Journal of Mathematical Modeling
Vol. 10, No. 2, 2022, pp. 299-313. Research Article

JMM

An efficient wavelet-based numerical method to solve nonlinear Fredholm integral equation of second kind with smooth kernel

Jyotirmoy Mouley[†][*], Birendra Nath Mandal[‡]

[†]*Department of Applied Mathematics, University of Calcutta, Kolkata, India*
[‡] *Physics and Applied Mathematics Unit, Indian Statistical Institute, Kolkata, India*
Email(s): jyoti87.cu.wavelet@gmail.com, bnm2006@rediffmail.com

Abstract. In this paper, a wavelet-based numerical algorithm is described to obtain approximate numerical solution of a class of nonlinear Fredholm integral equations of second kind having smooth kernels. The algorithm involves approximation of the unknown function in terms of Daubechies scale functions. The properties of Daubechies scale and wavelet functions together with one-point quadrature rule for the product of a smooth function and Daubechies scale functions are utilized to transform the integral equation to a system of nonlinear equations. The efficiency of the proposed method is demonstrated through three illustrative examples.

Keywords: Nonlinearity, Fredholm integral equation, Daubechies wavelet function, one-point quadratute rule.
AMS Subject Classification 2010: 45-XX, 45Gxx 45G10.

1 Introduction

Most of the natural phenomena are described by some nonlinear equations as they are intrinsically nonlinear in nature. In many cases, the nonlinear equations are approximated by some linear equations which are then solved either exactly or approximately. This linearization procedure works nicely upto certain extent. However, the presence of a small error in the input value may sometimes provides a large error in the output value so as to vitiate the process. Thus it is better to study the nonlinear system as such without making any linearization. Nonlinear integral equations arise in many areas of mathematical physics such as in scattering theory, stationary shapes of needle, emerging broken symmetry in space and time, neutron transport, traffic model and many more (see [3, 5, 20]), etc.

[*]Corresponding author.

Received: 4 September 2021/ Revised: 24 October 2021 / Accepted: 2 November 2021
DOI: 10.22124/JMM.2021.20512.1785

In this paper, an efficient numerical method has been established to obtain approximate numerical solution of a nonlinear Fredholm integral equation of second kind of the form

$$f(x)+\lambda\int_a^b \mathscr{L}(x,y,f(y))\mathrm{d}y = q(x), \quad a\le x\le b. \tag{1}$$

The main difficulty to solve this type equation is the presence of the unknown function $f(x)$ in the kernel of the integral equation. The above form of nonlinear integral equation is known as Urysohn integral equation. If the kernel function $\mathscr{L}(x,y,f(y))$ can be separated as $\mathscr{T}(x,y)g(y,f(y))$ ($g(.,.)$ is a different function of two aurguments) then the integral equation is known as Hammerstein integral equation. The forcing term $q(x)$ and the unknown function $f(x)$ both are assumed to be continuous on $[a,b]$. Also the function $\mathscr{L}(x,y,f(y))$ is continuous on $[a,b]\times[a,b]\times\mathbb{R}$. We also assume that the kernel $\mathscr{L}(x,y,f(y))$ satisfies the Lipschitz condition with respect to the third argument and Lipschitz constant γ satisfies the inequality $|\lambda|\gamma<1$ to ascertain the unique solution of the nonlinear integral equation [19].

It is well known that various numerical schemes have been proposed to solve a nonlinear integral equation in the literature from time to time. As our concern here is to employ wavelet method to solve an integral equation, some works in which mostly wavelets have been used are cited here. For example, both nonlinear Volterra and Fredholmn integral equations are solved numerically by employing Legendre wavelets by Mahamoudi [11]. Maleknejad et al. [12] constructed Petrov-Galerkin elements from piece-wise polynomials to solve a class of nonlinear Hammerstein equations. Ordokhani and Razzaghi [15] have used collocation method to solve nonlinear Fredholm Hammarstein integral equations using rationlized Haar functions. The Toeplitz matrix method was used by Abdou et al. [2] for Hammerstein integral equation. Discrete Legendre spectral projection method was used for Fredholm-Hammerstein equation by Das et al [6]. Sinc-collocation method was used to find approximate numerical solution of Hammerstein integral equation of mixed type by Hashemizadeh et al. [8]. An efficient algorithmic was employed by Abdolmaleki and Najafi [1] to solve Hammerstein integral equation in which use of the Taylor series expansion plays an important role. Very recently Chandrashekhar integral equation was solved by iterative method after transforming it to Hammerstein integral equation by Hermandez-veron et al [9].

In this paper, we use Daubechies wavelets to solve a class of nonlinear Fredholm integral equations. The raw images (the unknown coefficients used in the expansions of fnctions) are expressed in terms of products of weights and functional values at nodes of one-point quadrature rule. Using the properties of Daubechies scale and wavelet functions, the integral equation is transformed to a system of nonlinear equations. Though the expansion of raw images is possible using weights and nodes of M-(>1) point Gauss-Daubechies quadrature rule [18] for the product of smooth function and Daubechies scale function but it will be more strenuous in dealing with nonlinear terms in the integral equation as the corresponding system of nonlinear equations contain greater number of variables than the number of equations. As one-point quadrature formula has adequete precision, it plays an important role in dealing with nonlinear terms. The raw images which are involved in the expansion of the unknown function can be obtained solving the system of equations by any standard numerical methods such as Newton's method, Seidal iteration method, iteration method or by using softwere such as MATLAB, MATHEMATICA, Python etc.

The paper is organized as follows: Section 2 is devoted to a short survey on Daubechies scale and wavelet functions. The subsequent section deals with approximation of unknown function, forcing term and kernel function. The numerical method is described in Section 4 and error has been estimated in Section 5. Numerical illustration is given in Section 6 and a conclusion in Section 7.

2 A short survey on Daubechies scale and wavelet functions

Wavelets have diverse applications in different diciplines such as approximation theory, differential equation, integral equation, numerical analysis in mathematics, image compression in computer science, signal processing in electrical engineering and data analysis in statistics. Haar wavelet method has been applied by Beylkin et al. [4] for the first time to solve integral equations. In fact this provides a new direction of research dealing with integral and differential equations using wavelets. Here Daubechies wavelet basis is used in approximation of functions. Though the detail description of Daubechies scale and wavelet functions is available (cf. [7,13,14]), a short review is given below for the sake of completeness.

