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Existence, uniqueness and stability of fuzzy delay differential equations with local Lipschitz and linear growth conditions

Samira Siah Mansouri[†], **Morteza Gachpazan**[‡], **Nazanin Ahmady**[§∗], **Elham Ahmady**[£]

[†]*University of Applied Science and Technology, Center of Poolad Peechkar, Varamin, Iran*
[‡]*Department of Applied Mathematics, School of Mathematical Sciences, Ferdowsi University Of Mashhad, Mashhad, Iran*
[§]*Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran*
[£]*Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran*
Email(s): samira.mansouri91@yahoo.com, gachpazan@um.ac.ir, n.ahmadi@iauvaramin.ac.ir, e.ahmadi@qodsiau.ac.ir

Abstract. Fuzzy delay differential equation driven by Liu's process is a type of functional differential equations. In this paper, we are going to provide and prove a novel existence and uniqueness theorem for the solutions of fuzzy delay differential equation under Local Lipschitz and linear growth conditions. Also the stability of the solutions for fuzzy delay differential is investigated. Finally, to illustrate the main results we give some examples.

Keywords: Fuzzy delay differential equations, fuzzy Liu's process, existence and uniqueness.
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1 Introduction

Many of the natural phenomena around us do not have an instantaneous indirect effect from the moment of their events. For example, a sick person shows signs of an illness after a few days (or even weeks) she or he was infected.

In order to incorporate a model with time lag (the moment when the process takes place until the observation of the state is made), it is necessary to consider an extra term often called delay time. Delay differential equations are recently gaining more and more attention in the literature, due to this fact that the future state of a mathematical model for many fields of science (such as medicine, physics, ecology, biology, economics and etc) may depend on the present state as well as the past state. For example, Hale proposed the theory of delay differential equations [4]. On the other hand, when an ordinary differential

∗Corresponding author.
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equation is incorporated with other phenomena, such as randomness and delay, it is not natural to expect that the solution of the corresponding dynamic system behaves similar to that of ODEs. Therefore, we would be interested in investigating the theory of fuzzy differential equations in the sense of Liu's process.

Historically, the concept of fuzzy sets have been first investigated by Zadeh via membership function [16]. Liu introduced credibility theory and presented for the first time the concept of credibility measure to facilitate measuring of fuzzy events [8]. It is worthy to note that this measure is a powerful tool for dealing with fuzzy phenomena and is based on normality, monotonicity, self-duality, and maximality axioms.

Liu proposed the concept of fuzzy process [5]. A particular fuzzy process with stationary and independent increment is Liu's process, which is just like a stochastic process described by Brownian motion. Recently, many literatures have been published on the Liu's process and its applications in other sciences such as economics and optimal control [10, 13, 15]. Liu inspired by stochastic notions and Ito process introduced fuzzy differential equations which were driven by Liu's process for better understanding of the fuzzy phenomenon [8].

Qin and Li applied of fuzzy differential equation was driven by Liu's process to solve European pricing problem in fuzzy environment by [11]. Liu investigated the existence and uniqueness of solution to the fuzzy differential equations by employing Lipschitz and Linear growth conditions [14]. Afterward, Fei studied the uniqueness of solution to the fuzzy differential equations driven by Liu's process with non-Lipschitz coefficients [3].

Fuzzy delay differential equations (FDDEs) have a wide range of applications in real time applications of control theory, physics, ecology, economics, population study, inventory control, and the theory of nuclear reactors. Many authors proposed the new methods for solving FDDEs (see [2, 6, 12]). Regarding the importance of the existence and uniqueness of solution to FDDE driven by Liu's process, in this paper, we study existence and uniqueness of solution to FDDE driven by Liu's process. The main goal of this paper is to provide weaker conditions to study the existence and uniqueness of solution to the FDDE. To this end, we prove a new existence and uniqueness theorem under the Local Lipschitz and Linear growth conditions.

The rest of this paper is organized as follows. In Section 2, we will review some basic concepts about credibility theory, fuzzy variable, fuzzy process, and Liu's process. Then the fuzzy delay differential equation will be introduced in Section 3, according to Local Lipschitz and Linear growth conditions, a new existence and uniqueness theorem will be proved and finally we will study the stability theorm. Concluding results are given in Section 4.

2 Preliminaries

The emphasis in this section is mainly on introducing some concepts such as credibility measure, credibility space, fuzzy variables, independence of fuzzy variables, expected value, variance, fuzzy process, Liu's process, and stopping time.

