFs) depend on the distance from some center points [9, 10, 48]. Here we use Gaussian basis function $\varphi(r) = e^{-\varepsilon^2 r^2}$, where ε is the shape parameter which takes the arbitrary values.

The approximate expansion of $u(x)$ can be obtained by

$$u(x) \approx \sum_{i=1}^{N} \mu_i \varphi_i(r) = \sum_{i=1}^{N} \mu_i \varphi(\|x - x_i\|_2) = S_N(x), \tag{11}$$

where x_i, $i = 1,2,\dots,N$ are center points, $\|.\|_2$ is the Euclidean norm, μ_i are the coefficients and φ is the RBF function. The exponential convergence of RBF have been studied in [48].

We try to find a matrix D corresponding to the fractional differential operator ${}^C_aD^\alpha_x$ of function $u(x)$ which is approximated by the radial basis function $\varphi(r)$. By using Eq. (11) we get an RBF expansion of the function value u_j at each node x_j as follows

$$u_j = \sum_{i=1}^{N} \mu_i \varphi(\left\|x_j - x_i\right\|_2). \tag{12}$$

It can be shown that $A\Xi = \mathbf{U}$, where $A_{ij} = \varphi\left(\left\|x_j - x_i\right\|_2\right)$, $\Xi^T = [\mu_1,\dots,\mu_N]$ and $\mathbf{U}^T = [u(x_1),\dots,u(x_N)]$. By applying the differential operator ${}^C_aD^\alpha_x$ and ${}^C_xD^\alpha_b$ to Eq. (12) gives

$${}^C_aD^\alpha_x u(x)|_{x=x_j} = U^\alpha_{+j} = \sum_{i=1}^{N} \mu_i {}^C_aD^\alpha_x \varphi(\|x - x_i\|_2)|_{x=x_j}, \tag{13}$$

$$ {}_{x}^{C}D_{b}^{\alpha}u(x)|_{x=x_j} = U_{-j}^{\alpha} = \sum_{i=1}^{N}\mu_i {}_{x}^{C}D_{b}^{\alpha}\varphi(\|x-x_i\|_2)|_{x=x_j}, \tag{14} $$

and the matrix form of (13) and (14) are $B^{+}\Xi = U_{+}^{\alpha}$, $B^{-}\Xi = U_{-}^{\alpha}$, where $B_{ij}^{+} = {}_{a}^{C}D_{x}^{\alpha}\varphi\left(\|x-x_i\|_2\right)|_{x=x_j}$ and $B_{ij}^{-} = {}_{x}^{C}D_{b}^{\alpha}\varphi\left(\|x-x_i\|_2\right)|_{x=x_j}$.

Using the Sinc quadrature rule for fractional integrand of (7) we try to approximate each elements of the matrices $B^{+}$and B^{-}. By the change of variable

$$ s = \psi_{a,x}(\tau) = \frac{x-a}{2}\tanh(\frac{\pi}{2}\sinh(\tau)) + \frac{x+a}{2}, $$

and

$$ s = \psi_{x,b}(\tau) = \frac{b-x}{2}\tanh(\frac{\pi}{2}\sinh(\tau)) + \frac{b+x}{2}, $$

for $1 < \alpha \leqslant 2$, we obtain

$$ \begin{aligned} {}_{a}^{C}D_{x}^{\alpha}\varphi_i(x) &= {}_{a}I_{x}^{\alpha}[\frac{d^2}{dx^2}\varphi_i](x) = \frac{1}{\Gamma(2-\alpha)}\int_{a}^{x}\frac{\varphi''{}_i(s)}{(x-s)^{\alpha-1}}ds \\ &= \tfrac{(x-a)^{2-\alpha}}{\Gamma(2-\alpha)}\int_{-\infty}^{\infty}\frac{\pi\cosh(\tau)\varphi_i''(\psi_{a,x}(\tau))}{(1+e^{-\pi\sinh(\tau)})(1+e^{\pi\sinh(\tau)})^{2-\alpha}}d\tau, \\ {}_{x}^{C}D_{b}^{\alpha}\varphi_i(x) &= {}_{x}I_{b}^{\alpha}[\frac{d^2}{dx^2}\varphi_i](x) = \frac{1}{\Gamma(2-\alpha)}\int_{x}^{b}\frac{\varphi''{}_i(s)}{(s-x)^{\alpha-1}}ds \\ &= \tfrac{(b-x)^{2-\alpha}}{\Gamma(2-\alpha)}\int_{-\infty}^{\infty}\frac{\pi\cosh(\tau)\varphi_i''(\psi_{x,b}(\tau))}{(1+e^{-\pi\sinh(\tau)})^{2-\alpha}(1+e^{\pi\sinh(\tau)})}d\tau. \end{aligned} $$

