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Inverse spectral problems for arrowhead matrices

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Abstract. The problem of constructing a matrix by its spectral information is called inverse eigenvalue problem (IEP) which arises in a variety of applications. In this paper, we study an IEP for arrowhead matrices in different cases. The problem involves constructing of the matrix by some eigenvalues of each of the leading principal submatrices and one eigenpair. We will also investigate this problem and its variants in the cases of matrix entries being real, nonnegative, positive definite, complex and equal diagonal entries. To solve the problems, a new method to establish a relationship between the IEP and properties of symmetric and general form of matrices is developed. The necessary and sufficient conditions of the solvability of the problems are obtained. Finally, some numerical examples are presented.

Keywords: Inverse eigenvalue problem, arrowhead matrix, principal submatrix.
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1 Introduction

An IEP, is the problem of constructing a matrix by its spectral information. IEPs are categorized by the available eigen information of the matrix to be constructed. Chu and Golub in [4] gave a perfect characterization of IEPs. IEPs usually come with a practical background and its applications have been studied in various scientific fields such as control, image processing, graph theory, finite element method.

An arrowhead matrix is the following matrix:

$$B_n = \begin{bmatrix} h_1 & e_1 & e_2 & \cdots & e_{n-1} \\ f_1 & h_2 & 0 & \cdots & 0 \\ f_2 & 0 & h_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \cdots & 0 & h_n \end{bmatrix}, \tag{1}$$

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such that $e_i f_i \neq 0,\ i = 1,\dots,n-1$. Variants of this matrix are well studied in many literatures and they have many applications in orthogonal reduction method and other tridiagonalization processes, including Householder transformation, Givens rotations, the Rutishauser method, and so on, and it may also be used effectively to explore the arrow in graph theory [4]. Peng et al. in 2006 [11] studied arrowhead matrices to model nonlinear control systems.

In [3, 6, 7, 12–15, 19], recursive methods to solve different IEPs for different matrices was studied. For an $n \times n$ matrix, the graph of this matrix is a graph with n vertices such that off-diagonal entries of the matrix represents the edges. The entry $a_{ij} \neq 0$ if and only if there is an edge between vertices i and j. IEP of matrices of graphs is a famous class of IEPs and it has attracted the attention of many researchers in the past decade. Different variants of this problem for different graphs have been studied in the literature [1, 8, 9, 17, 18]. They recursively constructed real matrices by the given eigendata. Also Rundell and Sacks in [16] study an IEP for symmetric arrowhead matrices which correspond to a simple star graph. It is worthwhile to mention that matrices corresponding to undirected graphs are symmetric but matrices corresponding to directed graphs are not necessarily symmetric.

In this paper, we will investigate constructing of arrowhead matrices entries in different cases: real, nonnegative, equal diagonal entries positive definite and complex by an eigenpair of the matrix to be constructed with one or two eigenvalues of each of its leading principal submatrices by a recursive method. It appears in different applications including perturbation of control systems, directed star graphs and Sturm-Liouville.

The paper is organized as follows: In Section 2, the relationship between symmetric and general matrices and their IEPs are obtained. In Section 3, the conditions of existence of solutions in different cases mentioned above are achieved. In Section 4, we test the results by a numerical example. Finally, in Section 5, some concluding remarks are presented.

2 The relation between symmetric and general form of IEPs

In this section, we establish a relationship between matrices B_n defined as (1) and its similar symmetric form. By this similarity we can further extend the results proved in the symmetric case to the general form of matrices and IEPs. By considering that similar matrices have the same eigenvalues [5], we obtain similar and symmetric matrices for B_n as follows.