Suppose that Θ is a non-empty set and $\mathscr{P}$ is the power set of Θ. Each element of $\mathbf{A}$ in $\mathscr{P}$ is called an event. In order to present an axiomatic definition of credibility, it is necessary to assign a number $\mathbf{Cr}\{\mathbf{A}\}$ to each event $\mathbf{A}$ which indicates the credibility that $\mathbf{A}$ will occur. To ensure that the number $\mathbf{Cr}\{\mathbf{A}\}$ has certain mathematical properties which we intuitively expect to have a credibility, we accept the following four axioms [8]:

1. Axiom (Normality) $\mathbf{Cr}\{\Theta\} = 1$.
2. Axiom (Monotonicity) $\mathbf{Cr}\{\mathbf{A}\} \leq \mathbf{Cr}\{\mathbf{B}\}$ whenever $\mathbf{A} \subset \mathbf{B}$.
3. Axiom (Self-Duality) $\mathbf{Cr}\{\mathbf{A}\} + \mathbf{Cr}\{\mathbf{A}^c\} = 1$ for any event $\mathbf{A}$.
4. Axiom (Maximality) $\mathbf{Cr}\{\mathbf{U}_i \mathbf{A}_i\} = \sup_i \mathbf{Cr}\{\mathbf{A}_i\}$ for any events $\{\mathbf{A}_i\}$ with $\sup_i \mathbf{Cr}\{\mathbf{A}_i\} < 0.5$.

Definition 1. *[8] The set function $\mathbf{Cr}$ is called a credibility measure if it satisfies the normality, monotonicity, self-duality, and maximality axioms. A family $\mathscr{P}$ with these four properties is called a σ-algebra. The pair $(\Theta, \mathscr{P})$ is called a measurable space, and the elements of $\mathscr{P}$ is afterwards called $\mathscr{P}$-measurable sets instead of events.*

Definition 2. *[1] Let Θ be a nonempty set, $\mathscr{P}$ the power set of Θ, and $\mathbf{Cr}$ a credibility measure. The triple $(\Theta, \mathscr{P}, \mathbf{Cr})$ is called a credibility space.*

Let $(\Theta, \mathscr{P}, \mathbf{Cr})$ be a credibility space. A filtration is a family $\{\mathscr{P}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathscr{P}$ (i.e. $\mathscr{P}_t \subset \mathscr{P}_s \subset \mathscr{P}$ for all $0 \leq t < s < \infty$). The filtration is said to be right continuous if $\mathscr{P}_t = \bigcap_{s>t} \mathscr{P}_s$ for all $t \leq 0$. When the credibility space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathscr{P}_0$ contains all $\mathbf{Cr}$-null sets.

We also define $\mathscr{P}_\infty = \sigma(U_{t\geq 0}\mathscr{P}_t)$ (i.e. σ-algebra generated by $U_{t\geq 0}\mathscr{P}_t$) $\mathscr{P}$-measurable fuzzy variable is denoted by $\mathbf{L}^p(\Theta, \mathbf{R^d})$ that will be defined later. A process is called $\mathscr{P}$-adapted, if for all $t \in [0,t]$ the fuzzy variable $x(t)$ is $\mathscr{P}$-measurable.

Definition 3. *[1] A fuzzy variable is defined as a (measurable) function $\xi : (\Theta, \mathscr{P}, \mathbf{Cr}) \longrightarrow \mathbf{R}$.*

Definition 4. *[1] Let ξ be a fuzzy variable. Then the expected value of ξ is defined by*

$$\mathbf{E}[\xi] = \int_0^{+\infty} \mathbf{Cr}\{\xi \geq r\}\mathrm{dr} - \int_{-\infty}^0 \mathbf{Cr}\{\xi \leq r\}\mathrm{dr},$$

provided that at least one of the two integrals is finite. Furthermore, the variance is defined by $\mathbf{E}[(\xi - e)^2]$.

Let ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have

$$\mathbf{E}[a\xi + b\eta] = a\mathbf{E}[\xi] + b\mathbf{E}[\eta].$$

Definition 5. *[1] The credibility distribution $\mu(x)$ of a fuzzy variable ξ is defined by*

$$\mu(x) = \max\{1, 2\mathbf{Cr}(\xi = x)\}, \quad x \in \mathbf{R}.$$

Definition 6. *[7] A credibility distribution $\mu(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \mu(x) < 1$, and*

$$\lim_{x\to -\infty} \mu(x) = 0, \quad \lim_{x\to +\infty} \mu(x) = 1.$$

In addition, the inverse function $\mu^{-1}(\alpha)$ is called the inverse credibility distribution of ξ.

Definition 7. *[1] Let $\mathbf{T}$ be an index set and $(\Theta, \mathscr{P}, \mathbf{Cr})$ be a credibility space. A fuzzy process is a function from $\mathbf{T} \times (\Theta, \mathscr{P}, \mathbf{Cr})$ to the set of real numbers. That is, a fuzzy process $X_t(\theta)$ is a function of two variables such that the function $X_{t^*}(\theta)$ is a fuzzy variable for each t^*. For each fixed θ^*, the function $X_t(\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X_t(\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ.*

In the following expression, we use the notation $\mathbf{x}(t)$ instead of $\mathbf{x}_t(\theta)$. A fuzzy process is essentially a sequence of fuzzy variables indexed by time or space. As one of the most important types of fuzzy processes, the Liu's process is defined as follows.