By applying the quadrature rule (9) we can estimate the left and right Caputo fractional derivative of the function φ as

$$ \begin{aligned} {}_{a}^{C}D_{x}^{\alpha}\varphi_i(x) &= {}_{a}I_{x}^{\alpha}[\frac{d^2}{dx^2}\varphi_i](x) \approx \mathscr{I}_{M}^{+}[\frac{d^2}{dx^2}\varphi_i](x) \\ &= \tfrac{(x-a)^{2-\alpha}}{\Gamma(2-\alpha)}h\sum_{k=-M}^{M}\frac{\pi\cosh(kh)\varphi_i''(\psi_{a,x}(kh))}{(1+e^{-\pi\sinh(kh)})(1+e^{\pi\sinh(kh)})^{2-\alpha}} \\ &= \tfrac{(x-a)^{2-\alpha}}{\Gamma(2-\alpha)}h\sum_{k=-M}^{M}\frac{\pi\cosh(kh)\left(4\varepsilon^4\left(\psi_{a,x}(kh)-x_i\right)^2-2\varepsilon^2\right)\mathrm{e}^{-\varepsilon^2(\psi_{a,x}(kh)-x_i)^2}}{(1+e^{-\pi\sinh(kh)})(1+e^{\pi\sinh(kh)})^{2-\alpha}}, \end{aligned} \tag{15} $$

and

$$ \begin{aligned} {}_{x}^{C}D_{b}^{\alpha}\varphi_i(x) &= {}_{x}I_{b}^{\alpha}[\frac{d^2}{dx^2}\varphi_i](x) \approx \mathscr{I}_{M}^{-}[\frac{d^2}{dx^2}\varphi_i](x) \\ &= \tfrac{(b-x)^{2-\alpha}}{\Gamma(2-\alpha)}h\sum_{k=-M}^{M}\frac{\pi\cosh(kh)\varphi_i''(\psi_{x,b}(kh))}{(1+e^{-\pi\sinh(kh)})^{2-\alpha}(1+e^{\pi\sinh(kh)})} \\ &= \tfrac{(b-x)^{2-\alpha}}{\Gamma(2-\alpha)}h\sum_{k=-M}^{M}\frac{\pi\cosh(kh)\left(4\varepsilon^4\left(\psi_{x,b}(kh)-x_i\right)^2-2\varepsilon^2\right)\mathrm{e}^{-\varepsilon^2(\psi_{x,b}(kh)-x_i)^2}}{(1+e^{-\pi\sinh(kh)})^{2-\alpha}(1+e^{\pi\sinh(kh)})}. \end{aligned} \tag{16} $$

Then by considering (3), (5) and (6) at the grid point x_j we have

$$\begin{aligned} \tfrac{\partial^\alpha u(x)}{\partial_+ x^\alpha}|_{x=x_j} &= \tfrac{u'(a)}{\Gamma(2-\alpha)}(x_j-a)^{1-\alpha} + {}_a^C D_x^\alpha u(x)|_{x=x_j}, \quad j=1,\dots,N,\\ \tfrac{\partial^\alpha u(x)}{\partial_- x^\alpha}|_{x=x_j} &= \tfrac{u'(b)}{\Gamma(2-\alpha)}(b-x_j)^{1-\alpha} + {}_x^C D_b^\alpha u(x)|_{x=x_j}, \quad j=1,\dots,N, \end{aligned}$$

and the matrix forms are

$$\mathbf{U}_\alpha^+ = \Lambda^+ {\Phi_a^1}^T \Xi + B^+ \Xi, \tag{17}$$

$$\mathbf{U}_\alpha^- = \Lambda^- {\Phi_b^1}^T \Xi + B^- \Xi, \tag{18}$$

where

$$\Lambda^+ = \frac{1}{\Gamma(2-\alpha)} \begin{bmatrix} (x_1-a)^{1-\alpha} \\ (x_2-a)^{1-\alpha} \\ \vdots \\ (x_N-a)^{1-\alpha} \end{bmatrix}, \quad \Phi_a^1 = \begin{bmatrix} \varphi'(\|a-x_1\|_2) \\ \varphi'(\|a-x_2\|_2) \\ \vdots \\ \varphi'(\|a-x_N\|_2) \end{bmatrix},$$

$$\Lambda^- = \frac{1}{\Gamma(2-\alpha)} \begin{bmatrix} (b-x_1)^{1-\alpha} \\ (b-x_2)^{1-\alpha} \\ \vdots \\ (b-x_N)^{1-\alpha} \end{bmatrix}, \quad \Phi_b^1 = \begin{bmatrix} \varphi'(\|b-x_1\|_2) \\ \varphi'(\|b-x_2\|_2) \\ \vdots \\ \varphi'(\|b-x_N\|_2) \end{bmatrix}.$$

