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Numerical solution of DGLAP equations by the Tau spectral method

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Abstract. In this paper, a special kind of Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations is introduced. Since it is not possible to solve DGLAP integral equations analytically, the numerical solutions of these equations can be of interest. Here, the Tau spectral method is used for solving this integral equation and offer an approximate solution. Finally, this solution is compared with solution obtained experimentally for $Q_0^2 = 0.35 GeV^2$.

Keywords: Integral-differential equation, Tau method, DGLAP equation, parton, quark and nucleon.
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1 Introduction

Probing the internal structure of the nucleons is one of the important aims of particle physics. The nucleons are the bound systems that are consisted of a number of fundamental particles called partons. These particles are quarks and gluons. The partonic structure of the nucleon has been investigated by lepton-nucleon deep inelastic scattering (DIS) in experimental particle physics [4]. The parton distribution functions of the nucleon describe the probability for finding a parton with a specified longitudinal momentum fraction in DIS processes. The parton distribution functions (PDF) of the proton are dependent on two variables Q^2 and x. The variable Q^2 is related to the momentum 4-vector transfer q as $Q^2 = -q^2$ and $x = \frac{Q^2}{2p.q}$ is the momentum fraction which is carried by the parton in which p denotes the momentum of the nucleon.

There are many theoretical models for studying dependence of x to the parton distribution functions [4]. On the other hand, the dependence of the PDFs on Q^2 variable has been investigated in perturbative quantum chromodynamics (QCD) via Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [5]. The PDF of the proton have been evolved from initial scale, Q_0^2, to the higher Q^2 values applying the DGLAP evolution equation.

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These coupled integro differential equations are applied in both theoretical and experimental studies of particle physics. One can evolves the unpolarized and polarized PDF of the nucleon from initial scale Q_0^2 to higher Q^2 values using the DGLAP evolution equation [8]. Some kinds of DGLAP equations can be seen in [3, 9]

The present paper is devoted to the study of the numerical solution of one kind of the DGLAP equations by the Tau method. For this aim, at the first the desired DGLAP and Tau method are introduced. In Section 2, the numerical solution of DGLAP equation for the flavor nonsinglet and singlet distributions by Tau method are given. Finally, some numerical examples are presented.

1.1 DGLAP Equations

We study the Q^2 evolutions of the unpolarized PDF of the proton applying the DGLAP evolution equations. One kind of DGLAP equations for the flavor nonsinglet distribution is given by [4]:

$$\frac{dq_{Ns}(x,Q^2)}{d\log Q^2} = -\frac{\alpha_s}{2\pi}\int_x^1 \Big(P_{qq}(\frac{x}{y})q_{Ns}(y,Q^2)\Big)\frac{dy}{y},$$

in which

$$\alpha_s = \frac{12\pi}{(33-2n_f)\log(\frac{Q^2}{0.2GeV^2})},$$

denotes the runnig coupling constant at leading order (LO), $q_{Ns} = \sum_{i=1}^{n_f} q_i - \bar{q}_i$ is a unpolarized nonsinglet quark distribution function and P_{qq} is the splitting of quark-quark vertex, where n_f is the number of quark flavors and $\bar{q}_i$ is pod quarak. This splitting function is given as:

$$P_{qq}(z) = \frac{4}{3}\Big(\frac{1+z^2}{1-z}\Big),$$

at LO approximation [8].