Let

$$S_n = \begin{bmatrix} \gamma_1 & \psi_1 & \psi_2 & \cdots & \psi_{n-1} \\ \psi_1 & \gamma_2 & 0 & \cdots & 0 \\ \psi_2 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \psi_{n-1} & 0 & \cdots & 0 & \gamma_n \end{bmatrix}, \qquad D_n = diag(d_1, d_2, \dots, d_n).$$

Now we construct the symmetric similarity matrix as follows:

$$D_nB_nD_n^{-1} = \begin{bmatrix} h_1 & \frac{e_1d_1}{d_2} & \frac{e_2d_1}{d_3} & \cdots & \frac{e_{n-1}d_1}{d_n} \\ \frac{f_1d_2}{d_1} & h_2 & 0 & \cdots & 0 \\ \frac{f_1d_3}{d_2} & 0 & h_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{f_{n-1}d_n}{d_1} & 0 & \cdots & 0 & h_n \end{bmatrix}.$$

Hence, we should have

$$\frac{e_{i-1}d_1}{d_i} = \frac{f_{i-1}d_i}{d_1} \text{ or } d_i^2 = \frac{e_{i-1}d_1^2}{f_{i-1}}, \quad i = 2,\dots,n.$$

So d_i's, $i = 2,\dots,n$, are multipliers of d_1. Without loss of generality, let d_1 be an arbitrary nonzero real number, say $d_1 = 1$. Hence

$$d_i = \sqrt{\frac{e_{i-1}}{f_{i-1}}},\ i = 2,\dots,n,$$

and therefore,

$$\psi_i = \sqrt{e_i f_i},\ i = 1,\dots,n-1,$$

$$\gamma_i = h_i,\ i = 1,\dots,n,$$

then, $S_n = D_nB_nD_n^{-1}$ and as a result B_n and S_n are similar.

Let $Q_n = \det(\lambda I - B_n)$ be the characteristic polynomial of B_n defined in (1) and $B_j, j = 1,\dots,n$ be the jth leading principal submatrix of B_n. We have the following lemmas.

Lemma 1. *The sequence $Q_j(\lambda), j = 1,\dots,n$ satisfy the following recurrence relations:*

$$Q_1(\lambda) = \lambda - h_1,$$

$$Q_2(\lambda) = (\lambda - h_2)Q_1(\lambda) - e_1f_1,$$

$$Q_j(\lambda) = (\lambda - h_j)Q_{j-1}(\lambda) - e_{j-1}f_{j-1}\prod_{i=2}^{j-1}(\lambda - h_i),\ j = 3,\dots,n. \tag{2}$$

Proof. The recurrence relations can be verified by expanding the determinant. □

Lemma 2. *If $(\lambda^{(n)}, X_n)$ is an eigenpair of B_n, then $x_1 \neq 0$ and the components of eigenvector X_n are obtained as:*

$$x_i = \frac{f_{i-1}}{\lambda^{(n)} - h_i}x_1, \quad i = 2,\dots,n.$$

Proof. Since $(\lambda^{(n)}, X_n)$ is an eigenpair of B_n, so $B_nX_n = \lambda^{(n)}X_n$. We can obtain the result by expanding this relation. □

The following theorems are necessary in the sequel.

Theorem 1. *[2] Let the matrix $A \in \mathbb{R}^{n\times n}$ be symmetric. A is positive definite if and only if all of its eigenvalues are positive.*

We know that a symmetric matrix A is positive definite if for every nonzero vector X, $X^tAX > 0$ [5].

Theorem 2. *(Cauchy's interlacing theorem [10]). Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of an $n \times n$ real symmetric matrix A and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of an $(n-1) \times (n-1)$ principal submatrix of A, then*

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n.$$

Also, it can be inferred that all the eigenvalues of B_n are distinct and real numbers [12, 13].

The important point is that there are two unknown set of entries in IEP of symmetric matrices, but there are three unknown sets of entries in IEP of asymmetric matrices.

By regarding the Cauchy's interlacing theorem, we will solve IEPs for matrix B_n in the next section.

3 Main problems

In this section, the problems statement in different cases of real, nonnegative, equal diagonal entries positive definite and complex are presented and necessary conditions for solvability of them are proposd. We briefly call them AIEP.