By using (12) we can eliminate the vector $\Xi = A^{-1}\mathbf{U}$ in Eqs. (17) and (18) and obtain

$$\begin{aligned} \mathbf{U}_\alpha^+ &= \left(\Lambda^+ {\Phi_a^1}^T A^{-1} + B^+ A^{-1}\right)\mathbf{U} = \mathfrak{D}_+^\alpha \mathbf{U},\\ \mathbf{U}_\alpha^- &= \left(\Lambda^- {\Phi_b^1}^T A^{-1} + B^- A^{-1}\right)\mathbf{U} = \mathfrak{D}_-^\alpha \mathbf{U}. \end{aligned}$$

Then

$$\left(\mathfrak{D}_+^\alpha + \mathfrak{D}_-^\alpha\right)\mathbf{U} = \mathfrak{D}^\alpha \mathbf{U}. \tag{19}$$

In the sequel, we obtain the following theorem for the convergence of the presented method.

Theorem 1. *Let $S_N(x)$ be an approximation of the function $u(x)$ defined by (11). Then there exist constants K and C such that*

$$\left| \frac{\partial^\alpha u(x)}{\partial_+ x^\alpha} - \frac{\partial^\alpha \bar{u}(x)}{\partial_+ x^\alpha} \right| \le \frac{(b-a)^{2-\alpha}}{\Gamma(2-\alpha)} K h_\Omega^{l-2} c_u + C \exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})} \right).$$

Proof. Assume that

$$\tfrac{\partial^\alpha u(x)}{\partial_+ x^\alpha} = \tfrac{u'(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} + {}_a^C D_x^\alpha u(x),$$

$$\tfrac{\partial^\alpha \bar{u}(x)}{\partial_+ x^\alpha} = \tfrac{S'_N(a)}{\Gamma(2-\alpha)}(x-a)^{1-\alpha} + \mathscr{I}_M^+ \left[\tfrac{d^2}{dx^2} S_N\right](x).$$

According to Sections 2 and 3 and by using relation (21) we have

$$\left| \frac{\partial^\alpha u(x)}{\partial_+ x^\alpha} - \frac{\partial^\alpha \bar{u}(x)}{\partial_+ x^\alpha} \right| \le \frac{|x-a|^{1-\alpha}}{\Gamma(2-\alpha)} \left|u'(a) - S'_N(a)\right| + \left| {}_a^C D_x^\alpha u(x) - \mathscr{I}_M^+ \left[\frac{d^2}{dx^2} S_N\right](x) \right|, \tag{20}$$

and by using [34, Theorem 11.14, page 183] we have

$$|D^n u(x) - D^n S_N(x)| \leqslant K_1 h_\Omega^{l-n} |u|_{N_\varphi(\Omega)}, \quad n = 1, 2, \tag{21}$$

where N_φ is a Hilbert space corresponding to φ and $h_\Omega = \max_{x\in\Omega} \min_{1\leq i\leq N} \|x - x_i\|$ is the meshsize and K_1, l are positive constant. Then, by considering (10) for the second term in the right-hand side of (20) we obtain

$$\begin{aligned}
&\left|{}_a^C D_x^\alpha u(x) - \mathscr{I}_M^+\left[\frac{d^2}{dx^2} S_N\right](x)\right| \\
&\quad \leq \left|{}_a^C D_x^\alpha u(x) - \mathscr{I}_M^+\left[\frac{d^2}{dx^2} u\right](x)\right| + \left|\mathscr{I}_M^+\left[\frac{d^2}{dx^2} u\right](x) - \mathscr{I}_M^+\left[\frac{d^2}{dx^2} S_N\right](x)\right| \\
&\quad \leq C\exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})}\right) + \frac{|x-a|^{2-\alpha}}{\Gamma(2-\alpha)} h \sum_{k=-M}^{M} \frac{\pi\cosh(kh)\,|u''(\psi_{a,x}(kh)) - S''{}_N(\psi_{a,x}(kh))|}{(1+e^{-\pi\sinh(kh)})(1+e^{\pi\sinh(kh)})^{2-\alpha}} \\
&\quad \leq C\exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})}\right) + \frac{(b-a)^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)} \left(|u''(x) - S''{}_N(x)|_{L^\infty}\right) \\
&\quad \leq C\exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})}\right) + \frac{(b-a)^{2-\alpha}}{\Gamma(3-\alpha)} K_1 h_\Omega^{l-2} c_u,
\end{aligned}$$