The unpolarized flavor singlet DGLAP equation is written as [4]:

$$\frac{dq_s(x,Q^2)}{d\log Q^2} = -\frac{\alpha_s}{2\pi}\int_x^1 \Big(P_{qq}(\frac{x}{y})q_s(y,Q^2) + 2n_f P_{qg}(\frac{x}{y})g(y,Q^2)\Big)\frac{dy}{y},$$

where $q_s = \sum_{i=1}^{n_f} q_i + \bar{q}_i$ denotes singlet parton distribution, n_f is the number of quark flavors and P_{qg} is the splitting function of gluon to quark-antiquark pair [4]:

$$P_{qg}(z) = \frac{1}{2}\Big(z^2 + (1-z)^2\Big).$$

Finally, the DGLAP evolution equation for gluon distribution is given as [4]:

$$\frac{dg(x,Q^2)}{d\log Q^2} = -\frac{\alpha_s}{2\pi}\int_x^1 \Big(P_{gq}(\frac{x}{y})q_s(y,Q^2) + P_{gg}(\frac{x}{y})g(y,Q^2)\Big)\frac{dy}{y},$$

where g is the gluon distribution function and P_{gg} denotes the splitting function of gluon to gluon which has the following form at the LO approximation [4]:

$$P_{gg}(z) = 6\Big(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\Big).$$

1.2 Tau method

In 1981, Ortiz and Samara introduced the Tau method which is used to compute the numerical solutions of both linear and nonlinear normal differential equations with initial conditions using linearization approach [7]. In the last two decades, Ortiz and Samara have introduced several works on theoretical development of the Tau method and its practical applications. In fact, the Tau method offers a series of matrix operations instead of localization or other similar methods. Due to its high precision, the Tau method has been widespread used in teh numerical solving of differential equations. Hosseini and Ortiz investigated the applications of Tau method by first linearizing the non-linear equations before solving them using the Tau method [1]. Liu and Pan used the Tau method to solve normal differential equations in 1999 [6]. Dao and Khajeh solved a simple form of linear integral equations numerically using the Tau method in 1997 [2]. From 1999 onwards, several researchers have offered numerical solutions for integral and integro-differential equations using the Tau method [2, 5–7].

In the Tau method, a family of independent linear functions with finite dimensions is selected so that the exact solution to the equation can be approximated as a linear combination of these functions. Then, we offer conditions in order to minimize the approximation error. So, we have a system of linear equations or a Sylvester matrix equation. By solving the obtained system and determining the coefficients of linear combination, we can portrait the solution in a special function space with finite dimension [7].

To introduce the Tau method the following matrices are introduced: (see [7])

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mu = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 1. *[7] If $y_n(x) = \underline{a_n}\underline{X}$ with $\underline{a_n} = (a_0, a_1, \ldots, a_n, 0, \ldots)$ and $\underline{X} = \left(1, x, x^2, \ldots, x^n, \ldots\right)^T$, then*

1. $\dfrac{d^m}{dx^m} y_n(x) = \underline{a_n}\eta^m\underline{X}$,
2. $x^m y_n(x) = \underline{a_n}\mu^m\underline{X}$.

2 Main results

2.1 Solving DGLAP evolution equation for nonsinglet quark distribution function using the Tau method

Consider the DGLAP evolution equation for distribution function of nonsinglet quark with the following initial conditions:

$$\frac{dq_{Ns}(x,Q^2)}{d\log Q^2} = -\frac{\alpha_s}{2\pi}\int_x^1 \left(P_{qq}(\frac{x}{y})q_{Ns}(y,Q^2)\right)\frac{dy}{y}, \tag{1}$$

$$q_{Ns}(x,Q_0^2) = u(x,Q_0^2) - \bar{u}(x,Q_0^2) + d(x,Q_0^2) - \bar{d}(x,Q_0^2), \tag{2}$$

where u, $\bar{u}$, d and $\bar{d}$ are distribution functions for up quark, up antiquark, down quark and down antiquark, respectively. We aim to approximate the solution of this equation by considering n points of distribution

function for non-singular quark (2) using the Tau method. We assume that $\{\phi_i(x)\}_{i=0}^{n}$ and $\{\nu_j(Q^2)\}_{j=0}^{n}$ are two sets of standardized polynomials and an approximate solution for Eq. (1) is calculated as follows:

