Journal of Mathematical Modeling
Vol. 10, No. 1, 2022, pp. 39-. Research Article

JMM

On generalized Schur complement of matrices and its applications to real and integer matrix factorizations

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Abstract. We provide a general finite iterative approach for constructing factorizations of a matrix A under a common framework of a general decomposition $A = BC$ based on the generalized Schur complement. The approach applies a zeroing process using two index sets. Different choices of the index sets lead to different real and integer matrix factorizations. We also provide the conditions under which this approach is well-defined.

Keywords: Matrix factorization, generalizd Shcur complement, quadrant interlocking factorization, *WZ* factorization.
AMS Subject Classification 2010: 15A06, 15A23, 49M27.

1 Introduction

Solving a linear system $Ax = b$ is an important and common problem in engineering and scientific computations. A direct method for solving a dense linear system is to factorize the matrix A into some simpler matrices and then solve the corresponding simpler linear systems. The most known factorization is the *LU* factorization. The *WZ* factorization or butterfly factorization was proposed by Evans and Hatzopoulos [8] to factorize nonsingular matrices. It is a direct method for solving linear systems in parallel. It was proposed as an alternative to the *LU* factorization and was originally named as Quadrant Interlocking Factorization (QIF).

ABS methods comprise an extensive class of algorithms, first introduced by Abaffy, Broyden and Spedicato, for solving linear systems of equations [1], and later extended to nonlinear algebraic equations and optimization problems [16,17], linear Diophantine equations [5,6], linear Diophantine systems with controlling the growth of intermediate results [13], computing the Smith normal form [11] and the *WZ* factorization [10, 12]. Recently Raboky and Mahdavi-Amiri presented a generalization of the ABS algorithm which is more general to produce several other matrix factorizations [9].

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Received: 30 April 2021/ Revised: 20 May 2021/ Accepted: 21 May 2021
DOI: 10.22124/jmm.2021.19495.1675

As a useful tool, the (generalized) Schur complement has various important applications in many aspects of matrix theory, applied mathematics, and statistics [2, 3]. Smith [15] focused on interlacing eigenvalues of Schur complement and econometrics. The notion of Schur complement can be extended to that of generalized Schur complement [4], when a matrix is allowed to have a singular block. Ando [2] studied generalized Schur complement and Lyapunov stability. Wang [18] gave some inequalities for generalized Schur complement.

Let $\mathbb{R}$ and $\mathbb{R}^{m\times n}$ denote the set of real numbers and the set of $m\times n$ real matrices, respectively, and $[n]=\{1,\dots,n\}$. Let also $A\in\mathbb{R}^{m\times n}$, and $\alpha\subset[m]$ and $\beta\subset[n]$ be two index sets. We denote the cardinality of α by $|\alpha|$. $A[\alpha,\beta]$ denotes the $|\alpha|\times|\beta|$ submatrix of A composed of the rows specified by α and the columns specified by β. The generalized partitioned form of A based on the index sets α and β is a 2×2 block matrix, with the following blocks:

$$E_{\alpha,\beta}=A[\alpha,\beta],\ F_{\alpha,\beta}=A[\alpha,\beta^c],\ G_{\alpha,\beta}=A[\alpha^c,\beta],\ H_{\alpha,\beta}=A[\alpha^c,\beta^c], \tag{1}$$

where $\alpha^c=[m]-\alpha$ and $\beta^c=[n]-\beta$. We also denote $A[\alpha]$ for $A[\alpha,\alpha]$.

Corresponding to any nonsingular submatrix, the Schur complement can be formed not just to a leading principal submatrix. Let $|\alpha|=|\beta|$ and $A[\alpha,\beta]$ be nonsingular. Then, the generalized Schur complement of A corresponding to $A[\alpha,\beta]$ is defined as

$$S_{\alpha,\beta}=A[\alpha^c,\beta^c]-A[\alpha^c,\beta](A[\alpha,\beta])^{+}A[\alpha,\beta^c], \tag{2}$$

where $(A[\alpha,\beta])^{+}$ is the Moore$-$Penrose inverse of the matrix $A[\alpha,\beta]$ [19]. Obviously, if $A[\alpha,\beta]$ is nonsingular, then the Moore-Penrose inverse $(A[\alpha,\beta])^{+}$ equals $(A[\alpha,\beta])^{-1}$. Many other results on Schur complement can be extended to generalized Schur complement.