where $c_u = |u|_{N_\varphi(\Omega)}$ and finally we get

$$\begin{aligned}
\left|\frac{\partial^\alpha u(x)}{\partial_+ x^\alpha} - \frac{\partial^\alpha \bar{u}(x)}{\partial_+ x^\alpha}\right| &\leq \frac{(b-a)^{1-\alpha}}{\Gamma(2-\alpha)} K_1 h_\Omega^{l-1} c_u + C\exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})}\right) + \frac{(b-a)^{2-\alpha}}{\Gamma(3-\alpha)} K_1 h_\Omega^{l-2} c_u \\
&\leq \frac{(b-a)^{2-\alpha}}{\Gamma(2-\alpha)} K h_\Omega^{l-2} c_u + C\exp\left(\frac{-\pi d M}{\log(\frac{dM}{\beta})}\right).
\end{aligned}$$

So, the proof is complete. □

By applying the similar manner one can get the same result for the error bound of the left-hand side of the fractional derivative in equation (6).

4 The GRBF-Sinc approximation method for both the time and space fractional diffusion equations

If the given function $v(t)$ vanishes at $t = 0$, then the relationship between the Riemann-Liouville and Caputo and Grunwald fractional derivatives are as follows [47]

$${}_0^{RL}D_t^\beta v(t) = {}_0^C D_t^\beta v(t) = {}_0^G D_t^\beta v(t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t^\beta} \sum_{\ell=0}^{\lfloor \frac{t}{\Delta t}\rfloor+1} (-1)^\ell \binom{\beta}{\ell} v(t - (\ell-1)\Delta t), \tag{22}$$

where $0 < t \leq T$, $\Delta t = T/\tau$ and τ is the time grid number. By using the time partition $t_n = n\Delta t$ for $n = 0, \ldots, \tau$, we have the following shifted Grunwald finite difference discretization

$$\left[{}_0^C D_t^\beta u(x_j, t)\right]\Big|_{t=t_n} = \frac{1}{\Delta t^\beta \Gamma(-\beta)} \sum_{\ell=0}^{n+1} \frac{\Gamma(\ell-\beta)}{\Gamma(\ell+1)} u_j^{n-\ell+1} + o(\Delta t), \quad j = 1, \ldots, N. \tag{23}$$

Then by considering both the time and space fractional diffusion equation (1) and applying GRBF-Sinc approximation method in space dimension and shifted Grunwald finite difference discretization (23) in time dimension we obtain the following stage matrix equation

$$\begin{aligned}\left(2I+\frac{\Delta t^{\beta}}{2\cos\left(\frac{\pi\alpha}{2}\right)}\mathfrak{D}^{\alpha}\right)\mathbf{U}^{n+1} &= \left(2\beta I-\frac{\Delta t^{\beta}}{2\cos\left(\frac{\pi\alpha}{2}\right)}\mathfrak{D}^{\alpha}\right)\mathbf{U}^{n} \\ &\quad -\frac{2}{\Gamma(-\beta)}\sum_{\ell=2}^{n+1}\frac{\Gamma(\ell-\beta)}{\Gamma(\ell+1)}\mathbf{U}_{j}^{n-\ell+1}+\Delta t^{\beta}\mathbf{G}^{n+\frac{1}{2}}, \quad \mathbf{U}^{0}=0,\end{aligned} \tag{24}$$

where I is the identity matrix of order N, $\mathbf{G}^n$ is the time-dependent analogue of $\mathbf{G}$ at the time step which is defined as $\mathbf{G}^T = [g(x_1,t),\ldots,g(x_N,t)]$ and $n = 0,\ldots,\tau$. By solving equation (24) we can obtain the solution of problem (1)–(3) at the final time $t = T$.