$$\frac{dq_{Ns_n}(x,Q^2)}{d\log Q^2}+\frac{\alpha_s(Q^2)}{2\pi}\int_1^x\Big(P_{qq_n}(\frac{x}{y})q_{Ns_n}(y,Q^2)\Big)\frac{dy}{y}=0, \tag{3}$$

where q_{Ns_n} and P_{qq_n} are approximations for q_{Ns} and P_{qq}, respectively. So we have:

$$q_{Ns_n}(x,Q^2)\approx\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\phi_i(x)\nu_j(Q^2)=A\Psi(x,Q^2), \tag{4}$$

$$P_{qq_n}=\sum_{l=0}^{n}p_l(\frac{x}{y})^l, \tag{5}$$

where $A=(a_{00},\ldots,a_{0n},a_{10},\ldots,a_{1n},\ldots,a_{n0},\ldots,a_{nn})_{1\times(n+1)^2}$, and

$$\Psi(x,Q^2)=\begin{pmatrix}\left\{\begin{matrix}\phi_0(x)\nu_0(Q^2)\\ \vdots\\ \phi_0(x)\nu_n(Q^2)\end{matrix}\right.\\ \left\{\begin{matrix}\phi_1(x)\nu_0(Q^2)\\ \vdots\\ \phi_1(x)\nu_n(Q^2)\end{matrix}\right.\\ \vdots\\ \left\{\begin{matrix}\phi_n(x)\nu_0(Q^2)\\ \vdots\\ \phi_n(x)\nu_n(Q^2)\end{matrix}\right.\end{pmatrix}_{(n+1)^2\times1}.$$

By substituting Eqs. (4) and (5) in the right-hand side of Eq. (1), we have:

$$\int_1^x\frac{P_{qq_n}(\frac{x}{y})}{y}q_{Ns_n}(y,Q^2)dy=AP_{qq}\Psi(x,Q^2), \tag{6}$$

where P_{qq} is an $(n+1)^2\times(n+1)^2$ matrix obtained from the coefficients of Tylor series of P_{qq_n} in (5). According to Theorem 1, we can easily obtain that for each linear differential operator $\frac{df(x,Q^2)}{d\log Q^2}$, there exists a unique matrix Π satisfying the following equation:

$$\frac{df(x,Q^2)}{d\log Q^2}=A\Pi\Psi(x,Q^2). \tag{7}$$

The structure of the matrix Π is defined as $\Pi=2.3\ \mathrm{diag}\,(C,\ldots,C)_{(n+1)^2\times(n+1)^2}$ where C is a block matrix defined as $C=\mathrm{diag}\,(0,1,\ldots,n)_{(n+1)\times(n+1)}$.

By substituting Eqs. (6) and (7) in (1), we get:

$$A(\Pi+\frac{\alpha_s}{2\pi}P_{qq})=0. \tag{8}$$

On the other hand, based on the initial condition of matrix equation, we have:

$$AB_i = d_i, \quad i = 1, \ldots, k, \tag{9}$$

where

$$B_i = \left(1, Q_i^2, \ldots, (Q_i^2)^n, x_i, x_i Q_i^2, \ldots, x_i (Q_i^2)^n, x_i^n, \ldots, x_i^n (Q_i^2)^n\right)^T_{1\times(n+1)^2}.$$

Now, let B be a matrix whose columns are B_i and use the following equations:

$$\begin{cases} T = \Pi + \dfrac{\alpha_s}{2\pi} P_{qq}, \\ G_n = \left(B \;\; T_1 \;\; \cdots \;\; T_{(n+1)^2-k}\right), \\ g_n = (d_0 \;\; \cdots \;\; d_k \;\; 0 \;\; \cdots \;\; 0). \end{cases}$$

where T_j is the jth column of matrix T. Therefore, the system of equations (8) and (1) can be written as follows:

$$AG_n = g_n,$$

where G_n is an $(n+1)^2 \times (n+1)^2$ matrix and g_n is an $1 \times (n+1)^2$ vector. Solving this system of linear equations produces the unknown coefficients $\{a_{00}, \ldots, a_{0n}, \ldots, a_{n0}, \ldots, a_{nn}\}$, in approximation (4).