Definition 8. *[1] A fuzzy process $\mathbf{C}_t$ is said to be a Liu's process if*
1. $\mathbf{C}_0 = 0$,
2. $\mathbf{C}_t$ has stationary and independent increments,
3. Every increment $\mathbf{C}_{t+s} - \mathbf{C}_s$ is a normally distributed fuzzy variable with expected value **et** *and variance $\sigma^2 t^2$ whose membership function is*

$$\mu(x) = 2(1 + \exp(\tfrac{\pi|x-et|}{\sqrt{6}\sigma t}))^{-1}, \qquad -\infty < x < +\infty.$$

The parameters e and σ are called the drift and diffusion coefficients, respectively, in addition, Liu's process is said to be standard if $e = 0$ and $\sigma = 1$.

Based on Liu's process, Liu integral is defined as a fuzzy counterpart of Ito integral as follows.

Definition 9. *[8] Suppose that $x(t)$ is a fuzzy process and $\mathbf{C}_t$ is a standard Liu's process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of $x(t)$ with respect to $\mathbf{C}_t$ is

$$\int_a^b x(t) d\mathbf{C}_t = \lim_{\Delta \longrightarrow 0} \sum_{i=1}^{k} x(t_i) \cdot (\mathbf{C}_{t_{i+1}} - \mathbf{C}_{t_i}),$$

provided that the limit exists almost surely and is a fuzzy variable.

Theorem 1. *[1] Let $\mathbf{C}_t$ be a standard Liu's process, and $h(t,c)$ a continuously differentiable function. Define $x(t) = h(t, \mathbf{C}_t)$. Then we have the following chain rule*

$$dx(t) = \tfrac{\partial h}{\partial t}(t, \mathbf{C}_t)dt + \tfrac{\partial h}{\partial c}(t, \mathbf{C}_t)d\mathbf{C}_t,$$

which is called Liu formula.

Definition 10. *[8] The fuzzy variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be independent if*

$$\mathbf{Cr}\{\textstyle\bigcap_{i=1}^m \{\xi_i \in \mathbf{B}_i\}\} = \min_{1 \le i \le m} \mathbf{Cr}\{\xi_i \in \mathbf{B}_i\},$$

for any sets $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_m$ of real numbers.

Let us define a sequence of credibility stopping times.

Definition 11. *[9] A fuzzy variable $\tau : \Theta \to [0,\infty]$ (it may take the value ∞) is called an $\{\mathscr{P}_t\}$-stopping time (or simply, stopping time) if $\{\theta : \tau(\theta) \le t\} \in \mathscr{P}_t$ for any $t \ge 0$*

$$\begin{cases} \tau_k = \inf\{t \ge 0 : |x(t)| \ge k\}, \\ \sigma_1 = \inf\{t \ge 0 : w(x(t)) \ge 2\varepsilon\}, \\ \sigma_{2i} = \inf\{t \ge \sigma_{2i-1} : w(x(t)) \le \varepsilon\}, \quad i = 1,2,\dots, \\ \sigma_{2i+1} = \inf\{t \ge \sigma_{2i} : w(x(t)) \ge 2\varepsilon\}, \quad i = 1,2,\dots, \end{cases} \tag{1}$$

where throughout this paper we set $\inf \phi = \infty$.

Definition 12. *[9] If $X = \{X_t\}_{t\ge 0}$ is a measurable process and τ is a stopping time, then $\{X_{\tau\wedge t}\}_{t\ge 0}$ is called a stopped process of X.*

There are several useful inequalities for fuzzy variables, such as Hölder inequality and Chebyshev inequality. In the sequens, we introduce generalized inequalities for fuzzy variables.

Theorem 2. *(**Hölder Inequality**) [3] Let $\mathbf{p}$ and $\mathbf{q}$ be two positive real numbers with $\frac{1}{\mathbf{p}} + \frac{1}{\mathbf{q}} = 1$, ξ and η be independent fuzzy variables with*

$$\mathbf{E}\,[\,|\xi|^{\mathbf{p}}\,] \le +\infty \quad \textit{and} \quad \mathbf{E}\,[\,|\eta|^{\mathbf{q}}\,] \le +\infty.$$

We have

$$\mathbf{E}\,[\,|\xi\eta|\,] \le \sqrt[\mathbf{p}]{\mathbf{E}\,[\,|\xi|^{\mathbf{p}}\,]}\ \sqrt[\mathbf{q}]{\mathbf{E}\,[\,|\eta|^{\mathbf{q}}\,]}.$$

Theorem 3. *(**Chebychev**'s Inequality) [3] Let $\xi : \theta \to \mathbf{R}^n$ be a fuzzy variable such that $\mathbf{E}\,[\,|\xi|^{\mathbf{p}}\,] \le +\infty$ for some $\mathbf{p}$, $0 \le \mathbf{p} \le \infty$. Then the Chebychev's inequality is:*

$$\mathbf{Cr}[\,|\xi| \ge \lambda\,] \le \tfrac{1}{\lambda^{\mathbf{p}}}\mathbf{E}\,[\,|\xi|^{\mathbf{p}}\,] \ \textit{for all} \ \lambda \ge 0.$$

Finally we introduce some symbols that are used in next sections.