5 GRBF-Sinc approximation method for space fractional diffusion equations

In the case of $\beta = 1$, we can consider the following space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t}-p_1(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial_{+}x^{\alpha}}-p_2(x,t)\frac{\partial^{\alpha}u(x,t)}{\partial_{-}x^{\alpha}}=g(x,t),\ x\in(a,b),\ t\in(0,T], \tag{25}$$

with the boundary conditions

$$u(a,t)=u(b,t)=0,\ t\in(0,T], \tag{26}$$

and the initial condition

$$u(x,0)=u_0(x),\ x\in[a,b]. \tag{27}$$

To approximate the solution of the space fractional diffusion Eq. (25), we apply the GRBF-Sinc approximation method which was developed in the previous section for space dimension of (25) and discretize in time with a Crank-Nicolson scheme.

Let

$$\frac{\partial u(x_j,t)}{\partial t}=\frac{u_j^{n+1}-u_j^{n}}{\Delta t},$$

for $j = 1,\ldots,N$ and $\Delta t = T/\tau$, then we obtain following stage matrix equation

$$\underbrace{\left(2I-\Delta tP_1^{n+1}\mathfrak{D}_{+}^{\alpha}-\Delta tP_2^{n+1}\mathfrak{D}_{-}^{\alpha}\right)}_{\mathfrak{A}}\mathbf{U}^{n+1}=\underbrace{\left(2I+\Delta tP_1^{n}\mathfrak{D}_{+}^{\alpha}+\Delta tP_2^{n}\mathfrak{D}_{-}^{\alpha}\right)}_{\mathfrak{B}}\mathbf{U}^{n}+\underbrace{\left(\Delta t\mathbf{G}^{n+\frac{1}{2}}\right)}_{\mathfrak{Q}}, \tag{28}$$

where I is the identity matrix of order N and P_1^n, P_2^n and $\mathbf{G}^n$ are the time-dependent analogous to P_1, P_2 and $\mathbf{G}$ at the time step which is defined as $P_i = diag\left(p_i(x_1),\ldots,p_i(x_N)\right), i = 1,2$ and $\mathbf{G}^T = [g(x_1,t),\ldots,g(x_N,t)]$. Then we can rewrite the matrix equation (28) as follows

$$\mathbf{U}^{n+1}=\mathfrak{A}^{-1}\left(\mathfrak{B}\mathbf{U}^{n}+\mathfrak{G}\right),\quad \mathbf{U}^{0}=\begin{bmatrix}u_0(x_1)\\ u_0(x_2)\\ \vdots\\ u_0(x_N)\end{bmatrix},\ n=0,\ldots,\tau, \tag{29}$$

where $u_0(x)$ is the initial condition (27). By solving the system (29) we can obtain the numerical solution of the space fractional diffusion equations (25)–(27) at the final time $t = T$.

6 Numerical Illustration

The propose methods are applied on three examples to test the efficiency and accuracy of the purposed methods. We set $M = 50$, $d = 3.14/2$, $\beta = 2-\alpha$ and $h = \log(\frac{dM}{\beta})/M$. We solve these examples for various values of the shape parameter, $\varepsilon = 0.1, 0.01, 0.001$.

Example 1. We consider the space fractional diffusion problem (25)–(27) with the fractional derivative of order $\alpha = 1.8$, $p_1(t) = p_2(t) = \Gamma(1.2)t$,

$$g(x) = -e^{-t}x^2(2-x)^2 - 8te^{-t}\left(x^{\frac{1}{5}} + (2-x)^{\frac{1}{5}} - \frac{5}{2}\left(x^{\frac{6}{5}} + (2-x)^{\frac{6}{5}}\right) + \frac{25}{22}\left(x^{\frac{11}{5}} + (2-x)^{\frac{11}{5}}\right)\right),$$

and the initial condition $u_0(x) = x^2(2-x)^2$, $x \in (0,2)$. The exact solution is $u(x,t) = e^{-t}x^2(2-x)^2$ [47]. We apply the GRBF-Sinc method presented in Section 5 to this example for computing the solution at $T = 1$. The max-absolute errors in the solution verses the number of Chebyshev nodes N for $\alpha = 1.8$ at