2.2 Solving DGLAP evolution equation for singlet quark distribution function using the Tau method

Here, consider the following integral differential equation:

$$\frac{dq_s(x,Q^2)}{d\log Q^2} = -\frac{\alpha_s}{2\pi}\int_x^1 \left(P_{qq}(\frac{x}{y}) q_s(y,Q^2) + 2n_f P_{qg}(\frac{x}{y}) g(y,Q^2)\right)\frac{dy}{y}, \tag{10}$$

with initial conditions:

$$q_s(x,0.35) = u(x,0.35) + \bar{u}(x,0.35) + d(x,0.35) + \bar{d}(x,0.35) + s(x,0.35) + \bar{s}(x,0.35).$$

First, there exists a function q_{s_1} such that q_s can be considered as:

$$q_s(x,Q^2) = q_{s_1}(x,Q^2) + g(x,Q^2).$$

Therefore, the left-hand side of the equation (10) can be written as follows:

$$\frac{dq_s(x,Q^2)}{d\log Q^2} = \frac{dq_{s_1}(x,Q^2)}{d\log Q^2} + \frac{dg(x,Q^2)}{d\log Q^2}.$$

Consider the following approximations

$$\begin{aligned} g_n(x,Q^2) &= \sum_{i=0}^{n}\sum_{j=0}^{n} c_{ij}\phi_i(x)\nu_j(t) = C\Psi(x,Q^2), \\ q_{s_{n1}}(x,Q^2) &= \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}\phi_i(x)\nu_j(t) = A\Psi(x,Q^2), \end{aligned} \tag{11}$$

of g and q_{s_1}, respectively. So, the approximation of q_s can be considered as follows:

$$q_{s_n}(x,Q^2) = q_{s_{n1}}(x,Q^2) + g_n(x,Q^2) = (A+C)\Psi(x,Q^2). \tag{12}$$

Therefore, by replacing Eqs. (11), (12) and $n_f = 3$ in the right-hand of Eq. (10), we get:

$$\frac{\alpha_s}{2\pi}\int_1^x \Big(P_{qq}(\frac{x}{y})q_{s_n}(y,Q^2) + 2n_f P_{qg}(\frac{x}{y})g_n(y,Q^2)\Big)\frac{dy}{y} = \frac{\alpha_s}{2\pi}\Big(AP_{qq} + CP_{qq} + 6CP_{qg}\Big)\Psi(x,Q^2), \tag{13}$$

where P_{qq} and P_{qg} are $(n+1)^2 \times (n+1)^2$ matrices obtained from the coefficients of Tylor series of functions $P_{qq}(\frac{x}{y})$ and $P_{qg}(\frac{x}{y})$, respectively. On the other hand, based on Theorem 1 we have:

$$\begin{aligned} \frac{dq_{s_{n1}}(x,Q^2)}{dlogQ^2} &= A\Pi\Psi(x,Q^2), \\ \frac{dq_g(x,Q^2)}{dlogQ^2} &= C\Pi\Psi(x,Q^2). \end{aligned} \tag{14}$$

Take Eq. (14) in Eq. (10) we can write:

$$A(\Pi + \frac{\alpha_s}{2\pi}P_{qq})\Psi(x,Q^2) + C(\Pi + \frac{\alpha_s}{2\pi}P_{qq} + 6\frac{\alpha_s}{2\pi}P_{qg})\Psi(x,Q^2) = 0.$$

So, we have:

$$A(\Pi + \frac{\alpha_s}{2\pi}P_{qq}) + C(\Pi + \frac{\alpha_s}{2\pi}P_{qq} + 6\frac{\alpha_s}{2\pi}P_{qg}) = 0.$$