3.1 Real matrix

In this case, we construct the matrix B_n with real entries. The diagonal entries may be zero.

Problem 1 (AIEP1). *The real and distinct numbers*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

and real vector $X_n = (x_1, \ldots, x_n)^T$ are given. Find matrix B_n such that the numbers $\lambda_1^{(i)}, \lambda_i^{(i)}, i = 1, \ldots, n$ are minimum and maximum eigenvalues of B_i, respectively and $(\lambda_n^{(n)}, X_n)$ is an eigenpair of B_n.

In the next theorem , we obtain the necessary and sufficient conditions for the existence and uniqueness of a solution to AIEP1.

Theorem 3. *The AIEP1 has a unique solution if and only if $x_i \neq 0$, $i = 1, \ldots, n$.*

Proof. Let $x_i \neq 0$, $i = 1, \ldots, n$. It is easy to see that $h_1 = \lambda_1^{(1)}$. From Lemma 1, and noting that

$$Q_i(\lambda_1^{(i)}) = Q_i(\lambda_i^{(i)}) = 0,$$

we obtain

$$h_2 = \frac{\lambda_2^{(2)} Q_1(\lambda_2^{(2)}) - \lambda_1^{(2)} Q_1(\lambda_1^{(2)})}{Q_1(\lambda_2^{(2)}) - Q_1(\lambda_1^{(2)})}, \tag{3}$$

and

$$h_i = \frac{Q_{i-1}(\lambda_i^{(i)})\lambda_i^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}{Q_{i-1}(\lambda_i^{(i)}) \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)}) \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}, \quad i = 3, \ldots, n, \tag{4}$$

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$$\psi_{i-1}^2 = e_{i-1}f_{i-1} = \frac{(\lambda_i^{(i)}-\lambda_1^{(i)})Q_{i-1}(\lambda_1^{(i)})Q_{i-1}(\lambda_i^{(i)})}{Q_{i-1}(\lambda_i^{(i)})\prod_{j=2}^{i-1}(\lambda_1^{(i)}-h_j) - Q_{i-1}(\lambda_1^{(i)})\prod_{j=2}^{i-1}(\lambda_i^{(i)}-h_j)}, \quad i=2,\ldots,n.$$

Since $B_nX_n = \lambda_n^{(n)}X_n$, thus

$$f_{i-1} = \frac{(\lambda_n^{(n)}-h_i)x_i}{x_1}, \quad i=2,\ldots,n. \tag{5}$$

By substituting h_i and f_{i-1} in Eq. (2) and from Lemma 1, we obtain

$$e_1 = \frac{(\lambda_2^{(2)}-h_2)Q_1(\lambda_2^{(2)})}{f_1}, \tag{6}$$

$$e_{i-1} = \frac{(\lambda_i^{(i)}-h_i)Q_{i-1}(\lambda_i^{(i)})}{f_{i-1}\prod_{j=2}^{i-1}(\lambda_i^{(i)}-h_j)}, \quad i=3,\ldots,n. \tag{7}$$

Hence, there exists real solution to the problem if $e_jf_j > 0$, $j=1,\ldots,n-1$. Since

$$Q_{j-1}(\lambda) = (\lambda-\lambda_1^{(j-1)})(\lambda-\lambda_2^{(j-1)})\cdots(\lambda-\lambda_{j-1}^{(j-1)}),$$

therefore, we obtain

$$Q_{j-1}(\lambda_j^{(j)}) > 0 \ \text{ and } \ (-1)^{j-1}Q_{j-1}(\lambda_1^{(j)}) > 0.$$

So

$$(\lambda_j^{(j)}-\lambda_1^{(j)})(-1)^{j-1}Q_{j-1}(\lambda_1^{(j)})Q_{j-1}(\lambda_j^{(j)}) > 0.$$

Let

$$N_{j-1} = Q_{j-1}(\lambda_1^{(j)})Q_{j-2}(\lambda_j^{(j)}) - Q_{j-1}(\lambda_j^{(j)})Q_{j-2}(\lambda_1^{(j)}), \tag{8}$$

then by the same reasons $(-1)^{j-1}N_{j-1} > 0$ and so

$$e_jf_j > 0, \quad j=1,\ldots,n-1.$$

With regard to Lemma 1, we obtain a unique solution for AIEP1.