2.1 Two-scale relation

The set of functions $\left\{\phi_{jk}(x) = 2^{\frac{j}{2}}\phi(2^jx-k) : k \in \mathbb{Z}\right\}$ which is used in approximation of functions can be generated by repeated application of translation and transformation from a single function $\phi(x)$, known as scale function. The scale function with compact support $[0,2K-1]$ satisfies the following relation known as two-scale relation

$$\phi(x) = \sqrt{2}\sum_{l=0}^{2K-1} h_l \phi(2x-l). \tag{2}$$

This two-scale relation plays a key role in developing numerical algorithms as the explicit forms of Daubechies scale functions are not known. The wavelet function $\psi(x)$ satisfies the relation

$$\psi(x) = \sqrt{2}\sum_{l=0}^{2K-1} g_l \phi(2x-l). \tag{3}$$

This relation provides as a connector between the scale function $\phi(x)$ and wavelet function $\psi(x)$. Here h_l and g_l $(l = 0,1,\ldots,2K-1)$ are called low pass and high pass filter coefficients and they are linked by the relation $g_l = (-1)^l h_{2K-1-l}$.

Though the translation parameter k in the basis set $\{\phi_{jk}(x)\}$ belongs to $\mathbb{Z}$ but when one deals with finite interval $[a,b]$, a restriction on the selection of the values of k has to be imposed. In this case, three collections of k are considered namely Λ_j^L, Λ_j^I and Λ_j^R. The notation Λ_j is used as index set to mean $\Lambda_j = \Lambda_j^L \cup \Lambda_j^I \cup \Lambda_j^R$ [14].

2.2 One-point quadrature rule

The one-point quadrature rule in the interval $[a,b]$ for the interior scale function $\phi_{jk}^I(x)$ was carried out by Sweldens and Piessen (c.f. [16]) whereas for truncated scale function $\phi_{jk}^{L\ or\ R}(x)$ was carried out by Panja and Mandal [16]. The one-point quadrature rule for integrals involving the product of a smooth function $f(x)$ and Daubechies scale function is given by

$$\int_a^b f(x)\phi_{jk}^s(x)\mathrm{d}x = w_{jk}^s f\left(\bar{x}_{jk}^s\right), \quad s = L,I,R, \tag{4}$$

where

$$w^s_{jk} = \begin{cases} w^L_{jk} = \frac{1}{2^{\frac{j}{2}}} < x^0 >_{[a2^j-k, 2K-1]}, & \text{if } k \in \Lambda^L_j, \\ w^I_{jk} = \frac{1}{2^{\frac{j}{2}}}, & \text{if } k \in \Lambda^I_j, \\ w^R_{jk} = \frac{1}{2^{\frac{j}{2}}} < x^0 >_{[0, b2^j-k]}, & \text{if } k \in \Lambda^R_j, \end{cases} \tag{5}$$

and

$$\bar{x}^s_{jk} = \begin{cases} \bar{x}^L_{jk} = \dfrac{k + \frac{<x>_{[a2^j-k,2K-1]}}{<x^0>_{[a2^j-k,2K-1]}}}{2^j}, & \text{if } k \in \Lambda^L_j, \\ \bar{x}^I_{jk} = \dfrac{k + < x >}{2^j}, & \text{if } k \in \Lambda^I_j, \\ \bar{x}^R_{jk} = \dfrac{k + \frac{<x>_{[0,b2^j-k]}}{<x^0>_{[0,b2^j-k]}}}{2^j}, & \text{if } k \in \Lambda^R_j. \end{cases} \tag{6}$$

Here Λ^I_j and $\Lambda^{L \text{ or } R}_j$ (see [14]) are the collection of translation parameter k for the interior scale function $\phi^I_{jk}(x)$ and truncated scale function $\phi^{L\ or\ R}_{jk}(x)$. $< x^m >_{[a2^j-k,2K-1]} = \int_{a2^j-k}^{2K-1} x^m \phi(x) \mathrm{d}x$ and $< x^m >_{[0,b2^j-k]} = \int_0^{b2^j-k} x^m \phi(x) \mathrm{d}x$ are the partial moments for truncated scale function. The valus of these partial moments are tabulated in Table-3a and Table-3b for Dau-3 scale function in [17]. $< x^m > = \int_{\mathbb{R}} x^m \phi(x) \mathrm{d}x$ is the full moment for interior scale function which can be calculated by the formula (see [10])

$$< x^m > = \frac{1}{2^{m-1}} \frac{1}{\sqrt{2}} \sum_{n=0}^{m-1} \frac{m!}{n!(m-n)!} \left(\sum_{l=1}^{2K-1} h_l l^{m-n} \right) < x^n > . \tag{7}$$

Moreover, the partial and full moments satisfy the relation

$$< x^m >_{[a2^j-k,2K-1]} + < x^m >_{[0,b2^j-k]} = < x^m > . \tag{8}$$

3 Approximation of functions in Daubechis wavelet basis

The unknown function $f(x)$ and the known forcing term $q(x)$ of the integral equation are approximated in terms of Daubechies scale functions as

$$f(x) \approx \sum_{k \in \Lambda_j} f^s_{jk} \phi^s_{jk}(x), \tag{9}$$

and

$$q(x) \approx \sum_{k \in \Lambda_j} q^s_{jk} \phi^s_{jk}(x). \tag{10}$$

The raw images f^s_{jk} and q^s_{jk} in the expansion of $f(x)$ and $q(x)$ are obtained after multiplying both sides of (9) and (10) by $\phi^s_{jk}(x)$ and then integrating over $[a,b]$ as

$$f^s_{jk} = \begin{cases} f^L_{jk} = \sum_{l=a2^j-2K+2}^{a2^j-1} (N^L)^{-1}_{k-a2^j, l-a2^j} w^L_{jl} f(\bar{x}^L_{jl}), & \text{if } k \in \Lambda^L_j, \\ f^I_{jk} = w^I_{jk} f(\bar{x}^I_{jk}), & \text{if } k \in \Lambda^I_j, \\ f^R_{jk} = \sum_{l=b2^j-2K+2}^{b2^j-1} (N^L)^{-1}_{k-b2^j, l-b2^j} w^R_{jl} f(\bar{x}^R_{jl}) & \text{if } k \in \Lambda^R_j; \end{cases} \tag{11}$$

and

$$q^s_{jk}=\begin{cases} q^L_{jk}=\sum_{l=a2^j-2K+2}^{a2^j-1}(N^L)^{-1}_{k-a2^j,l-a2^j}w^L_{jl}q(\bar{x}^L_{jl}), & \text{if } k\in\Lambda^L_j,\\ q^I_{jk}=w^I_{jk}q(\bar{x}^I_{jk}), & \text{if } k\in\Lambda^I_j,\\ q^R_{jk}=\sum_{l=b2^j-2K+2}^{b2^j-1}(N^L)^{-1}_{k-b2^j,l-b2^j}w^R_{jl}q(\bar{x}^R_{jl}), & \text{if } k\in\Lambda^R_j.\end{cases} \tag{12}$$