Now, consider the following system of equations:

$$\begin{cases} A(\Pi + \frac{\alpha_s}{2\pi}P_{qq}) = 0, \\ C(\Pi + \frac{\alpha_s}{2\pi}P_{qq} + 6\frac{\alpha_s}{2\pi}P_{qg}) = 0. \end{cases} \tag{15}$$

On the other hand, based on the initial conditions for gluon and singlet quark, we have:

$$\begin{cases} AB_i = d_i, & i = 1,\ldots,k, \\ CB_i = e_i, & i = 1,\ldots,k. \end{cases} \tag{16}$$

Set $T = \Pi + \frac{\alpha_s}{2\pi}P_{qq}$ and $S = \Pi + \frac{\alpha_s}{2\pi}P_{qq} + 6\frac{\alpha_s}{2\pi}P_{qg}$. Therefore, based on Eqs. (15) and (16), the following systems are obtained:

$$AU_n = u_n, \quad CW_n = w_n,$$

where U_n and W_n are $(n+1)^2 \times (n+1)^2$ matrices and u_n and w_n are an $1 \times (n+1)^2$ vectors as follows:

$$\begin{aligned} U_n &= \Big(B \;\; T_1 \;\; \cdots \;\; T_{(n+1)^2-k}\Big), \\ W_n &= \Big(B \;\; S_1 \;\; \cdots \;\; S_{(n+1)^2-k}\Big), \\ u_n &= (d_0, d_1, \ldots, d_k, 0, \ldots, 0), \\ w_n &= (e_0, e_1, \ldots, e_k, 0, \ldots, 0). \end{aligned} \tag{17}$$

Solving them provides us with the unknown coefficients $\{a_{00}, a_{01}, \ldots, a_{n0}, \ldots, a_{nn}\}$ and $\{c_{00}, c_{01}, \ldots, c_{n0}, \ldots, c_{nn}\}$ in (11).

3 Numerical example

In this section, we give a numerical example. The computations were done in MATLAB 2018a. Also, the obtained results were compared with the function of initial conditions obtained experimentally.

Example 1. Consider the integro differential equation for nonsinglet quark distribution function:

$$\frac{dq_{Ns}(x,Q^2)}{d\log Q^2} = \frac{-\alpha_s(Q^2)}{2\pi}\int_1^x \left(p_{qq}(\frac{x}{y})q_{Ns}(y,Q^2)\right)\frac{dy}{y},$$

with the initial conditions:

$$\begin{cases} u(x) = 1.53461x^{-0.46792}(1-x)^{1.43212}(1-3.38053x+2.98227x^{0.5}), \\ \bar{u}(x) = 2.85227x^{-0.05840}(1-x)^{5.30056}(1+1.65916x-2.31355x^{0.5}), \\ d(x) = 6.78582x^{-0.12584}(1-x)^{0.34160}(1+0.97882x-1.96756x^{0.5}), \\ \bar{d}(x) = 4.84634x^{-0.05340}(1-x)^{5.35315}(1+1.68065x-2.32755x^{0.5}). \end{cases}$$

The solution of this example is approximated using 11 points of initial nonsinglet quark function (2) using the Tau method. We assume that $\{\phi_i(x)\}_{i=0}^n$ and $\{v_j(Q^2)\}_{j=0}^n$ are the sets of standardized polynomials $\{1,x,x^2,\ldots,x^n\}$ and $\{1,Q,Q^2,\ldots,Q^n\}$, respectively. The approximate function is shown by $q_{Ns_n}(x,0.35)$. The CPU time for determining of the coefficients of $q_{Ns_n}(x,0.35)$ is $5.056251s$. Finally this function and the function $xq_{Ns_n}(x,0.35)$ are compared to the functions $q_{Ns}(x,0.35)$ and $q_{Ns}(x,0.35)$ obtained experimentally. This comparison is shown in Figures 1 and 2.