Let P_α and P_β denote the partial permutation matrices to move the rows and columns of A, respectively, to locate $A[\alpha,\beta]$ into the upper left corner of A. Letting

$$A_{\alpha,\beta}=P_\alpha^T A P_\beta=\begin{pmatrix} A[\alpha_1,\beta_1] & \cdots & A[\alpha_s,\beta_s]\\ \vdots & \ddots & \vdots\\ A[\alpha_s,\beta_1] & \cdots & A[\alpha_s,\beta_s]\end{pmatrix}, \tag{3}$$

we have

$$A_{\alpha,\beta}=\begin{pmatrix} A[\alpha,\beta] & A[\alpha,\beta^c]\\ A[\alpha^c,\beta] & A[\alpha^c,\beta^c]\end{pmatrix}=\begin{pmatrix} E_{\alpha,\beta} & F_{\alpha,\beta}\\ G_{\alpha,\beta} & H_{\alpha,\beta}\end{pmatrix}. \tag{4}$$

If $A(\alpha,\beta)$ is invertible, then

$$A_{\alpha,\beta}=\begin{pmatrix} I & 0\\ G_{\alpha,\beta}E_{\alpha,\beta}^{-1} & I\end{pmatrix}\begin{pmatrix} E_{\alpha,\beta} & 0\\ 0 & S_{\alpha,\beta}\end{pmatrix}\begin{pmatrix} I & E_{\alpha,\beta}^{-1}F_{\alpha,\beta}\\ 0 & I\end{pmatrix}. \tag{5}$$

Let $E_{\alpha,\beta}$ be invertible. Then, A is invertible if and only if $S_{\alpha,\beta}$ is invertible. If $\alpha=\beta$, then there exists a permutation matrix P such that

$$P^TAP=\begin{pmatrix} A[\alpha] & A[\alpha,\alpha^c]\\ A[\alpha^c,\alpha] & A[\alpha^c]\end{pmatrix}. \tag{6}$$

Definition 1. *We say that $\alpha=\{\alpha_1,\dots,\alpha_s\}$ is an index set of $[n]$ if and only if $\alpha_i\subset[n]$, for $i=1,\dots,s$, $\alpha_i\bigcap\alpha_j=\emptyset$, whenever $i\neq j$, and $\cup_{i=1}^s\alpha_i=[n]$.*

An index set is a set whose members are formed as sets of members of another set. The index set α can be seen as a block permutation vector, where the elements of the sets α_i are written in ascending order. For example, $\alpha = \{\{1,6\},\{2,5\},\{3,4\}\}$ is an index set [6].

Let $A \in \mathbb{R}^{m\times n}$, and α and β be two index sets of $[n]$. Assume that $A(:,b)$ denotes the $m \times |b|$ submatrix of A containing the columns specified by b and $A(\alpha_i,\beta_j)$ denotes the $|\alpha_i| \times |\beta_j|$ submatrix of A composed of the rows specified by α_i and the columns specified by β_j. If $\alpha_i = \beta_j$, then $A(\alpha_i,\beta_j)$ is a principal submatrix of A, and if $\alpha_i = \beta_j = \{1,\dots,i\},\ 1 \le k \le \min\{m,n\}$, then $A(\alpha_i,\beta_j)$ is a leading principal submatrix of A.

Definition 2. *Let $A \in \mathbb{R}^{n\times n}$, and $\alpha = \{\alpha_1,\dots,\alpha_s\}$ and $\beta = \{\beta_1,\dots,\beta_s\}$ be two index sets of $[n]$, $A(\alpha_i,\beta_j)$, $1 \le i,j \le s$ denote the (i,j)th block of A and*

$$A(\cup_{i=1}^{k}\alpha_i,\cup_{j=1}^{k}\beta_j) = \begin{pmatrix} A[\alpha_1,\beta_1] & \cdots & A[\alpha_1,\beta_k] \\ \vdots & \ddots & \vdots \\ A[\alpha_k,\beta_1] & \cdots & A[\alpha_k,\beta_k] \end{pmatrix}, \quad k = 1,\dots,s.$$

We say A is (α,β)-block strongly nonsingular if and only if $A(\cup_{i=1}^{k}\alpha_i,\cup_{i=1}^{k}\beta_i)$, for $k = 1,\dots,s$, are nonsingular.