The kernel $\mathscr{L}(x,y,f(y))$ is a function of several variables in which the unknown function $f(y)$ is present in nonlinear form. The kernel function $\mathscr{L}(x,y,f(y))$ is approximated in Daubechies scale functions basis as

$$\mathscr{L}(x,y,f(y))\approx\sum_{k'\in\Lambda^j}\sum_{k\in\Lambda^j}\mathscr{L}^{s's}_{j:k'k}\phi^{s'}_{jk'}(x)\phi^s_{jk}(y). \tag{13}$$

The raw images $\mathscr{L}^{s's}_{j:k'k}$ are evaluated from the equation (13) by successively multiplying by $\phi^s_{jk}(y)$, $\phi^{s'}_{jk'}(x)$ and integrating with respect to x and y over $[a,b]$, and are given by

$$\mathscr{L}^{s's}_{j:k'k}=\begin{cases} \mathscr{L}^{LL}_{j:k'k}=\sum_{l_1=a2^j-2K+2}^{a2^j-1}\sum_{l_2=a2^j-2K+2}^{a2^j-1}(N^L)^{-1}_{k'-a2^j,l_1-a2^j}(N^L)^{-1}_{k-a2^j,l_2-a2^j} \\ \quad w^L_{jl_1}w^L_{jl_2}\mathscr{L}(\bar{x}^L_{jl_1},\bar{y}^L_{jl_2},f(\bar{y}^L_{jl_2})), & \text{if } k\in\Lambda^L_j,k'\in\Lambda^L_j,\\ \mathscr{L}^{IL}_{j:k'k}=\sum_{l=a2^j-2K+2}^{a2^j-1}(N^L)^{-1}_{k-a2^j,l-a2^j}w^L_{jl}w^I_{jk'}\mathscr{L}(\bar{x}^I_{jk'},\bar{y}^L_{jl},f(\bar{y}^L_{jl})), & \text{if } k\in\Lambda^L_j,k'\in\Lambda^I_j,\\ \mathscr{L}^{LL}_{j:k'k}=\sum_{l_1=b2^j-2K+2}^{b2^j-1}\sum_{l_2=a2^j-2K+2}^{a2^j-1}(N^R)^{-1}_{k'-b2^j,l_1-b2^j}(N^L)^{-1}_{k-a2^j,l_2-a2^j}, \\ \quad w^R_{jl_1}w^L_{jl_2}\mathscr{L}(\bar{x}^R_{jl_1},\bar{y}^L_{jl_2},f(\bar{y}^L_{jl_2})), & \text{if } k\in\Lambda^L_j,k'\in\Lambda^R_j,\\ \mathscr{L}^{LI}_{j:k'k}=\sum_{l=a2^j-2K+2}^{a2^j-1}(N^L)^{-1}_{k'-a2^j,l-a2^j}w^L_{jl}w^I_{jk}\mathscr{L}(\bar{x}^L_{jl},\bar{y}^L_{jk},f(\bar{y}^L_{jk})), & \text{if } k\in\Lambda^I_j,k'\in\Lambda^L_j,\\ \mathscr{L}^{II}_{j:k'k}=w^I_{jk'}w^I_{jk}\mathscr{L}(\bar{x}^I_{jk'},\bar{y}^I_{jk},f(\bar{y}^I_{jk})), & \text{if } k\in\Lambda^I_j,k'\in\Lambda^I_j,\\ \mathscr{L}^{RI}_{j:k'k}=\sum_{l=b2^j-2K+2}^{b2^j-1}(N^R)^{-1}_{k'-b2^j,l-b2^j}w^R_{jl}w^I_{jk}\mathscr{L}(\bar{x}^R_{jl},\bar{y}^I_{jk},f(\bar{y}^I_{jk})), & \text{if } k\in\Lambda^I_j,k'\in\Lambda^R_j,\\ \mathscr{L}^{LR}_{j:k'k}=\sum_{l_1=a2^j-2K+2}^{a2^j-1}\sum_{l_2=b2^j-2K+2}^{b2^j-1}(N^L)^{-1}_{k'-a2^j,l_1-a2^j}(N^R)^{-1}_{k-b2^j,l_2-b2^j}, \\ \quad w^L_{jl_1}w^R_{jl_2}\mathscr{L}(\bar{x}^L_{jl_1},\bar{y}^R_{jl_2},f(\bar{y}^R_{jl_2})), & \text{if } k\in\Lambda^R_j,k'\in\Lambda^L_j,\\ \mathscr{L}^{IR}_{j:k'k}=\sum_{l=b2^j-2K+2}^{b2^j-1}(N^R)^{-1}_{k-b2^j,l-b2^j}w^R_{jl}w^I_{jk'}\mathscr{L}(\bar{x}^I_{jk'},\bar{y}^R_{jl},f(\bar{y}^R_{jl})), & \text{if } k\in\Lambda^R_j,k'\in\Lambda^I_j,\\ \mathscr{L}^{RR}_{j:k'k}=\sum_{l_1=b2^j-2K+2}^{b2^j-1}\sum_{l_2=b2^j-2K+2}^{b2^j-1}(N^R)^{-1}_{k'-b2^j,l_1-b2^j}(N^R)^{-1}_{k-b2^j,l_2-b2^j} \\ \quad w^R_{jl_1}w^R_{jl_2}\mathscr{L}(\bar{x}^R_{jl_1},\bar{y}^R_{jl_2},f(\bar{y}^R_{jl_2})), & \text{if } k\in\Lambda^R_j,k'\in\Lambda^R_j.\end{cases} \tag{14}$$

$(N^s)^{-1}_{u,v}$ are the entries of the inverse of the matrix formed by the normalization coefficients $N^s_{u,v}$ $(s=L,I,R)$. The numerical values of both $(N^s)^{-1}_{u,v}$ and $N^s_{u,v}$ $(s=L,I,R)$ are available in [17].

4 The numerical method

Substituting the approximate form of the unknown function $f(x)$, forcing term $q(x)$ and the kernel function $\mathscr{L}(x,y,f(y))$ in (1), we obtain

$$\sum_{k\in\Lambda_j} f^s_{jk}\phi^s_{jk}(x)+\lambda\sum_{k'\in\Lambda^j}\sum_{k\in\Lambda^j}\mathscr{L}^{s's}_{j:k'k}\phi^{s'}_{jk'}(x)\int_a^b\phi^s_{jk}(y)\mathrm{d}y=\sum_{k\in\Lambda_j}q^s_{jk}\phi^s_{jk}(x). \tag{15}$$

Setting $\Gamma^s_{j:k}=\int_a^b\phi^s_{jk}(y)\mathrm{d}y$, the equation (15) is written as

$$\sum_{k\in\Lambda_j} f^s_{jk}\phi^s_{jk}(x)+\lambda\sum_{k'\in\Lambda^j}\sum_{k\in\Lambda^j}\mathscr{L}^{s's}_{j:k'k}\phi^{s'}_{jk'}(x)\Gamma^s_{j:k}=\sum_{k\in\Lambda_j}q^s_{jk}\phi^s_{jk}(x). \tag{16}$$