Conversely, assume that there exists a solution to AIEP1 and X_n be an eigenvector of B_n. By Lemma 2, the component x_1 is nonzero, so we obtain $x_i \neq 0$, $i=1,\ldots,n$ which completes the proof. □

The following algorithm is presented to solve AIEP1.

Algorithm 1 (Solving AIEP1)

Input:

Real numbers $\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}$, and the

real vector $X_n = (x_1, \ldots, x_n)^T$.

If one of $x_i = 0$, $i = 1, \ldots, n$, Stop. (Problem 1 can not be solved by this algorithm)

Set $\quad h_1 = \lambda_1^{(1)}$,

Set

$\qquad h_2 = \frac{\lambda_2^{(2)} Q_1(\lambda_2^{(2)}) - \lambda_1^{(2)} Q_1(\lambda_1^{(2)})}{Q_1(\lambda_2^{(2)}) - Q_1(\lambda_1^{(2)})}$,

$\qquad f_1 = \frac{(\lambda_n^{(n)} - h_2) x_2}{x_1}$,

$\qquad e_1 = \frac{(\lambda_2^{(2)} - h_2) Q_1(\lambda_2^{(2)})}{f_1}$.

Set

$\qquad Q_1(\lambda) = \lambda - h_1$,

$\qquad Q_2(\lambda) = (\lambda - h_2) Q_1(\lambda) - e_1 f_1$,

For $i = 3, \ldots, n$

$\qquad h_i := \frac{Q_{i-1}(\lambda_i^{(i)}) \lambda_i^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)}) \lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}{Q_{i-1}(\lambda_i^{(i)}) \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)}) \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}$,

$\qquad f_{i-1} := \frac{(\lambda_n^{(n)} - h_i) x_i}{x_1}$,

$\qquad e_{i-1} := \frac{(\lambda_i^{(i)} - h_i) Q_{i-1}(\lambda_i^{(i)})}{f_{i-1} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}$.

$\qquad Q_i(\lambda) := (\lambda - h_i) Q_{i-1}(\lambda) - e_{i-1} f_{i-1} \prod_{j=2}^{i-1} (\lambda - h_j)$.

end for.

Output: B_n.

3.2 Nonnegative matrices

In this case, we construct the matrix B_n such that all its entries are nonnegative numbers.

Problem 2 (AIEP2). *The real and distinct numbers*

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

and real vector $X_n = (x_1, \ldots, x_n)^T$ are given. Find matrix B_n with nonnegative entries such that the numbers $\lambda_1^{(i)}, \lambda_i^{(i)}, i = 1, \ldots, n$ are minimum and maximum eigenvalues of B_i, respectively and $(\lambda_n^{(n)}, X_n)$ is an eigenpair of B_n.

Theorem 4. *Problem 2 has a unique solution if and only if*

$$\lambda_1^{(1)} > 0, \tag{9}$$

$$\lambda_i^{(i)} \prod_{j=2}^{i-1} \frac{(\lambda_1^{(i)} - h_j)}{(\lambda_i^{(i)} - h_j)} > \lambda_1^{(i)} \frac{Q_{i-1}(\lambda_1^{(i)})}{Q_{i-1}(\lambda_i^{(i)})}, \quad i = 2, \ldots, n, \tag{10}$$

$$x_1 x_i > 0, \; i = 1, \cdots, n. \tag{11}$$

Proof. Let from Theorem 3 the matrix B_n exists with the form of its solutions as given in Eqs. (3), (4), (5), (6), and (7) with nonnegative entries. We find the conditions under which the solution is nonnegative. First, $h_1 = \lambda_1^{(1)} > 0$.