Notation 1. $\ell^{\mathbf{p}}(\theta,\mathbf{R}^{\mathbf{d}})$: the family of $\mathbf{R}^{\mathbf{d}}$-valued fuzzy variables ξ with $\mathbf{E}|\xi|^{\mathbf{p}} < \infty$.

Notation 2. $\ell^{\mathbf{p}}([a,b],\mathbf{R}^{\mathbf{d}})$: the family of $\mathbf{R}^{\mathbf{d}}$-valued $\mathscr{P}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b |f(t)|^{\mathbf{p}}\mathrm{d}t < \infty$ almost surely.

Notation 3. $M^{\mathbf{p}}([a,b],\mathbf{R}^{\mathbf{d}})$: the family of processes $\{f(t)\}_{a\le t\le b}$ in $\ell^{\mathbf{p}}([a,b],\mathbf{R}^{\mathbf{d}})$ such that $\int_a^b |f(t)|^{\mathbf{p}}dt < \infty$.

Notation 4. $\ell^{\mathbf{p}}(\mathbf{R}_+,\mathbf{R}^{\mathbf{d}})$: the family of processes $\{f(t)\}_{t>0}$ such that for every $T > 0$, $\{f(t)\}_{a\le t\le T} \in \ell^{\mathbf{p}}([0,T],\mathbf{R}^{\mathbf{d}})$.

Remark 1. *A Liu's process $\mathbf{C}$ is Lipschitz-continuous, that is so, for any given $\theta \in \Theta$, there exists $\mathbf{K}(\theta) > 0$ such that for all $t,s \ge 0$,*

$$|\mathbf{C}_{t_\theta} - \mathbf{C}_{s_\theta}| \le |t-s|.$$

Lemma 1. *[1] Suppose that $\mathbf{C}_t$ is a standard Liu's process and $x(t)$ is a fuzzy process on $[a,b]$ with respect to t. If $\mathbf{K}(\Theta) > 0$ is the Lipschitz constant for path $\mathbf{C}_{t_\theta}$ with $\theta \in \Theta$ fixed, then we have*

$$\left| \int_a^b x(t)d\mathbf{C}_t \right| \le \mathbf{K}(\theta) \int_a^b |x(t)|dt.$$

3 Fuzzy delay differential equation

We consider the following FDDEs:

$$\begin{cases} dx(t) = f(x(t-\tau),t)dt + g(x(t-\tau),t)d\mathbf{C}_t, & t \geq \tau, \\ x(t) = \phi(t), & t < \tau, \end{cases} \tag{2}$$

where $\mathbf{C}_t$ is a standard Liu's process, $\tau \geq 0$ and denote by $\mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$. $\mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$ is Banach space with norm $\|\phi\| = \sup|\phi(t)|$. Also for two functions f and g we have $f,g : \mathbf{C}([-\tau,\mathbf{R}],\mathbf{R}) \times \mathbf{R} \to \mathbf{R}$.

For a fuzzy process $x : \mathbf{C}([-\tau,\mathbf{R}],\mathbf{R}) \times \Theta \to \mathbf{R}$, denote a fuzzy segment process of $x(t)$ given by $x_t(\tau,\theta) = x((t-\tau),\theta)$ for $t \in [0,T]$ and $\theta \in \Theta$. $x(t)$ is called a fuzzy process with delay (or memory) of the fuzzy process x at moment $t \geq 0$.

3.1 Existence and uniqueness of the solution for FDDEs

In this section, we will consider existence and uniqueness of the solution For FDDEs. First we consider two conditions as follows:

(I) Lipschitz condition: there exists a positive number $\mathbf{L}$ such that:

$$|f(x(t),t) - f(y(t),t)| + |g(x(t),t) - g(y(t),t)| \leq \mathbf{L}|x(t) - y(t)|.$$

(II) Linear growth condition: there exists a positive number $\mathbf{L}$ such that:

$$|f(x(t),t)| + |g(x(t),t)| \leq \mathbf{L}|1 + x(t)|.$$

Now, we introduce the mapping $\mathbf{A}$ on $\mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$ for $x(t) \in \mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$ as follows;

$$\mathbf{A}(x(t)) = \phi(0) + \int_0^t f(x(s-\tau),s)ds + \int_0^t g(x(s-\tau),s)d\mathbf{C}_s, \tag{3}$$

where $t \in [0,T], \tau \geq 0$.