$\Gamma^s_{j:k}$ can be evaluated using the one-point quadrature rule. It is evident that the values of $\Gamma^s_{j:k}$ are identical with w^s_{jk}, given by (5). Multiplying both sides of the equation (16) by $\phi^s_{jk_1}(x)$ $(k_1=a2^j-2K+2,\dots,b2^j-1)$ and integrating over $[a,b]$, we find

$$\sum_{k\in\Lambda_j} f^s_{jk}N^s_{kk_1}+\lambda\sum_{k'\in\Lambda^j}\sum_{k\in\Lambda^j}\mathscr{L}^{s's}_{j:k'k}\Gamma^s_{j:k}N^{s'}_{k'k_1}=\sum_{k\in\Lambda_j}q^s_{jk}N^s_{kk_1}. \tag{17}$$

As the raw images f^s_{jk}, q^s_{jk} and $\mathscr{L}^{s's}_{j:k'k}$ are expressed using the weights and nodes of one-point quadrature rule, so equation (17) represents a nonlinear system of $(b-a)2^j+2K-2$ number of equations in $f(\bar{x}^s_{jk})$ $(k=a2^j-2K+2,\dots,b2^j-1)$. The raw images f^s_{jk} can be evaluated only when the values of $f(\bar{x}^s_{jk})$ can be found by solving the nonlinear system (17) by any standard numerical method such as Newton's method, Broyden's method, iteration method, Seidal iteration or by using softwere such as MATLAB, MATHEMATICA, Python etc. The unknown function $f(x)$ then can be evaluated at any point in the interval $[a,b]$ using the expression (9).

As a system of nonlinear equations may have more than one solution, one important question arises at this stage is: Can the system (17) be solved to determine the required solution? In every standard numerical method such as Newton's method, Broyden's method, Iteration method, Seidal iteration method, suitable initial approximation is needed which is closed to the required solution. A wrong choice of initial approximation may lead to three situations - the method may diverge; a large number of iteration may be needed; the method may converge to different solution of the system. The code "NSolve[expr, vars]" can be used in MATHEMATICA to find all the possible approximations of the solutions of the system including complex solutions. Again, the code "NSolve[expr, vars, Real]" finds the solution over the domain of real numbers. These two codes are suitable for the system containing maximum three or four variables. For the system containing more variables, the simulation code does not give the required results. The code "FindRoot[$\{equ_1,equ_2,\dots,equ_n\},\{x_1,x_1(0)\},\{x_2,x_2(0)\},\dots,\{x_n,x_n(0)\}$]" enumerate the solution of a system of n number of equations having n number of variables with initial approximation $[x_1(0),x_2(0),\dots,x_n(0)]$. It is evident that the system (17) can be solved to find the required solution if a suitable initial approximation can be found.

It is a somewhat complicated task to determine an initial approximation of the solution of the system (17), as the nature of $f(x)$ is completely unspecified except its continuity. As $f(x)$ is defined on a closed and bounded interval $[a,b]$, so the images of f form a bounded set. The nodes $\bar{x}^s_{jk}$ $(k=a2^j-2K+2,a2^j-2K+3,\dots,b2^j-1)$ for one-point quadrature rule belongs to (a,b). We consider two different starting points of the solution, one is $\{q(a)+\varepsilon, q(a)+2\varepsilon, q(a)+3\varepsilon, \ldots, q(a)+((b-a)2^j+2K-2)\varepsilon)\}$ and the other is $\{q(a)-\varepsilon, q(a)-2\varepsilon, q(a)-3\varepsilon, \ldots, q(a)-((b-a)2^j+2K-2)\varepsilon)\}$ of trial and error method. Taking these two starting points, we measure CPU time for solving nonlinear system for each case. The initial approximation which corresponds to lower CPU time is taken as starting point. For different choices of $\varepsilon(\geq 0)$, we get different initial approximations. Sometime, the forcing term $q(x)$ may not be defined at the starting point $x=a$ of the domain $[a,b]$. In that case $x=a+\delta$ $(\delta>0,\text{very small})$ is to be considered instead of $x=a$. Though the starting point for applying "FindRoot[ ]" code in MATHEMATICA can be guessed here by trial and error method but it has no theoretical basis. The practical aspect of such selection is verified by taking a large number of illustrative examples.

5 Error estimation

As the equation (17) represents a system of $(b-a)2^j+2K-2$ number of nonlinear equations in $f(\bar{x}^s_{jk})$ $(k=a2^j-2K+2,\ldots,b2^j-1)$, we assume that the system can be written as

$$G_i(f(\bar{x}^L_{ja2^j-2K+2}),\ldots,f(\bar{x}^I_{ja2^j}),\ldots,f(\bar{x}^R_{jb2^j-1}))=0\;,(i=1,2,\ldots(b-a)2^j+2K-2). \tag{18}$$

Setting $f(\bar{x}^s_{jk})=\eta^j_{k-(a2^j-2K+1)}$, the system (18) can be written as

$$G_i(\eta^j_1,\eta^j_2,\ldots,\eta^j_{(b-a)2^j+2K-2})=0. \tag{19}$$

We assume that $(\xi^j_1,\xi^j_2,\ldots,\xi^j_{(b-a)2^j+2K-2})$ is the approximation to the exact results $(\zeta^j_1,\zeta^j_2,\ldots,\zeta^j_{(b-a)2^j+2K-2})$ at resolution level j. If $\Delta\xi^j_i$ $(i=1,2,\ldots,(b-a)2^j+2K-2)$ is the error in the deremination of ξ^j_i, then $\xi^j_i+\Delta\xi^j_i(=\zeta^j_i)$ satisfies

$$G_i(\xi^j_1+\Delta\xi^j_1,\xi^j_2+\Delta\xi^j_2,\ldots,\xi^j_{(b-a)2^j+2K-2}+\Delta\xi^j_{(b-a)2^j+2K-2})=0. \tag{20}$$

Taylor series expansion about $(\xi^j_1,\xi^j_2,\ldots,\xi^j_{(b-a)2^j+2K-2})$ gives

$$\begin{aligned} G_i(\xi^j_1,\xi^j_2+,\ldots,\xi^j_{(b-a)2^j+2K-2})+\left(\sum_{i=1}^{(b-a)2^j+2K-2}\Delta\xi^j_i\frac{\partial}{\partial\xi^j_i}\right)G_i(\xi^j_1,\xi^j_2+,\ldots,\xi^j_{(b-a)2^j+2K-2})\\ +\frac{1}{2!}\left(\sum_{i=1}^{(b-a)2^j+2K-2}\Delta\xi^j_i\frac{\partial}{\partial\xi^j_i}\right)^2 G_i(\xi^j_1,\xi^j_2+,\ldots,\xi^j_{(b-a)2^j+2K-2})+\cdots=0. \end{aligned} \tag{21}$$