Set

$$n_{i-1} = Q_{i-1}(\lambda_i^{(i)})\lambda_i^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j).$$

Since B_n and S_n are similar matrices, hence they have the same eigenvalues and the Cauchy's theorem guarantees the following inequality

$$\lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)}.$$

From Theorem 3 and Eq. (8), for $i = 2, \ldots, n$,

$$h_i = \frac{Q_{i-1}(\lambda_i^{(i)})\lambda_i^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)}{Q_{i-1}(\lambda_i^{(i)}) \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)}) \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j)} = \frac{(-1)^{i-1} n_{i-1}}{(-1)^{i-1} N_{i-1}} > 0,$$

where $(-1)^{i-1} N_{i-1} > 0$, so $(-1)^{i-1} n_{i-1} > 0$ and as a result Eq. (10) holds. In addition, $f_i > 0$, by considering $f_{i-1} = \frac{(\lambda_n^{(n)} - h_i)x_i}{x_1}$, $i = 2, \ldots, n$, it concludes Eq. (11) .

Conversely, assume that Eqs. (9), (10) and (11) hold. From Eq. (9), $\lambda_1^{(1)} = h_1 > 0$. From condition (10) we see that

$$(-1)^{i-1} n_i = (-1)^{i-1} Q_{i-1}(\lambda_i^{(i)})\lambda_i^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) + (-1)^{i-2} Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_i^{(i)} - h_j) > 0,$$

by the Cauchy's theorem and condition (9), $0 < \lambda_1^{(1)} < \lambda_j^{(j)}$. By the proof of Theorem 3 we obtain $Q_{i-1}(\lambda_i^{(i)}) > 0$ and $(-1)^{i-1} Q_{i-1}(\lambda_1^{(i)}) > 0$, so $h_j = \frac{n_{j-1}}{N_{j-1}} > 0$. If $x_1 x_i > 0$, then $f_{i-1} = \frac{(\lambda_n^{(n)} - h_i)x_i}{x_1} > 0$, $i = 2, \ldots, n$. From Theorem 3, $e_i f_i > 0$, so $e_j > 0$ and it completes the proof. □

3.3 Equal diagonal entries

We construct the matrix B_n such that diagonal entries have the same values by one eigenpair and maximum eigenvalue of each leading principal submatrix.

Problem 3 (AIEP3). *Real distinct numbers $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n-1)} < \lambda^{(n)}$, and real vector $X_n = (x_1, \ldots, x_n)^T$ are given. Construct the matrix B_n such that $\lambda^{(j)}$ is the maximum eigenvalue of B_j and $(\lambda^{(n)}, X_n)$ is an eigenpair of B_n and diagonal entries of B_n have the same value.*

Theorem 5. *There exists a real solution to Problem 3 if and only if $x_i \neq 0$, $i = 1, \ldots, n$.*

Proof. Let $x_i \neq 0$, $i = 1, \ldots, n$, and the value of diagonal entries be a real number h. It is clear that $h_i = h = \lambda_1$. From $Q_j(\lambda^{(j)}) = 0$, by Lemma 1, we get

$$e_1 f_1 = (\lambda^{(2)} - h) Q_1(\lambda^{(2)}),$$

$$\psi_{j-1}^2 = e_{j-1}f_{j-1} = \frac{(\lambda^{(j)}-h)Q_{j-1}(\lambda^{(j)})}{\prod_{i=2}^{j-1}(\lambda^{(j)}-h)} = \frac{Q_{j-1}(\lambda^{(j)})}{(\lambda^{(j)}-h)^{j-3}}, \quad j=3,\ldots,n.$$