Lemma 2. *For any $\theta \in \Theta$, and $x(t,\theta) \in \mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$, we have $A(x(t,\theta)) \in \mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$.*

Proof. Let $s_1, s_2 \in [0,T]$ and $s_1 > s_2$. We have

$$\begin{aligned} |\mathbf{A}(x(s_1,\theta)) - \mathbf{A}(x(s_2,\theta))| &= |\int_{s_2}^{s_1} f(x(s-\tau),s)ds + \int_{s_2}^{s_1} g(x(s-\tau),s)d\mathbf{C}_s(\theta)| \\ &\leq \int_{s_2}^{s_1} |f(x(s-\tau),s)|ds + \int_{s_2}^{s_1} |g(x(s-\tau),s)d\mathbf{C}_s(\theta)| \\ &\leq \int_{s_2}^{s_1} |f(x(s-\tau),s)|ds + \mathbf{K}(\theta)\int_{s_2}^{s_1} |g(x(s-\tau),s)|ds \\ &\leq \mathbf{L}(1 + \|x(t,\tau)\|)(1 + \mathbf{K}(\theta))(s_1 - s_2). \end{aligned}$$

Thus $|\mathbf{A}(x(s_1,\theta)) - \mathbf{A}(x(s_2,\theta))| \to 0$ as $|s_1 - s_2| \to 0$. So, we have that $\mathbf{A}(x(t))$ is sample-continuous on $\mathbf{C}([-\tau,\mathbf{R}],\mathbf{R})$. □

Lemma 3. *There exists $c > 0$ such that for any $t \in [0,T)$, the FDDE (2) has a unique solution on the interval $[t,t+c]$ (setting $t+c = \mathbf{T}$ if $t+c > \mathbf{T}$) if the coefficients $f(x(t),t)$ and $g(x(t),t)$ satisfy* (I) *and* (II).

Proof. Let $c > 0$ and $\beta(\theta) = \mathbf{L}(1+\mathbf{K}(\theta))$ such that $c.\beta(\theta) \in (0,1)$. For $t \in [0,T)$, define

$$\mathbf{A}(x(\xi)) = x(t) + \int_t^{\xi} f(x(s-\tau),s)ds + \int_t^{\xi} g(x(s-\tau),s)d\mathbf{C}_s, \quad \xi \in [t,t+c]. \tag{4}$$

Lemma 2 implies that $\mathbf{A}(x(\xi)) \in \mathbf{C}[t,t+c]$ for $x(\xi) \in C[t,t+c]$. For any $\xi \in [t,t+c]$, we have

$$\begin{aligned}
&|\mathbf{A}(x(\xi,\theta)) - \mathbf{A}(y(\xi,\theta))| \\
&\quad = \left| \int_t^{\xi} [f(x(s-\tau),s) - f(y(s-\tau),s)]ds + \int_t^{\xi} [g(x(s-\tau),s) - g(y(s-\tau),s)]d\mathbf{C}_s(\theta) \right| \\
&\quad \le \int_t^{\xi} |f(x(s-\tau),s) - f(y(s-\tau),s)|ds + \int_t^{\xi} |g(x(s-\tau),s) - g(y(s-\tau),s)d\mathbf{C}_s(\theta)| \\
&\quad \le \int_t^{\xi} |f(x(s-\tau),s) - f(y(s-\tau),s)|ds + \mathbf{K}(\theta) \int_t^{\xi} |g(x(s-\tau),s) - g(y(s-\tau),s)|ds \\
&\quad \le \mathbf{L}(1+\mathbf{K}(\theta)) \int_t^{\xi} |x((s-\tau),\theta) - y((s-\tau),\theta)|ds \\
&\quad \le \mathbf{L}(1+\mathbf{K}(\theta)) \int_t^{\xi} \sup_{t \le s \le t+c} |x((s-\tau),\theta) - y((s-\tau),\theta)|ds \\
&\quad \le \mathbf{L}(1+\mathbf{K}(\theta))c \|x((s-\tau),\theta) - y((s-\tau),\theta)\|.
\end{aligned}$$

If we set $A(x(\xi,\theta)) = x(t)$ for $\xi \in [t,t+c]$, then

$$\|\mathbf{A}(x(\xi,\theta)) - \mathbf{A}(y(\xi,\theta))\| \le \beta(\theta)\|x((s-\tau),\theta) - y((s-\tau),\theta)\|.$$

Then, $\mathbf{A}$ is a contraction mapping on $\mathbf{C}[t,t+c]$. Thus by the well-known Banach fixed point theorem we have a unique fixed point $x(t,\theta)$ which satisfies (3) for $[t,t+c]$. Thus, (2) has a unique solution on the interval $[t,t+c]$. □

Theorem 4. *The FDDE (2) has a unique solution on the interval $[0,T]$ if $f(x(t-\tau),t)$ and $g(x(t-\tau),t)$ satisfy* (I) *and* (II).