As $\Delta\xi^j_i$ $(i=1,2,3,\ldots,(b-a)2^j+2K-2)$ are very small, so neglecting the second and higher power of $\Delta\xi^j_i$, we get

$$G_i(\xi^j_1,\xi^j_2+,\ldots,\xi^j_{(b-a)2^j+2K-2})+\left(\sum_{i=1}^{(b-a)2^j+2K-2}\Delta\xi^j_i\frac{\partial G_i}{\partial\xi^j_i}\right)=0. \tag{22}$$

The above system can be written in matrix form as

$$\tilde{\mathbf{J}}\boldsymbol{\Delta}=-\tilde{\mathbf{F}}, \tag{23}$$

where

$$\tilde{\mathbf{J}} = \begin{pmatrix} \frac{\partial G_1}{\partial \xi_1^j} & \frac{\partial G_1}{\partial \xi_2^j} \cdots & \frac{\partial G_1}{\partial \xi_{(b-a)2^j+2K-2}^j} \\ \frac{\partial G_2}{\partial \xi_1^j} & \frac{\partial G_2}{\partial \xi_2^j} \cdots & \frac{\partial G_2}{\partial \xi_{(b-a)2^j+2K-2}^j} \\ \vdots & \vdots \quad \cdots & \vdots \\ \frac{\partial G_{(b-a)2^j+2K-2}}{\partial \xi_1^j} & \frac{\partial G_{(b-a)2^j+2K-2}}{\partial \xi_2^j} \cdots & \frac{\partial G_{(b-a)2^j+2K-2}}{\partial \xi_{(b-a)2^j+2K-2}^j} \end{pmatrix}_{(\xi_1^j, \xi_2^j, \ldots, \xi_{(b-a)2^j+2K-2}^j)}, \tag{24}$$

and

$$\boldsymbol{\Delta} = \begin{pmatrix} \Delta\xi_1^j \\ \Delta\xi_2^j \\ \vdots \\ \Delta\xi_{(b-a)2^j+2K-2}^j \end{pmatrix}, \quad \tilde{\mathbf{F}} = \begin{pmatrix} G_1(\xi_1^j, \xi_2^j, \ldots, \xi_{(b-a)2^j+2K-2}^j) \\ G_2(\xi_1^j, \xi_2^j, \ldots, \xi_{(b-a)2^j+2K-2}^j) \\ \vdots \\ G_{(b-a)2^j+2K-2}(\xi_1^j, \xi_2^j+, \ldots, \xi_{(b-a)2^j+2K-2}^j) \end{pmatrix}. \tag{25}$$

The matrix $\boldsymbol{\Delta}$ which contains errors $\Delta\xi_i^j$ as its elements is determined as $\boldsymbol{\Delta} = -\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{F}}$ and its norm is determined from the inequality

$$\|\boldsymbol{\Delta}\| = \left\|\tilde{\mathbf{J}}^{-1}\tilde{\mathbf{F}}\right\| \leq \left\|\tilde{\mathbf{J}}^{-1}\right\| \left\|\tilde{\mathbf{F}}\right\|. \tag{26}$$

Here, $\|.\|$ indicates a suitable norm such as Euclidean norm, maximum norm, spectral norm, etc.

6 Numerical illustration

The method is illustrated numerically by considering the following three examples.

Example 1. Consider the nonlinear integral equation [1]

$$f(x) - \int_0^1 xy[f(y)]^2 \mathrm{d}y = -\frac{1}{4}(x - 4e^x + xe^2), \quad 0 \leq x \leq 1. \tag{27}$$

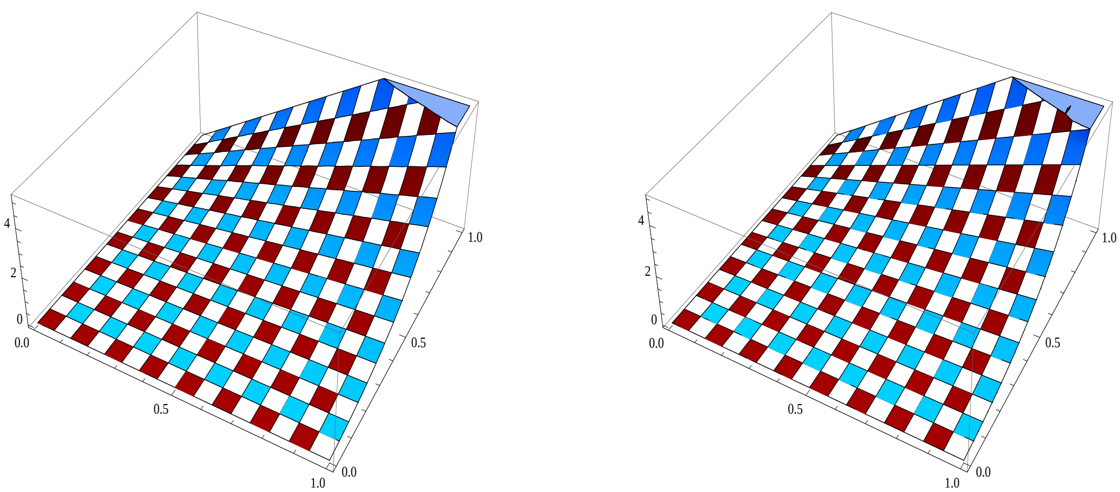
This integral equation is of the Hammerstein type and has the exact solution $f(x) = e^x$. The system (17) corresponding to this example is solved here by using the code "FindRoot[ ]" in MATHEMATICA with $\{q(a)+\varepsilon, q(a)+2\varepsilon, \ldots, q(a)+((b-a)2^j+2K-2)\varepsilon\}$ as the starting point. $\varepsilon = 0.2$ and $\varepsilon = 0.15$ are chosen for the level $j = 4$ and $j = 5$ respectively and $K = 3$ is selected in both resolution level. In Table 1, a comparison between the exact and approximate solutions are shown for Example 1 at the intermediate points $\frac{s}{8}$ $(s = 0, 1, 2, \ldots, 8)$ for $j = 4, 5$. In Fig. 1, the exact and approximate forms of the kernel are depicted for Example 1. Actually, Fig. 1, shows the efficiency of Daubechies scale functions as basic building block in the approximation of functions of several variables. In Fig. 2, the exact and approximate forms of the unknown function $f(x)$ are depicted for Example 1.