By the Cauchy's theorem, we have $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(n-1)} < \lambda^{(n)}$, so $\lambda^{(j)} - h > 0,\ j=2,\ldots,n$, $Q_{j-1}(\lambda^{(j)}) > 0$. Therefore, $e_{j-1}f_{j-1} > 0, j=2,\ldots,n$. From Lemma 2, $f_{i-1} = \frac{(\lambda^{(n)}-h)x_j}{x_1}$, $i=2,\ldots,n$, so

$$e_1 = \frac{(\lambda^{(2)}-h)Q_1(\lambda^{(2)})}{f_1},$$

and

$$e_{j-1} = \frac{Q_{j-1}(\lambda^{(j)})}{f_{j-1}(\lambda^{(j)}-h)^{j-3}}, \quad j=3,\ldots,n.$$

Conversely, if there is a real solution to the problem, then it is easy to see that $x_i \neq 0$, $i=1,\ldots,n$. □

3.4 Symmetric positive definite matrix

The conditions for the existence of a positive definite solution is given in Theorem 1. Since a positive definite matrix is symmetric, so $e_i = f_i$. Therefore, we just need to set the following condition on the input set to obtain a positive definite solution with

$$0 < \lambda_1^{(n)} < \lambda_1^{(n-1)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

and from Algorithm 1 we get

$$h_1 = \lambda_1^{(1)}, \tag{12}$$

$$e_1{}^2 = (\lambda_2^{(2)} - h_2)Q_1(\lambda_2^{(2)}), \tag{13}$$

$$h_i = \frac{Q_{i-1}(\lambda_i^{(i)})\lambda_i^{(i)}\prod_{j=2}^{i-1}(\lambda_1^{(i)}-h_j) - Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)}\prod_{j=2}^{i-1}(\lambda_i^{(i)}-h_j)}{Q_{i-1}(\lambda_i^{(i)})\prod_{j=2}^{i-1}(\lambda_1^{(i)}-h_j) - Q_{i-1}(\lambda_1^{(i)})\prod_{j=2}^{i-1}(\lambda_i^{(i)}-h_j)}, \quad i=2,\ldots,n, \tag{14}$$

$$e_{i-1}{}^2 = \frac{(\lambda_i^{(i)}-h_i)Q_{i-1}(\lambda_i^{(i)})}{\prod_{j=2}^{i-1}(\lambda_i^{(i)}-h_j)}, \quad i=3,\ldots,n. \tag{15}$$

Without loss of generality, we take the maximum as e_i.

3.5 Complex matrix

If any of the eigenvalues or any component of the vector $X_n = (x_1,\ldots,x_n)^T$ are complex, then it is possible that some entries of B_n become complex. In this case, by eliminating the inequality in Problem 1, we get the same solutions.

Corollary 1. *$2n-1$ distinct scalars $\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_1^{(n)}, \lambda_2^{(n)}$ and vector $X_n = (x_1, \ldots, x_n)^T$ are given. Arrowhead matrix B_n of the form (1) with entries*

$$
\begin{aligned}
h_1 &= \lambda_1^{(1)}, \\
h_2 &= \frac{\lambda_1^{(2)} Q_1(\lambda_1^{(2)}) - \lambda_2^{(2)} Q_1(\lambda_2^{(2)})}{Q_1(\lambda_1^{(2)}) - Q_1(\lambda_2^{(2)})}, \\
h_i &= \frac{Q_{i-1}(\lambda_2^{(i)})\lambda_2^{(i)} \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)})\lambda_1^{(i)} \prod_{j=2}^{i-1} (\lambda_2^{(i)} - h_j)}{Q_{i-1}(\lambda_2^{(i)}) \prod_{j=2}^{i-1} (\lambda_1^{(i)} - h_j) - Q_{i-1}(\lambda_1^{(i)}) \prod_{j=2}^{i-1} (\lambda_2^{(i)} - h_j)}, \quad i = 3, \ldots, n, \\
f_{i-1} &= \frac{(\lambda_n^{(n)} - h_i)x_i}{x_1}, \quad i = 2, \ldots, n, \\
e_{i-1} &= \frac{(\lambda_2^{(i)} - h_i) Q_{i-1}(\lambda_2^{(i)})}{f_{i-1} \prod_{j=2}^{i-1} (\lambda_2^{(i)} - h_j)}, \quad i = 2, \ldots, n,
\end{aligned}
$$

exists if and only if $x_i \neq 0$, $i = 1, \ldots, n$.