Proof. Suppose that $[0,c],[c,2c],\ldots,[kc,T]$ are the subsets of $[0,T]$. For any subset $[(i-1)c,ic]$ and by using Lemma 3 it follows that the FDDE (2) has a unique solution $x^i(t,\theta)$ on the interval $[(i-1)c,ic]$ for $i = 1,2,\ldots,k+1$ and setting $(k+1)c = T$. Therefore, the FDDE (2) has a unique solution $x(t)$ on the interval $[0,T]$ by setting

$$x(t,\theta) = \begin{cases} x^1(t,\theta), & t \in [0,c], \\ \quad \vdots & \\ x^k(t,\theta), & t \in [(k-1)c,kc], \\ x^{k+1}(t,\theta), & t \in [kc,T]. \end{cases}$$

□

Example 1. Consider a l-dimensional linear FDDE

$$\begin{cases} dx(t) = [nx(t) + \bar{n}x(t-\tau)]dt + \sum_{k=1}^{p}[m_k x(t) + \bar{m}_k x(t-\tau)]d\mathbf{C}_{t_k}, & t_0 \leq t, \\ x(\theta) = \xi, & t_0 - \tau \leq \theta \leq t_0. \end{cases}$$

We consider two parts:
Part 1: On $t \in [t_0, t_0 + \tau]$, the linear FDDE becomes a linear FDE

$$\begin{cases} dx(t) = [nx(t) + \eta_1(t)]dt + \sum_{k=1}^{p}[m_k x(t) + \mu_{k_1}(t)]d\mathbf{C}_{t_k}, \\ x(t_0) = \xi(0), \end{cases}$$

where $\eta_1 = \bar{n}\xi(t-\tau)$ and $\mu_{k_1}(t) = \bar{m}_k\xi(t-\tau)$. This linear FDE has the explicit solution

$$x(t) = \zeta_1(t)[\xi(0) + \int_{t_0}^{t} \zeta_1^{-1}(s)(\bar{n}\xi(s-\tau) - \sum_{k=1}^{p} m_k\bar{m}_k\xi(s-\tau))ds + \sum_{k=1}^{p}\int_{t_0}^{t}\zeta_1^{-1}(s)\bar{m}_k\xi(s-\tau)d\mathbf{C}_{s_k}],$$

where

$$\zeta_1(t) = \exp[(n - \frac{1}{2}\sum_{k=1}^{p} m_k^2)(t-t_0) + \sum_{k=1}^{p} m_k(\mathbf{C}_{t_k} - \mathbf{C}_{t_{0k}})].$$

Part 2: On $t \in [t_0 + \tau, t_0 + 2\tau]$, the linear FDDE becomes a linear FDE

$$\begin{cases} dx(t) = [nx(t) + \eta_2(t)]dt + \sum_{k=1}^{p}[m_k x(t) + \mu_{k_2}(t)]d\mathbf{C}_{t_k}, \\ x(t_0) = x(t_0 + \tau), \end{cases}$$

where $\eta_2 = \bar{n}\xi(t-\tau)$ and $\mu_{k_2}(t) = \bar{m}_k\xi(t-\tau)$. This linear FDE has the explicit solution

$$x(t) = \zeta_2(t)[x(t_0+\tau) + \int_{t_0+\tau}^{t} \zeta_2^{-1}(s)(\bar{n}\xi(s-\tau) - \sum_{k=1}^{p} m_k\bar{m}_k\xi(s-\tau))ds + \sum_{k=1}^{p}\int_{t_0+\tau}^{t}\zeta_2^{-1}(s)\bar{m}_k\xi(s-\tau)d\mathbf{C}_{s_k}],$$

where

$$\zeta_2(t) = \exp[(n - \frac{1}{2}\sum_{k=1}^{p} m_k^2)(t-t_0-\tau) + \sum_{k=1}^{p} m_k(\mathbf{C}_{t_k} - \mathbf{C}_{t_0-\tau_k})].$$

Iterating this process over the intervals $[t_0 + 2\tau, t_0 + 3\tau]$ etc. we obtain the explicit solution for the linear FDDE.

3.2 Stability theorem for FDDEs

Regarding the fact that if a physical examination is repeated twice with the same conditions, how much measurements are conducted carefully, the initial imposed conditions will not be exactly the same. it is expected that the results of two physical examinations have a litter difference which each other. This means that very small changes in initial conditions are expected to cause only small changes in answer or in other words, the answer of mathematical model is stable. In this section, we will propose a concept of stability for a FDDE in the sense of fuzzy measure. A sufficient condition will also be derived for a FDDE be stable.

Corollary 1. *Let $\mathbf{C}_t$ be a fuzzy process on a credibility space $(\Theta, \mathscr{P}, \mathbf{Cr})$. Then there exists a fuzzy variable $\mathbf{K}$ such that $\mathbf{K}(\theta)$ is a Lipschitz constant of the sample path $\mathbf{C}_{t_\theta}$ for each θ and*

$$\lim_{x\to+\infty} \mathbf{Cr}\{\theta \in \Theta | \mathbf{K}(\theta) \leq x\} = 1.$$