Example 2. Consider the nonlinear integral equation [12]

$$f(x) - \int_0^1 \frac{4xy + \pi x \sin(\pi y)}{(f(y))^2 + y^2 + 1} \mathrm{d}y = \sin(\frac{\pi}{2}x) - 2x\ln(3), \quad 0 \leq x \leq 1. \tag{28}$$

Table 1: Exact and approximate solutions of Example 1.

x	Exact solution	Approximate $j=4$	solution $j=5$
0	1	0.988488	0.988028
1/8	1.1331485	1.112890	1.133145
2/8	1.2840254	1.284001	1.284022
3/8	1.4549914	1.454964	1.454988
4/8	1.6487231	1.648698	1.648717
5/8	1.8682460	1.868210	1.868241
6/8	2.1170000	2.116960	2.116995
7/8	2.3988753	2.399263	2.398869
1	2.7182818	2.718243	2.718283



(a) Kernel's exact form (b) Kernel's approximate form $(K=3, j=5)$

Figure 1: Comparison of the exact and approximate form of kernel for Example 1.

This nonlinear integral equation is of Uryshon type having nonseparable kernel and it has exact solution $f(x)=\sin(\frac{\pi}{2}x)$.

For Example 2, $\{q(a)+\varepsilon, q(a)+2\varepsilon, \dots, q(a)+((b-a)2^j+2K-2)\varepsilon)\}$ is taken as starting point with $\varepsilon=0.006$ for $j=4$ and $\varepsilon=0.0003$ for $j=5$. In Table 2, the approximate solution together with exact solution are given at nine different intermediate dyadic points taking $K=3$ at resolution level $j=4$ and $j=5$. Also in Fig. 3 and Fig. 4, the approximate and analytical forms of kernel function and unknown function are compared graphically for Example 2. From Fig. 3, it is clear that there is some fluctuation in the kernel's approximate values near $x=1$ and $y=1$.

Example 3. Consider the Hammerstein integral equation of mixed type [8]

$$f(x)=q(x)+\int_0^1 \mathscr{L}(x,y,f(y))\mathrm{d}y, \quad 0<x\leq 1, \tag{29}$$

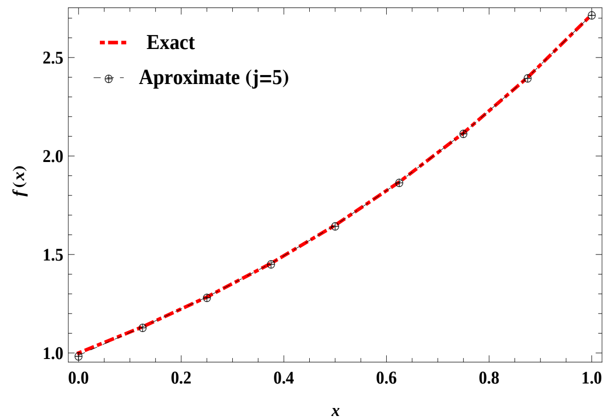


Figure 2: Exact and approximate solutions of Example 1.

Table 2: Exact and approximate solutions of Example 2.

x	Exact solution	Approximate $j=4$	solution $j=5$
0	0	0.001443	0.000722
1/8	0.19509032	0.197702	0.195100
2/8	0.38268343	0.382758	0.382693
3/8	0.55557023	0.555639	0.555579
4/8	0.70710678	0.707167	0.707114
5/8	0.83146961	0.831519	0.831475
6/8	0.92387953	0.923916	0.923884
7/8	0.98078528	0.980381	0.980788
1	1.00000000	0.999968	0.999989

where

$$q(x) = -\frac{x^4}{6} - \frac{x^6}{4} + \ln(x), \tag{30}$$

and

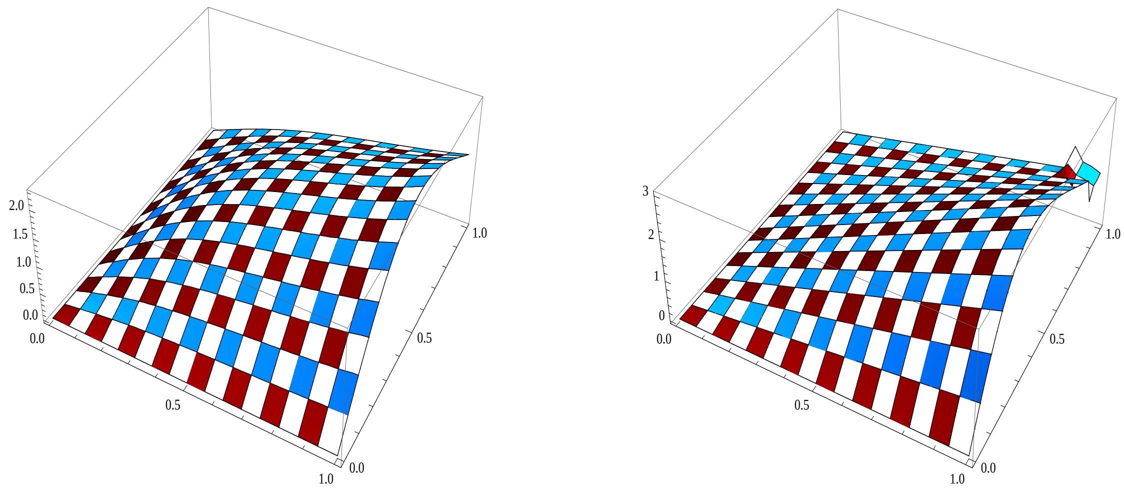
$$\mathscr{L}(x,y,f(y)) = \sum_{i=1}^{2} \mathscr{L}_i(x,y,f(y)) \tag{31}$$

with

$$\mathscr{L}_1(x,y,f(y)) = x^4 y^4 e^{f(y)};\ \mathscr{L}_2(x,y,f(y)) = x^6 y \{f(y)\}^2. \tag{32}$$

The exact solution of this mixed type nonlinear integral equation is $f(x) = \ln(x)$. The integral equation is called mixed type as the kernel $\mathscr{L}(x,y,f(y))$ can be expressed as the sum of two different kernels $\mathscr{L}_1(x,y,f(y))$ and $\mathscr{L}_2(x,y,f(y))$.

As the forcing function $q(x) = -\frac{x^4}{6} - \frac{x^6}{4} + ln(x)$ is not defined at $x = a = 0$, so we consider the starting point as $\{q(0.001)+\varepsilon, q(0.001)+2\varepsilon, \ldots, q(0.001)+((b-a)2^j+2K-2)\varepsilon\}$ with $\varepsilon = 0.30$ for $j = 4$ and $\{q(0.0008)+\varepsilon, q(0.0008)+2\varepsilon, \ldots, q(0.0008)+((b-a)2^j+2K-2)\varepsilon\}$ with $\varepsilon = 0.5$ for $j = 5$ for using “FindRoot[ ]” simulation code. In Table 3, the approximate and exact solution of Example 3 are



(a) Kernel's exact form (b) Kernel's approximate form $(K = 3, j = 5)$

Figure 3: Comparison of the exact and approximate form of the kernel for Example 2.