4 Numerical Results

In this section, we present the numerical results of the problem AIEP1. The computational results are provided by MATLAB software.

Example 1. To compare the computations accuracy, we define the matrix B_n for $k = 1, \ldots, n$, $n = 1, \ldots, 50$ as follows:

$$
h_k = -\frac{1}{k(k+1)}, \quad e_k = 3 - \frac{1}{k}, \qquad f_k = 2 + \frac{1}{k}.
$$

The extremal eigenvalues of each leading principal submatrices and the eigenvector of B_n are computed, then they are set as inputs of Algorithm 1. We call the output of Algorithm 1 and its extremal eigenvalues, $\tilde{\lambda}_1^{(n)}$, $\tilde{\lambda}_n^{(n)}$, respectively and compute relative error

$$
e_{\lambda^{(n)}} = \frac{\left\| \lambda^{(n)} - \tilde{\lambda}^{(n)} \right\|_2}{\left\| \lambda^{(n)} \right\|_2},
$$

with $\lambda^{(n)} = (\lambda_n^{(n)}, \lambda_1^{(n)})$ and $\tilde{\lambda}^{(n)} = (\tilde{\lambda}_n^{(n)}, \tilde{\lambda}_1^{(n)})$. Figure 1 shows variation of this error.

Example 2. In AIEP2, we define the matrix B_n for $k = 1, \ldots, n$, $n = 1, \ldots, 50$ as follows:

$$
h_k = k, \quad e_k = \frac{1}{k}, \quad f_k = \sin^2(k).
$$

In AIEP3, this matrix B_n for $k = 1, \ldots, n$, $n = 1, \ldots, 50$ is defined as the following form:

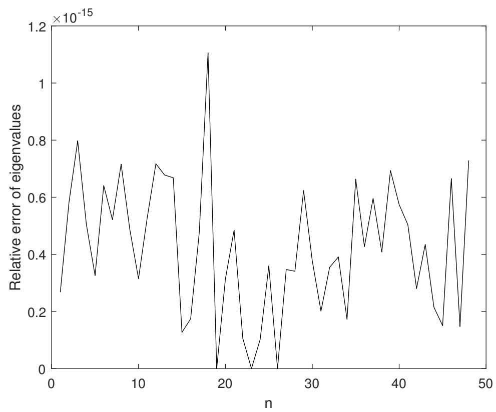

Figure 1: Relative error for eigenvalues in AIEP1.

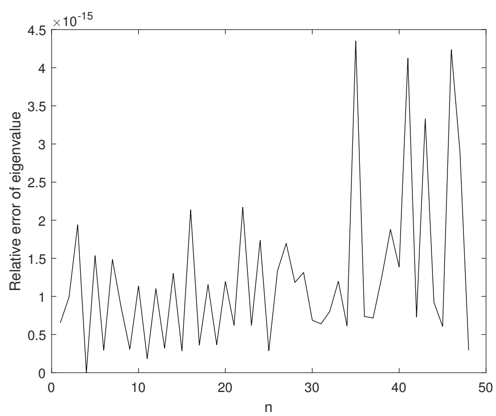

Figure 2: Relative error for eigenvalues in AIEP2.

$$h_k = 10, \quad e_k = \frac{k}{k+1}, \quad f_k = \sqrt{k}.$$

Also, in corollary 1, the matrix B_n for $k = 1, \dots, n,\ n = 1, \dots, 50$ has the following definition:

$$h_k = \frac{1}{k}, \quad e_k = e^{-2k} i, \quad f_k = -10 + \frac{ki}{10}.$$

By similar definition of Example 1 for $e_{\lambda^{(n)}}$, Figures 2, 3 and 4 show variations of the errors.