Proof. Define

$$\mathbf{K}(\theta) = \sup_{0<t<s} \frac{|\mathbf{C}_{t_\theta} - \mathbf{C}_{s_\theta}|}{t-s},$$

for each sample path $\mathbf{C}_{t_\theta}$. Then $\mathbf{K}$ is a fuzzy variable on $(\Theta, \mathscr{P}, \mathbf{Cr})$, and

$$|\mathbf{C}_{t_\theta} - \mathbf{C}_{s_\theta}| \leq \mathbf{K}(\theta)|t-s|,$$

for any $t_1, t_2 \in \mathbf{R}^+$. That is, $\mathbf{K}(\theta)$ is a Lipschitz constant of the sample path $\mathbf{C}_{t_\theta}$. Given $x \in \mathbf{R}$, if

$$\left|\frac{d\mathbf{C}_t}{dt}(\theta)\right| \leq x, \qquad \forall\ t > 0,$$

then we have

$$|\mathbf{C}_{t_\theta} - \mathbf{C}_{s_\theta}| \leq \int_t^s \left|\frac{d\mathbf{C}_t}{dt}(\theta)\right| dt \leq x(s-t),$$

for any $t_1, t_2 \in \mathbf{R}^+$, which means that

$$\sup_{0<t<s} \frac{|\mathbf{C}_{t_\theta} - \mathbf{C}_{s_\theta}|}{s-t} \leq x.$$

Thus, we obtain

$$\mathbf{Cr}\{\theta \in \Theta | \mathbf{K}(\theta) \leq x\} = \mathbf{Cr}\{\sup \frac{\mathbf{C}_s - \mathbf{C}_t}{s-t} \leq x\},$$
$$\mathbf{Cr}\{|\frac{d\mathbf{C}_t}{dt}| \leq x, \quad \forall\ t > 0\} \leq 2\phi(x) - 1,$$

where ϕ is the credibility distribution of standard normal fuzzy variable $\mathbf{N}(0,1)$. Since $\phi(x) \to 1$ *as* $x \to +\infty$, we have

$$\lim_{x\to+\infty} \mathbf{Cr}\{\theta \in \Theta | \mathbf{K}(\theta) \leq x\} = 1. \qquad \square$$

Corollary 2. *Let $\mathbf{C}_t$ be a fuzzy process on a credibility space $(\Theta, \mathscr{P}, \mathbf{Cr})$, and $K(\theta)$ be the least Lipschitz constant of the sample path $\mathbf{C}_{t_\theta}$. Then*

$$\mathbf{Cr}\{\mathbf{K} < +\infty\} = 1.$$

Proof. We will prove by a contradiction method. Assume that $\mathbf{Cr}\{\mathbf{K} < +\infty\} < 1$. Then there exists a positive number $\zeta > 0$ such that $\mathbf{Cr}\{\mathbf{K} < +\infty\} \leq 1 - \zeta$, so we have $\mathbf{Cr}\{\mathbf{K} \leq x\} \leq 1 - \zeta$ for any $x \in \mathbf{R}$. It means that

$$\lim_{x \to +\infty} \mathbf{Cr}\{\theta \in \Theta | \mathbf{K}(\theta) \leq x\} \leq 1 - \zeta,$$

which contradicts that

$$\lim_{x \to +\infty} \mathbf{Cr}\{\theta \in \Theta | \mathbf{K}(\theta) \leq x\} = 1.$$

The corollary is thus verified. □

Let x_t and y_t be two solutions of the following FDDE,

$$\begin{cases} dx(t) = f(x(t-\tau), t)dt + g(x(t-\tau), t)d\mathbf{C}_t, & t \in [0, T], \\ x(t_0) = \xi(0). \end{cases} \tag{5}$$

Then the FDDE (5) is said to be stable if for any given number $\xi > 0$, there exists a real number δ such that

$$\mathbf{Cr}\{|x(t) - y(t)| > \varepsilon\} \leq \varepsilon,$$

or

$$\lim_{|x_0 - y_0| \to 0} \mathbf{Cr}\{|x(t) - y(t)| > \varepsilon\} = 0, \quad \forall \ t > 0,$$

holds for any $t \geq 0$ provided $|x_0 - y_0| \leq \delta$.

Example 2. Consider the following FDDE

$$\begin{cases} dx(t) = \mu(t-\tau)dt + \sigma d\mathbf{C}_t, \\ x(t) = 1, \end{cases}$$

where μ and σ are constants, and $\mathbf{C}_t$ is a standard Liu's process. If $0 \leq t \leq \tau$, then $-\tau \leq t - \tau \leq 0$, and so $x(t-\tau) = 1$. Therefore, for $0 \leq t \leq \tau$, we have to solve the following fuzzy differential equation

$$\begin{cases} dx(t) = \mu dt + \sigma d\mathbf{C}_t, \\ x(0) = 1. \end{cases} \tag{6}$$

By solving this equation we obtain

$$x(t) = x(0) + \int_0^t \mu dt + \sigma \mathbf{C}_t,$$

and so $x(t) = x_0 + \mu t + \sigma \mathbf{C}_t$. Since the two solutions (6) are

$$x(t) = x_0 + \mu t + \sigma \mathbf{C}_t,$$
$$y(t) = y_0 + \mu t + \sigma \mathbf{C}_t,$$

we have $|x(t) - y(t)| = |x(0) - y(0)|$. Then for any given $\varepsilon > 0$, taking $\varepsilon = \delta$, we have

$$\mathbf{Cr}\{|x(t) - y(t)| \geq \varepsilon\} = |x(0) - y(0)| = 0 < \varepsilon,$$

for any $0 \leq t \leq \tau$. Hence, the FDDE is stable.