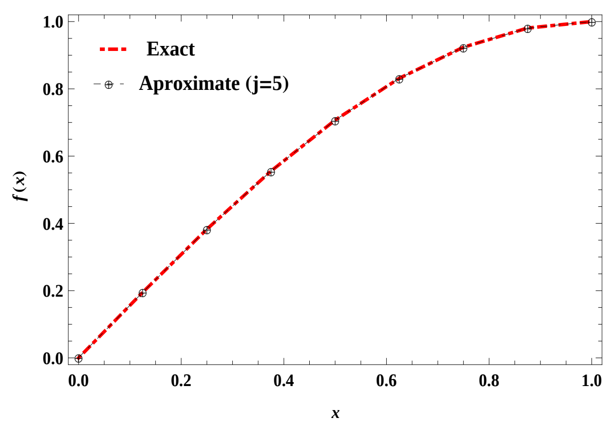


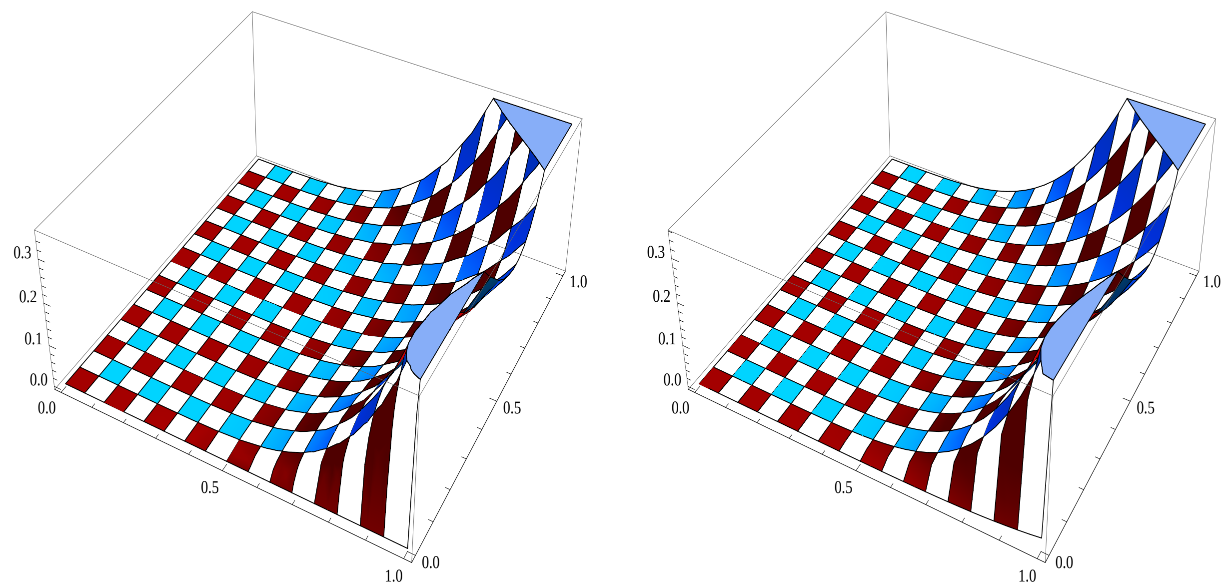
Figure 4: Exact and approximate solutions of Example 2.

displayed taking $K = 3$ and $j = 4, 5$. In Fig 5 and Fig 6, a comparison between the exact and approximate form of the kernel function and unknown function of Example 3 are shown graphically.

In Table 4, different types of norms $\|\mathbf{\Delta}\|_p$ $(p = 1, 2, \infty)$ at $j = 4, 5$ of the matrix $\mathbf{\Delta}$ are exhibited for the three examples. Here, $\|\mathbf{\Delta}\|_1 = \sum_{i=1}^{(b-a)2^j+2K-2} |\Delta\xi_i^j|$ is known as maximum absolute column sum norm whereas $\|\mathbf{\Delta}\|_2 = \left(\sum_{i=1}^{(b-a)2^j+2K-2} |\Delta\xi_i^j|^2\right)^{\frac{1}{2}}$ is the Euclidean norm or Frobenius norm. Also $\|\mathbf{\Delta}\|_\infty$ is the maximum absolute row sum norm and is defined by $\|\mathbf{\Delta}\|_\infty = \max_{1 \le i \le (b-a)2^j+2K-2} |\Delta\xi_i^j|$. The bar diagrams obtained by taking data from Table 4 are displayed in Fig. 7. The bar diagrams for each example describe the reverse proportional relation between norm of the error matrix $\mathbf{\Delta}$ and resolution level j. Observing the bar diagrams, it is clear that the norm of the error matrix decreases more rapidly for Example 1. From Table 4 and Fig. 7, it can be concluded that the errors in the calculation of the

Table 3: Exact and approximate solutions of Example 3.

x	Exact solution	Approximate $j=4$	solution $j=5$
$1/8$	-2.07944154	-2.087967	-2.086196
$2/8$	-1.38629436	-1.393049	-1.386774
$3/8$	-0.98082925	-0.982164	-0.980952
$4/8$	-0.69314718	-0.693627	-0.693195
$5/8$	-0.47000363	-0.470229	-0.470028
$6/8$	-0.28768207	-0.287805	-0.287696
$7/8$	-0.13353139	-0.133820	-0.133540
1	0.00000000	-0.000099	-0.000014



(a) Kernel's exact form (b) Kernel's approximate form $(K=3, j=5)$

Figure 5: Comparison of exact and approximate form of the kernel for Example 3.

unknowns of the nonlinear system of equations can be minimized taking relatively higher resolution level j.

7 Conclusion

Nonlinear integral equations are used to model various physical phenomena but they are generally difficult to solve in closed form. Due to this reason, an efficient wavelet-based numerical scheme has been proposed to find approximate numerical solution of nonlinear Fredholm integral equation of second kind having smooth kernels. One of the major advantages of this method is that the scheme can be applied for any kind of smooth kernel including separable as well as nonseparable kernels. Also, complicated oper-

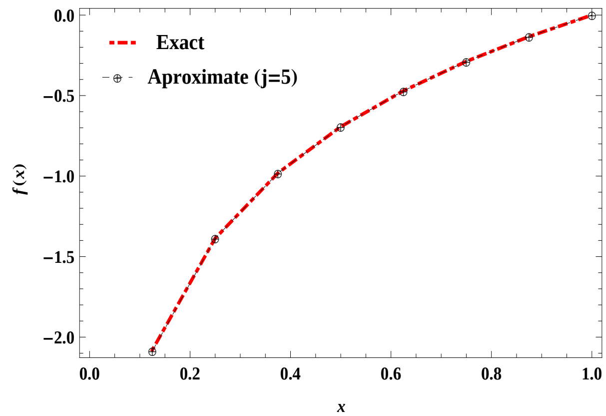


Figure 6: Exact and approximate solutions of Example 3.