Definition 3. *Let $A \in \mathbb{R}^{n\times n}$, and $\alpha = \{\alpha_1,\dots,\alpha_s\}$ and $\beta = \{\beta_1,\dots,\beta_s\}$ be two index sets of $[n]$ such that $|\alpha_i| = |\beta_i|$. The matrix A is called row (α,β)-block diagonally dominant matrix (with respect to the matrix norm $||\cdot||$) if for $i = 1,\dots,s$, the matrices $A(\alpha_i,\beta_i)$ are nonsingular, and*

$$\sum_{j=1, j\neq i}^{s} ||A(\alpha_i,\beta_i)^{-1}A(\alpha_i,\beta_j)|| \le 1,\ i = 1,\dots,s. \tag{7}$$

If a strict inequality holds in (7), then A is called row (α,β)-block strictly diagonally dominant (with respect to the matrix norm $||\cdot||$).

Let A be an (α,β)-block strictly diagonally dominant matrix. Then, the $A(\cup_{i=1}^{k}\alpha_i,\cup_{i=1}^{k}\beta_i)$ are nonsingular, for $k = 1,\dots,s$, and A is an (α,β)-strongly nonsingular matrix.

Definition 4. *Let $A \in \mathbb{R}^{n\times n}$, and $\alpha = \{\alpha_1,\dots,\alpha_s\}$ and $\beta = \{\beta_1,\dots,\beta_s\}$ be two index sets of $[n]$ such that $|\alpha_i| = |\beta_i|$. The matrix A is called (α,β)-block symmetric matrix if the $A(\alpha_i,\beta_i)$ are symmetric and $A(\alpha_i,\beta_j) = A^T(\beta_j,\alpha_i)$, for $i,j = 1,\dots,s$.*

Remark 1. *A is (α,β)-block strongly nonsingular, (α,β)-block strictly diagonally dominant, (α,β)-block symmetric positive definite, if and only if $A_{\alpha,\beta}$ is block strongly nonsingular, block strictly diagonally dominant, block symmetric positive definite, respectively.*

Now, consider the following definition.

Definition 5. *$A \in \mathbb{Z}^{n\times n}$ is a unimodular matrix if and only if $|det(A)| = 1$.*

Note that A is unimodular if and only if A^{-1} is unimodular.

Definition 6. *Let $A \in \mathbb{Z}^{n\times n}$, and $\alpha = \{\alpha_1,\dots,\alpha_s\}$ and $\beta = \{\beta_1,\dots,\beta_s\}$ be two index sets of $[n]$. We say that A is (α,β)-block strongly unimodular if and only if $A(\cup_{i=1}^{k}\alpha_i,\cup_{i=1}^{k}\beta_i)$, for $k = 1,\dots,s$, are unimodular.*

2 Generalized Schur complement and matrix factorization

Here, we provide an iterative approach based on the generalized Schur complement for constructing factorizations of $A = BC$. The approach applies a zeroing process using two index sets. Let $A_1 = A$. At the kth step, the method calculates $A_{k+1}(\alpha_k^c, \beta_k) = 0$ as follows:

$$A_{k+1}(\alpha_k^c, \beta_k^c) = A_k(\alpha_k^c, \beta_k^c) - A_k(\alpha_k^c, \beta_k) A_k(\alpha, \beta_k)^{-1} A_k(\alpha, \beta_k^c).$$

We note that different choices of the index sets lead to different real and integer matrix factorizations. Now, we are ready to present the following algorithm.

Algorithm 1 Generalized Schur complement algorithm.