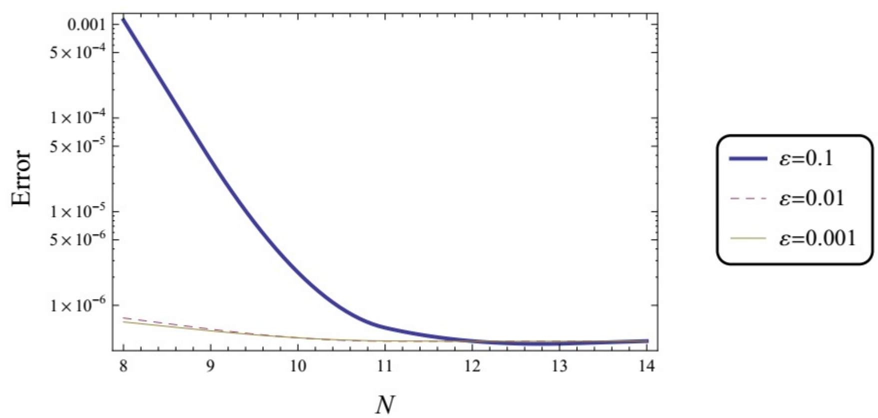

Figure 1: Graph of max-absolute error in solving Example 1.

$T = 1$ plotted in Figure 1, shows that resulting error in the solution of equation (25) using GRBF and DE-Sinc quadrature method at $T = 1$ with $\varepsilon = 0.1, 0.01, 0.001$, $\tau = 128$ and number of Chebyshev nodes $8 \le N \le 14$ decreases when ε decreases and N increases.

The approximate solution and error in solving Example 1 for $\alpha = 1.8$ by applying the GRBF-Sinc method with $\varepsilon = 0.001$, $N = 12$ and $\tau = 128$ have been plotted in Figure 2. The comparison of max-absolute errors of our method and the finite difference methods presented in [47], is tabulated in Table 1 for different values of N and M as the spatial and time grids, respectively.

Example 2. Consider the space fractional diffusion problem (25)–(27) with fractional derivative of order $\alpha = 1.5$ and

$$p_1(x) = -\cos\left(\frac{\pi(x+1)}{6}\right), \quad p_2(x) = -\sin\left(\frac{\pi(x+1)}{6}\right),$$

and

$$\begin{aligned} g(x) &= -32e^{-t}x^3(1-x)^3 - \tfrac{256e^{-t}}{105\sqrt{\pi}}\Big[\left(105x^{\frac{3}{2}} - 504x^{\frac{5}{2}} + 720x^{\frac{7}{2}} - 320x^{\frac{9}{2}}\right)\cos\left(\tfrac{\pi(1+x)}{6}\right) \\ &\quad + \left(1+23x-264x^2+560x^3-320x^4\right)\sqrt{1-x}\sin\left(\tfrac{\pi(1+x)}{6}\right)\Big]. \end{aligned}$$

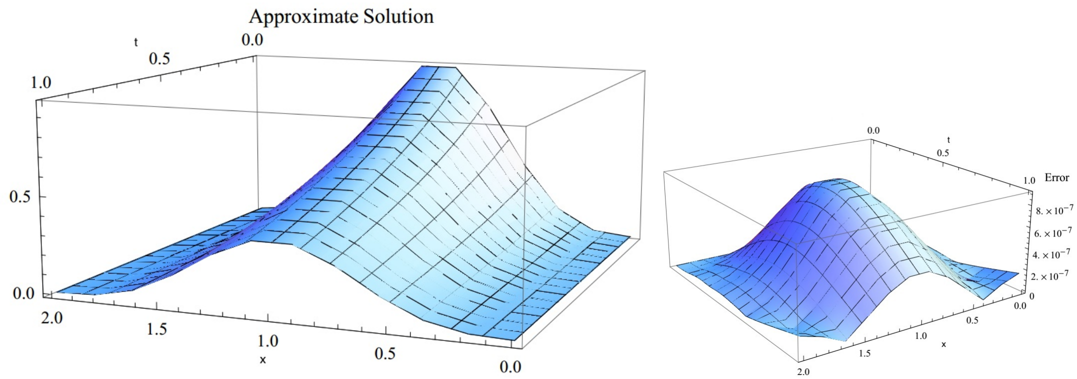

Figure 2: Graphs of approximate solution and error in the solution of Example 1 for $\alpha = 1.8$.

Table 1: Comparison of max-absolute errors of the GRBF-Sinc method with $\alpha = 1.8, \varepsilon = 0.01, 0.001, M = 50, \tau = 128$ and the Levinson and Superfast methods [47] for Example 1.