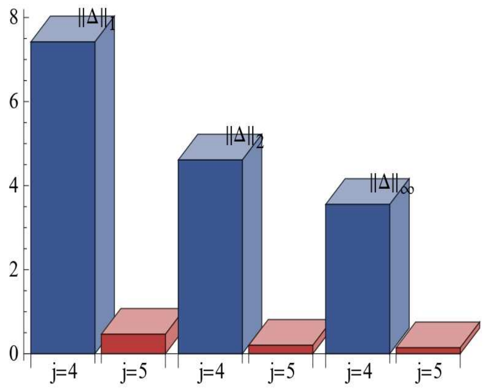
Table 4: Different type of the norms of error matrix $\mathbf{\Delta}$.

Resolution level (j)	$\|\mathbf{\Delta}\|_1$	$\|\mathbf{\Delta}\|_2$	$\|\mathbf{\Delta}\|_\infty$
Example 1			
4	7.41903	4.60869	3.55184
5	0.46210	0.19992	0.14166
Example 2			
4	0.21996	0.11745	0.08203
5	0.05814	0.03145	0.02156
Example 3			
4	0.01337	0.00779	0.00634
5	0.00883	0.00445	0.00321

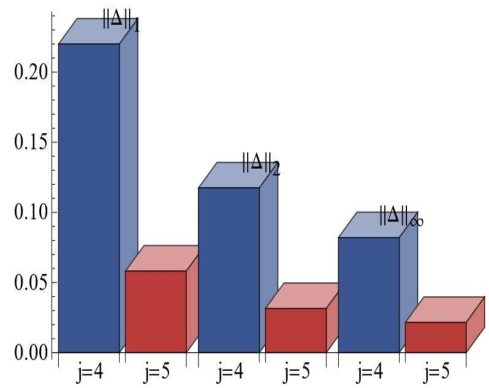
ations do not arise in the present scheme. The idea of the scheme can also be implemented for nonlinear integral equations having singular kernels but appropriate modification is needed. Inclusion of three illustrative examples shows that the present method is quite efficient to provide satisfactory results. The results can be improved by taking higher resolution level which is reflected from our numerical data. It is better to select moderate $(j = 3,4,5)$ resolution level as for higher resolution level, more computational time will be required in solving nonlinear system of equations.

Acknowledgement

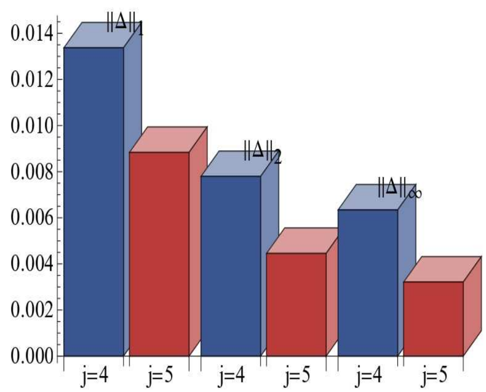
JM acknowledges financial support from University Grants Commission, New Delhi, for the award of research fellowship (File no. 16-9(june2017/2018(NET/CSIR))).



(a) Example 1.



(b) Example 2.



(c) Example 3.

Figure 7: Comparison of different norms $\|\mathbf{\Delta}\|_p$ $(p=1,2,\infty)$ at $j=4,5$.

References

[1] E. Abdolmaleki, H. Saberi Najafi, *An efficient algorithmic method to solve Hammerstein integral equations and application to a functional differential equation*, Adv. Mech. Eng. **9** (2017) 1-8.

[2] M.A. Abdou, M.M. El-Borai, M.M. El-Kojok, *Toeplitz matrix method and nonlinear integral equation of Hammerstein type*, J. Comput. Appl. Math. **223** (2009) 765–776.

[3] Y.B. Band, Y. Avishai, *Quantum Mechanics with Applications to Nanotechnology and Information Science*, Academic Press, 2013.

[4] G. Beylkin, R. Coifman, V. Rokhlin, *Fast wavelet transform and numerical algorithms I*, Communi. pure Appl. Math. **167** (1991) 141–183.

[5] S. Chandrasekhar, *Radiative Transfer*, Courier corporation, 2013.

[6] P. Das, G. Nelakanti, G. Long, *Discrete Legendre spectral projection methods for Fredholm Hammerstein integral equations*, J. Comput. Appl. Math. **278** (2015) 293–305.

[7] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992

[8] E. Hashemizadeh, M. Rostami, *Numerical solution of Hammerstein integral equation of mixed type using the Sinc-collocation method*, J. Comput. Appl. Math. **279** (2015) 31–39.

[9] M.A. Hernndez-Vern, E. Martnez E, S. Singh, *On the Chandrasekhar integral equation* Comp. Math. Methods 2021, DOI: https://doi.org/10.1002/cmm4.1150.

[10] B.M. Kessler, G.L. Payne, W.W. Polyzou, *Wavelet Notes*, 2003, https://arxiv.org/abs/nucl-th/0305025.

[11] Y. Mahmoudi, *Wavelet Galerkin method for numerical solution of nonlinear integral equation*, Appl. Math. Comput. **167** (2005) 1119–1129.

[12] K. Maleknejad, M. Karami, M. Rabbani, *Using the Petrov-Galerkin elements for solving Hammerstein integral equations*, Appl. Math. Comput. **172** (2006) 831–845.

[13] J. Mouley, M.M. Panja, B.N. Mandal, *Numerical solution of an integral equation arising in the problem of cruciform crack using Daubechies scale function*, Math. Sci **14** (2020) 21–27.

[14] J. Mouley, M.M. Panja, B.N. Mandal, *Approximate solution of Abel integral equation in Daubechies wavelet basis*, Cubo, A Math. J. **23** (2021) 21–27.

[15] Y. Ordokhani, M. Razzaghi, *Solution of Nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationlized Haar functions*, Appl. Math. Lett. **21** (2008) 4–9.

[16] M.M. Panja, B.N. Mandal, *A note on one-point quadrature formula for Daubechies scale function with partial support*, Appl. Math. Comput. **218** (2011) 4147–4151.

[17] M.M. Panja, B.N. Mandal, *Evaluation of singular integrals using Daubechies scale function*, *Adv. Comput. Math. Appl.* **3** (2012) 64–75.

[18] M.M. Panja , B.N. Mandal, *Gauss-type quadrature rule with complex nodes and weights for integrals involving Daubechies scale functions and wavelets*, J. Comput. Appl. Math. **290** (2015) 609–632.

[19] C. Richards, *Nonlinear integral equations and their solutions*, Boise State University, 2016.

[20] M. Suzuki, *Emerging broken symmetry in space and time, in Thermal Field Theories and Their Applications*, eds. H. Ezawa, T. Arimitsu and Y. Hashimoto (Elsevier, Amsterdam), (1991) 5-15.