Input: $A \in \mathbb{R}^{n\times n}$, $\alpha = \{\alpha_1, \dots, \alpha_s\}$ and $\beta = \{\beta_1, \dots, \beta_s\}$ two index sets of $[n]$ such that $|\alpha_i| = |\beta_i|$.
Output: The matrix factorization $A = BC$.
Let $B = I_n$.
for $k = 1, \dots, s$ **do**
  Compute $E_k = A[\alpha_k, \beta_k],\ F_k = A[\alpha_k, \beta_k^c],\ G_k = A[\alpha_k^c, \beta_k],\ H_k = A[\alpha_k^c, \beta_k^c]$.
  Set $S_k = H_k - G_k E_k^{-1} F_k$.
  Let $A[\alpha_k^c, \beta_k] = 0,\ \ A[\alpha_k^c, \beta_k^c] = S_k$, and $B[\alpha_k^c, \alpha_k] = G_k E_k^{-1}$.
end for
Set $C = A$.

If $A(\alpha_k, \beta_k)$ is singular for an index k, then $A(\alpha_{k+1}, \beta_{k+1})$ is singular. Now, according to (5) we have the next result.

Theorem 1. *(Factorization theorem) Let $A \in \mathbb{R}^{n\times n}$. The generalized Schur complement algorithm (Algorithm 1) can be performed on A to produce the matrix factorization $A = BC$ if and only if A is an (α, β)-block strongly nonsingular matrix.*

Proof. A is (α, β)-block strongly nonsingular if and only if E_k are nonsingular, for $k = 1, \dots, s$. This complete the proof. □

Corollary 1. *Let $A \in \mathbb{Z}^{n\times n}$, and $\alpha = \{\alpha_1, \cdots, \alpha_s\}$ and $\beta = \{\beta_1, \cdots, \beta_s\}$ be two index sets. If A is (α, β)-block strongly unimodular, then Algorithm 1 produces an integer matrix factorizatin $A = BC$, where B and C are integer matrices.*

3 Matrix factorizatios

Let $A \in \mathbb{R}^{n\times n}$, and $\alpha = \{\alpha_1, \dots, \alpha_s\}$ and $\beta = \{\beta_1, \dots, \beta_s\}$ be two index sets of n. Then, Algorithm 1 computes a matrix factorization $A = BC$ such that $B(\alpha(i), \beta(i)) = I$. In the kth step, Algorithm 1 performs a partial zeroing such that the elements of the submatrix A_k correspond to the columns specified by β_k and the rows specified by α_k^c.

Here, we consider different choices for α and β such that all the blocks $A(\alpha_i, \beta_j)$ are of size 1×1 and present the associated matrix factorizations. First, we define two index sets. Let $J = \{j_1, \dots, j_n\}$, such that

$$j_i = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd,} \\ n - \frac{i}{2} + 1, & \text{if } i \text{ is even.} \end{cases} \tag{8}$$

We define the index set $K = \{k_1, \ldots, k_n\}$ as follows. If n is an even number, define

$$k_i = \begin{cases} \frac{n}{2} - \frac{i+1}{2} + 1, & \text{if } i \text{ is odd,} \\ \frac{n}{2} + \frac{i}{2}, & \text{if } i \text{ is even,} \end{cases} \tag{9}$$

and if n is an odd number, define

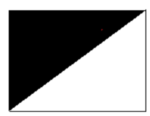
$$k_i = \begin{cases} \frac{n+1}{2} - \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{n+1}{2} + \frac{i-1}{2}, & \text{if } i \text{ is odd.} \end{cases} \tag{10}$$

The following cases are noted:

Case (1): For $\alpha_i = \beta_i = i$, B is a lower triangular and C is an upper triangular matrix.

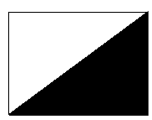
Case (2): For $\alpha_i = \beta_i = n-i+1$, B is an upper triangular and C is a lower triangular matrix.