Example 3. For positive definite case, we consider

$$\begin{aligned} \lambda_1^{(1)} &= 3,\ \lambda_1^{(2)} = 2.25,\ \lambda_2^{(2)} = 3.2,\ \lambda_1^{(3)} = 2,\ \lambda_3^{(3)} = 3.8, \\ \lambda_1^{(4)} &= 1.5,\ \lambda_4^{(4)} = 4,\ \lambda_1^{(5)} = 1,\ \lambda_5^{(5)} = 4.5, \end{aligned}$$

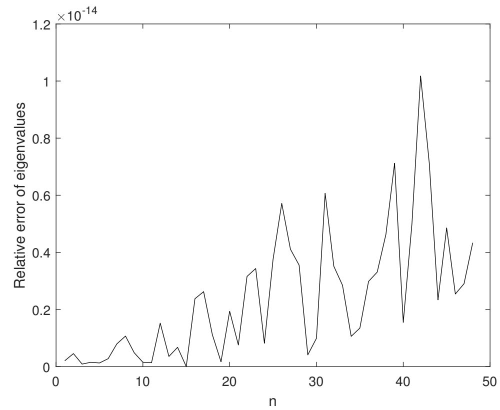

Figure 3: Relative error for eigenvalues in AIEP3.

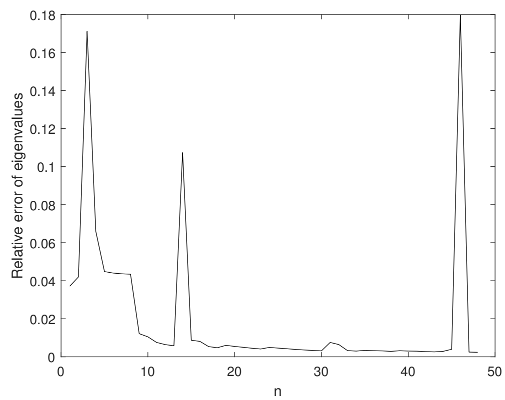

Figure 4: Relative error for the eigenvalues in Corollary 1.

By applying Eqs. (12), (13), (14) and (15) we get

$$A_5 = \begin{bmatrix} 3.0000 & 0.3873 & 0.7809 & 0.7878 & 1.2496 \\ 0.3873 & 2.4500 & 0 & 0 & 0 \\ 0.7809 & 0 & 2.9148 & 0 & 0 \\ 0.7878 & 0 & 0 & 2.1813 & 0 \\ 1.2496 & 0 & 0 & 0 & 2.4836 \end{bmatrix}.$$

We compute the spectra of all of the principal submatrices of A_5 to verify the results:

$$\sigma(A_5) = \{\mathbf{1.0000}, 2.2594, 2.4530, 2.8172\mathbf{4.5000}\},$$
$$\sigma(A_4) = \{\mathbf{1.5000}, 2.3887, 2.6574, \mathbf{4.0000}\},$$
$$\sigma(A_3) = \{\mathbf{2.0000}, 2.5648, \mathbf{3.8000}\},$$
$$\sigma(A_2) = \{\mathbf{2.2500}, \mathbf{3.2000}\},$$
$$\sigma(A_1) = \{\mathbf{3.0000}\}.$$

5 Conclusion

In this study, we have considered partially described inverse eigenvalue problems of arrowhead matrices under different conditions. Such problems are important when all spectral information is unavailable. The necessary and sufficient conditions for the existence of solution in different cases were obtained by the similar symmetric matrix. Then, the necessary and sufficient conditions for the solvability of AIEPs were discussed. The problems were studied under different cases: matrix entries being real and nonnegative, equal entries on diameter solution and complex matrix.

Matrices with complex entries are used in perturbation of one-dimensional Schrodinger equations. We also considered the case in which the matrix entries are nonnegative and it is applied in nonlinear control perturbation problem. Numerical algorithms were provided for the problems and illustrative examples were also given to test the algorithm.

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