Now, using this approximation, the values of the nonsinglet quark distribution function can be computed in Q^2 values, which is singificant for the values $Q^2 = 1GeV^2$, $Q^2 = 4GeV^2$ and $Q^2 = 10GeV^2$ in Table 1.

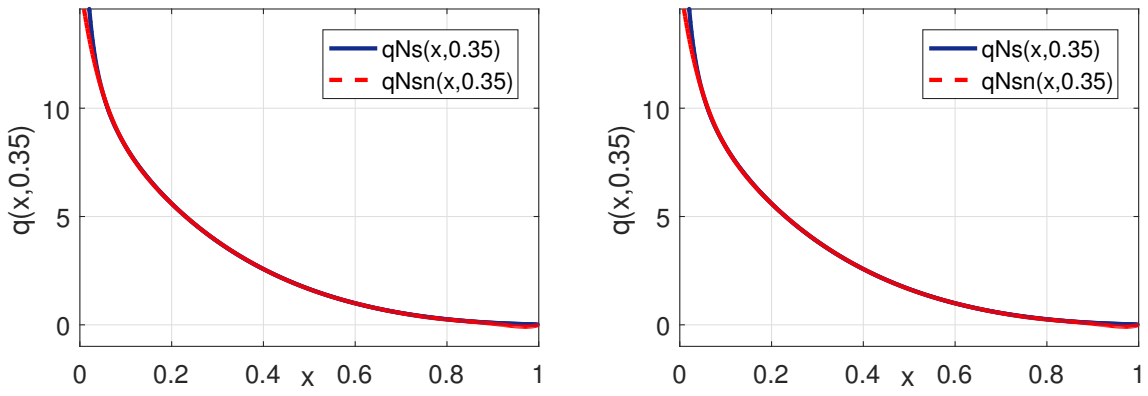

Figure 1: Comparison of charts of the $q_{Ns}(x,0.35)$ with $q_{Ns_n}(x,0.35)$.

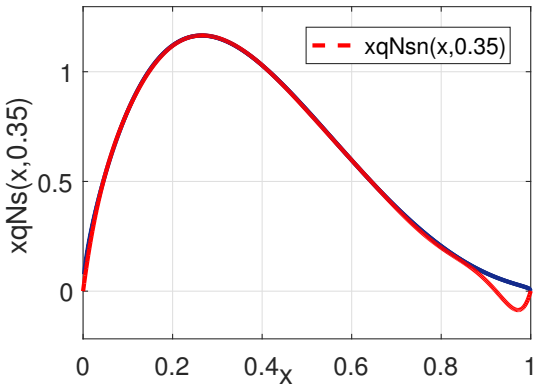

Figure 2: Comparison of charts of the $xq_{Ns}(x,0.35)$ with $xq_{Ns_n}(x,0.35)$.

Table 1: Numerical results for Example 1.

x	0.1	0.3	0.5	0.7	0.9
$Q^2 = 1GeV^2$	$-6.0591e+04$	$-8.3147e+04$	$-1.6747e+05$	$-5.2500e+05$	$-1.7647e+06$
$Q^2 = 4GeV^2$	$2.2333e+09$	$3.3330e+09$	$8.5565e+09$	$3.3987e+10$	$1.2706e+11$
$Q^2 = 10GeV^2$	$9.4033e+12$	$1.4567e+13$	$4.2416e+13$	$1.8583e+14$	$7.2073e+14$

4 Conclusion

Based on the solutions resulting from this method and analysis of the example provided, it was appeared that the results of numerical solution of DGLAP equation for distribution function of nonsinglet quark using the Tau approximation method for $n = 11$ and using $Q_0^2 = 0.35GeV^2$ was compatible with the initial distribution function of nonsinglet quark in $Q_0^2 = 0.35GeV^2$. However, as could be seen, there was some errors at the start and end of the interval which was due to singularity of the function at the start and end of the interval. Since all calculations have been carried out in a total space, decreasing the interval size can certainly lead to more accurate solution.

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