Case (3): For $\alpha_i = i, \beta_i = n-i+1$, B is lower triangular and C has the following structure:



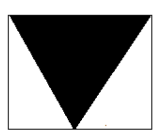
C

Case (4): For $\alpha_i = n-i+1, \beta_i = i$, B is upper triangular and C has the following structure:



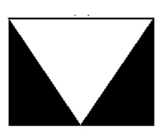
C

Case (5): For $\alpha_i = i, \beta_i = j_i$, B is lower triangular and C has the following structure:



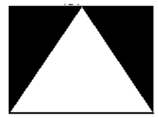
C

Case (6): For $\alpha_i = n-i+1, \beta_i = k_i$, B is upper triangular and C has the following structure:



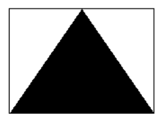
C

Case (7): For $\alpha_i = i, \beta_i = k_i$, B is lower triangular and C has the following structure:



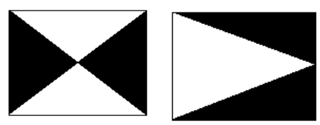
C

Case (8): For $\alpha_i = n-i+1, \beta_i = j_i$, B is upper triangular and C has the following structure:



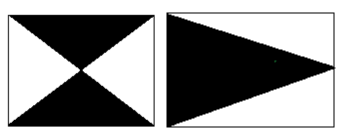
C

Case (9): For $\alpha_i = j_i, \beta_i = i$, we have:



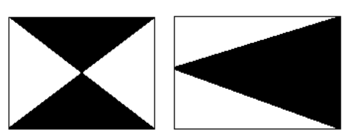
B C

Case (10): For $\alpha_i = k_i, \beta_i = n-i+1$, we have:



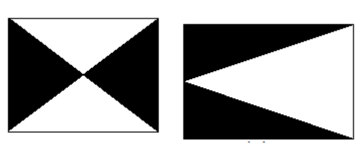
B C

Case (11): For $\alpha_i = k_i, \beta_i = i$, we have:



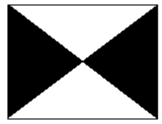
B C

Case (12): For $\alpha_i = j_i, \beta_i = n-i+1$, we have:



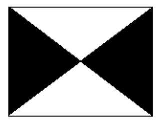
B C

Case (13): For $\alpha_i = j_i, \beta_i = j_i$, we have a *WZ* factorization:

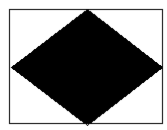
B C

Case (14): For $\alpha_i = k_i, \beta_i = k_i$, we have a *ZW* factorization:

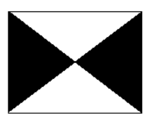
B C

Case (15): For $\alpha_i = k_i, \beta_i = j_i$, we have:

B C

Case (16): For $\alpha_i = j_i, \beta_i = k_i$, we have:

B C

Below, we give a MATLAB code for Algorithm 1, where $A \in \mathbb{R}^{n\times n}$, and $t = \{t_1, \dots, t_n\}$ and $b = \{b_1, \dots, b_n\}$ are two index sets and all the blocks $A(\alpha_i, \beta_i)$ are of size 1×1, for $i = 1, \dots, n$.

```
function [B,C]=Genschur(A,t,b)
clc
[n,n]=size(A);
B=eye(n);
C=zeros(n);
for k=1:n-1
    E=zeros(1,1);F=zeros(1,n-k);G=zeros(n-k,1);H=zeros(n-k,n-k);
    E=A(t(k),b(k));
    G=A(t(k+1:n),b(k));
    F=A(t(k),b(k+1:n));
    H=A(t(k+1:n),b(k+1:n));
    B(t(k+1:n),t(k))=G*inv(E);
    S=H-G*inv(E)*F;
    A(t(k+1:n),b(k))=0;
    A(t(k+1:n),b(k+1:n))=S;
end
C=A;
end
```

4 Quadrant interlocking factorization

A direct method called the *WZ* factorization, used for solving linear systems of equations $Ax = b$, was introduced by Evans and Hotzopoulos [8]. Let A be an $n \times n$ nonsingular matrix. The *WZ* factorization of [7] expresses A as $A = WZ$, with W and Z having the following forms:

$$W = \begin{pmatrix} \bullet & \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \bullet & \bullet \\ \bullet & \circ & \circ & \circ & \bullet \end{pmatrix}, \; Z = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \circ \\ \circ & \circ & \bullet & \circ & \circ \\ \circ & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix}, \tag{11}$$

where the empty bullets stand for zero and the other bullets stand for possible nonzeros.