GRBF-Sinc method			The Methods in [47]			
N	$\varepsilon = 0.01$	$\varepsilon = 0.001$	N	M	$Levinson$	$Superfast$
8	7.319×10^{-7}	6.643×10^{-7}	2^{13}	2^{12}	1.327×10^{-5}	1.327×10^{-5}
10	4.499×10^{-7}	4.498×10^{-7}	2^{14}	2^{13}	6.650×10^{-6}	6.648×10^{-6}
12	4.134×10^{-7}	4.134×10^{-7}	2^{15}	2^{14}	3.343×10^{-6}	3.338×10^{-6}

The initial condition is $u_0(x) = 32x^3(1-x)^3, \quad x \in (0,1)$. The exact solution is $u(x,t) = 32e^{-t}x^3(1-x)^3$ [45].

We have applied our method presented in Section 5 to this example and obtained the computed solution at the time $T = 1$. The max-absolute errors in the computed solution plotted in Figure 3 with $\varepsilon = 0.1, 0.01, 0.001$, $\tau = 128$ and number of the Chebyshev nodes $8 \le N \le 14$. It can be seen that we can obtain notable results with small numbers of basis functions.

Figure 4 shows the approximate solution and error in solving Example 2 for $\alpha = 1.5$ by applying the GRBF-Sinc method with $\varepsilon = 0.01, N = 12$ and $\tau = 128$.

We compare our results by the results of the Contour integral method given in [45] in Table 2 and J denotes the spatial grid number and N denotes the number of quadrature nodes.

Table 2: Comparison of max-absolute errors of the GRBF-Sinc method with $\alpha = 1.5, \varepsilon = 0.01, 0.001, M = 50, \tau = 128$ and method [45] for Example 2.

GRBF-Sinc method			Contour integral		
N	$\varepsilon = 0.01$	$\varepsilon = 0.001$	J	N	$Error$
8	1.795×10^{-4}	2.246×10^{-6}	1024	10	3.472×10^{-4}
10	2.891×10^{-7}	2.879×10^{-7}	2048	11	1.756×10^{-4}
12	2.761×10^{-7}	2.761×10^{-7}	4096	11	8.822×10^{-5}

Example 3. Let us consider both the time and the space fractional diffusion equation (1)–(3) with the

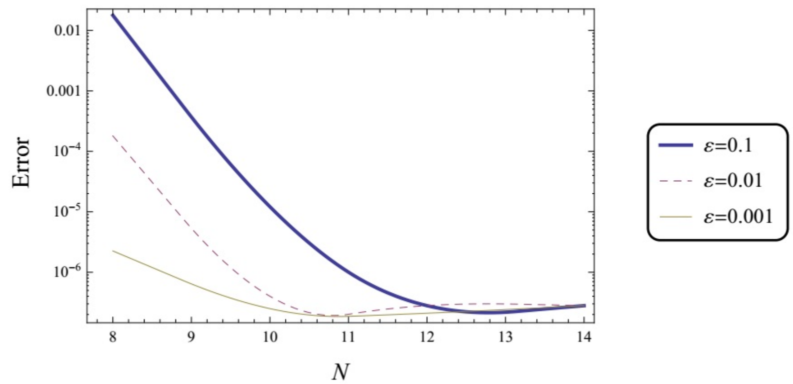

Figure 3: Graph of max-absolute error for Example 2 for $\alpha = 1.5$ at $T = 1$ with respect to the number of the Chebyshev nodes N by applying the GRBF-Sinc method with $\varepsilon = 0.1, 0.01, 0.001$ and $\tau = 128$.

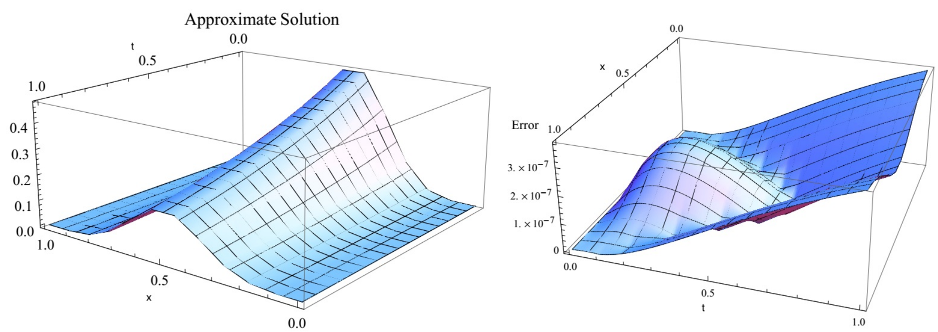

Figure 4: Graph of the approximate solution and error in solving Example 2 for $\alpha = 1.5$ by applying the GRBF-Sinc method with $\varepsilon = 0.01$, $N = 12$ and $\tau = 128$.