Now, if $\tau \leq t \leq 2\tau$, then $0 \leq t - \tau \leq \tau$, and so $x(t-\tau) = 1 + \mu(t-\tau) + \sigma \mathbf{C}_{t-\tau}$. Therefore, for $\tau \leq t \leq 2\tau$, we have to solve the following fuzzy differential equation

$$\begin{cases} dx(t) = \mu[1 + \mu(t-\tau) + \sigma \mathbf{C}_{t-\tau}] + \sigma d\mathbf{C}_t, \\ x(\tau) = 1 + \mu(\tau) + \sigma \mathbf{C}_\tau. \end{cases} \tag{7}$$

Solution of this equation is

$$x(t) = x(\tau) + \int_\tau^t \mu[1 + \mu(s-\tau) + \sigma \mathbf{C}_{s-\tau}]ds + \sigma \mathbf{C}_t,$$

and so $x(t) = x_0 + \mu^2 \frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma \int_\tau^t \mathbf{C}_{s-t} ds$. Since the two solutions (7) are

$$x(t) = x_0 + \mu^2 \frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma \int_\tau^t \mathbf{C}_{s-t} ds,$$
$$y(t) = y_0 + \mu^2 \frac{(t-\tau)^2}{2} + \mu(t-\tau) + \mu\sigma \int_\tau^t \mathbf{C}_{s-t} ds,$$

we have $|x(t) - y(t)| = |x(0) - y(0)|$. Then for any given $\varepsilon > 0$, taking $\varepsilon = \delta$, we have

$$\mathbf{Cr}\{|x(t) - y(t)| \geq \varepsilon\} = |x(0) - y(0)| = 0 < \varepsilon,$$

for any $\tau \leq t \leq 2\tau$. Hence the FDDE (7) is stable.

Theorem 5. *The FDDE (5) is stable if the coefficients $f(x,t)$ and $g(x,t)$ satisfy* (I) *and* (II).

Proof. Assume that $x(t)$ and $y(t)$ are two solutions of the FDDE (5) with different initial values $x(0)$ and $y(0)$, respectively. Then

$$x(t) = x(0) + \int_\tau^t f(x(s-\tau), s)ds + \int_\tau^t g(x(s-\tau), s)d\mathbf{C}_s,$$
$$y(t) = y(0) + \int_\tau^t f(y(s-\tau), s)ds + \int_\tau^t g(y(s-\tau), s)d\mathbf{C}_s.$$

For a Lipschitz continuous sample path $\mathbf{C}_{t_\theta}$, we have

$$\begin{aligned} &|x(t,\theta) - y(t,\theta)| \\ &\quad \leq |x(0) - y(0)| + \int_\tau^t |f(x(s-\tau), s) - f(y(s-\tau), s)|ds + \int_\tau^t |g(x(s-\tau), s) - g(y(s-\tau), s)||d\mathbf{C}_s| \\ &\quad \leq |x(0) - y(0)| + \int_\tau^t \mathbf{L}(s)|x(s,\theta) - y(s,\theta)|ds + \int_\tau^t \mathbf{L}(s)\mathbf{K}(\theta)|x(s,\theta) - y(s,\theta)|ds \\ &\quad = |x(0) - y(0)| + \int_\tau^t \mathbf{L}(s)(1 + \mathbf{K}(\theta))|x(s,\theta) - y(s,\theta)|ds. \end{aligned}$$

Thus by the Grownwall's equality we have

$$|x(t,\theta) - y(t,\theta)| \leq |x(0) - y(0)| + \exp(1+\mathbf{K}(\theta)) \int_{\tau}^{\infty} \mathbf{L}(s)ds.$$

For any given $\varepsilon > 0$, there exists a real number $\mathbf{H}$, such that $\mathbf{Cr}\{\theta | \mathbf{K}(\theta) \leq \mathbf{H}\} \geq 1 - \varepsilon$. We consider

$$\delta = \exp(-(1+\mathbf{H}) \int_{\tau}^{\infty} \mathbf{L}(s)ds)\varepsilon.$$

Then for any positive real number t provided that $|x(0) - y(0)| \leq \delta$ and $\mathbf{K}(\theta) \leq \mathbf{H}$ we have $|x(t,\theta) - y(t,\theta)| \leq \varepsilon$. In other words,

$$\lim_{|x_0 - y_0| \to 0} \mathbf{Cr}\{|x(t) - y(t)| > \varepsilon\} = 0,$$

so the FDDE is stable. □

4 Conclusion

In this work, emphasis was given to provide weaker conditions to study the existence and uniqueness of solution to the fuzzy delay differential equations. We proved a new existence and uniqueness theorem under the Local Lipschitz and Linear growth conditions. Finally, we proposed the sufficient condition of stability for the fuzzy delay differential equations.

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