Reisz fractional [3]

$$ {}_0^C D_t^{\beta} u(x,t) - \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} = g(x,t), \ x \in (a,b), \, t \in (0,T], \tag{30} $$

where $1 < \alpha < 2$ and $0 < \beta < 1$ and

$$ \begin{aligned} g(x) \ = & \tfrac{2}{\Gamma(3-\beta)} t^{2-\beta} x^2 (1-x)^2 + \tfrac{t^2}{\cos(\frac{\pi\alpha}{2})\Gamma(5-\alpha)} (12(x^{4-\alpha} + (1-x)^{4-\alpha}) - 6(4-\alpha)(x^{3-\alpha} + (1-x)^{3-\alpha}) \\ & + (3-\alpha)(4-\alpha)(x^{2-\alpha} + (1-x)^{2-\alpha})), \end{aligned} $$

and the exact solution is $u(x) = t^2 x^2 (1-x)^2$. The approximate and the exact solution and the resulting error in solution (30) for $\alpha = 1.5$ and $\beta = 0.5$ using the GRBF and DE-Sinc quadrature method with $\varepsilon = 0.01$, $\tau = 100$ and number of equispaced nodes $N = 14$ have been plotted in Figure 5.

The comparison of the errors in the grid point of the GRBF-Sinc method and the trapezoidal scheme with finite difference method presented in [3], for $\beta = 0.8$ and $\alpha = 1.2, 1.4, 1.6$, is tabulated in Table 3. It is mentioned that, N' and τ' denote the spatial and time grids, respectively. Here, we have set $\tau = \tau' = 1000$ to compare our mathod presented in Section 4.

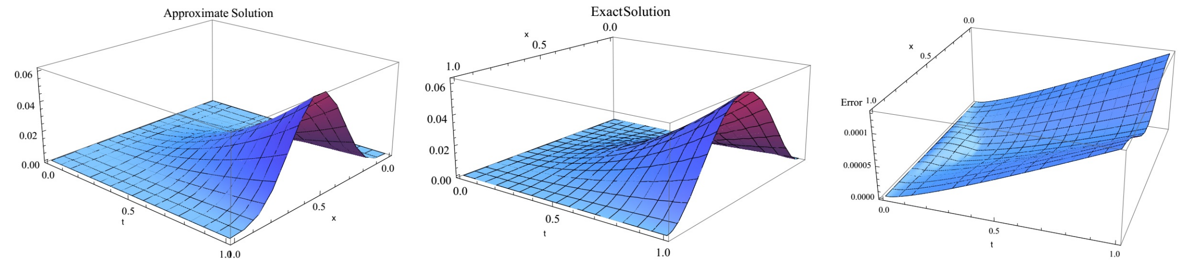

Figure 5: Graph of the approximate solution and the error in solution of Example 3 for $\alpha = 1.5$, $\beta = 0.5$ by applying the GRBF-Sinc method with $\varepsilon = 0.01$, $N = 14$ and $\tau = 100$.

Table 3: Comparison of the errors of the GRBF-Sinc method with $\beta = 0.8$, $\varepsilon = 0.001$, $M = 50$, $\tau = \tau' = 1000$ and the Trapezoidal scheme [3] for Example 3.

GRBF-Sinc method				Trapezoidal scheme [3]			
N	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	N'	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$
5	1.025×10^{-5}	6.912×10^{-6}	9.115×10^{-6}	5	1.550×10^{-3}	2.290×10^{-3}	3.269×10^{-3}
8	9.210×10^{-6}	5.394×10^{-6}	5.289×10^{-6}	10	3.374×10^{-4}	5.035×10^{-4}	7.436×10^{-4}
10	1.487×10^{-6}	3.782×10^{-6}	2.802×10^{-6}	20	7.808×10^{-5}	1.132×10^{-4}	1.694×10^{-4}
12	1.130×10^{-6}	2.601×10^{-6}	2.207×10^{-6}	40	1.889×10^{-5}	2.603×10^{-5}	3.865×10^{-5}
14	1.023×10^{-6}	2.052×10^{-6}	2.036×10^{-6}	80	4.514×10^{-6}	5.888×10^{-6}	8.585×10^{-6}

7 Conclusion

In the present work, a combination of the Sinc and the GRBF has been used for the time-space fractional diffusion equation with Reisz fractional derivative (in Section 4), and also for the space fractional diffusion equation (in Section 5). Our presented approach is able to give a quite accurate approximate solution. The comparison with recent existing methods in illustrated examples justifies the efficiency and accuracy of our presented approach.

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