The transpose of a *W-matrix* is called a *Z-matrix* and vise versa. A matrix which is both a *Z-* and a *W-matrix* is called an *X-matrix*.

Note: Without loss of generality, we assume that A is an $n \times n$ nonsingular matrix and n is even.

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$. A has a WZ factorization if and only if the nested submatrices $A([1:k, n-k+1:n],[1:k, n-k+1:n])$ are invertible, for $k = 1, \dots, \frac{n}{2}$.*

Proof. See proof for Theorem 2 of [14]. □

The cases (13) and (14) above compute the *WZ* and the *ZW* factorizations by 1×1 blocks. Next, we compute the factorizations using 2×2 blocks which is suitable for parallel computing.

Let $A \in \mathbb{R}^{n\times n}$ and $\alpha = \beta = \{k, n-k+1\}$, for $k = 1, \dots, \frac{n}{2}$. If A is (α, β)-block strictly nonsingular matrix, then by Theorem 2 A has a *WZ* factorization. Furtheremore, with the index sets, Algorithm 1 produces a *WZ* factorization for A.

Below we give a MATLAB code for Algorithm 1, where $A \in \mathbb{R}^{n\times n}$, $\alpha_k = \beta_k = \{k, n-k+1\}$, and all the blocks $A(\alpha_k, \beta_k)$ are of size 2×2, for $k = 1, \dots, \frac{n}{2}$.

```
function [B,C]=GenschurWZ(A)
% This program produces B as a W-matrix  and C as a Z-matrix
clc
[n,n]=size(A);
B=eye(n);
C=zeros(n);
for k=1:n/2
    E=A([k,n-k+1],[k,n-k+1]);
    G=A([k+1:n-k],[k, n-k+1] );
    F=A([k,n-k+1],[k+1:n-k]);
    H=A([k+1:n-k],[k+1:n-k]);
    B([k+1:n-k],[k,n-k+1])=G*inv(E);
    S=H-G*inv(E)*F;
    A([k+1:n-k],[k,n-k+1])=0;
    A([k+1:n-k],[k+1:n-k])=S;
end
C=A;
end
```

Remark 2. *Let $A \in \mathbb{Z}^{n\times n}$. If the nested submatrices $A(1:k, n-k+1:n, 1:k, n-k+1:n)$ are unimodular, for $k = 1,\dots,\frac{n}{2}$, then A has an integer WZ factorization.*

Theorem 3. *Let $A \in \mathbb{R}^{n\times n}$. A has a ZW factorization if and only if the nested submatrices $A([\frac{n}{2}-k+1 : \frac{n}{2}+k], [\frac{n}{2}-k+1 : \frac{n}{2}+k])$ are invertible, for $k = 1,\dots,\frac{n}{2}$.*

Proof. See proof for Theorem 7 of [10]. □

Consider two equal index sets $\alpha = \{\alpha_1,\dots,\alpha_{\frac{n}{2}}\}$ and $\beta = \{\beta_1,\dots,\beta_{\frac{n}{2}}\}$ such that $\alpha_k = \beta_k = \{\frac{n}{2}-k+1, \frac{n}{2}+k\}$. Let $A \in \mathbb{R}^{n\times n}$ be an (α,β)-block strictly nonsingular matrix. Upon an application of Algorithm 1, a *ZW* factorization for A is produced.

Below, we give a MATLAB code for Algorithm 1, where $A \in \mathbb{R}^{n\times n}$, $\alpha_k = \beta_k = \{\frac{n}{2}-k+1, \frac{n}{2}+k\}$, and all the blocks $A(\alpha_k,\beta_k)$ are of size 2×2, for $k = 1,\dots,\frac{n}{2}$.

```
function [B,C]=GenschurZW(A)
% This program produces B as a Z-matrix and C as a W-matrix
clc
[n,n]=size(A);
B=eye(n);
C=zeros(n);
for k=1:n/2
    E=A([n/2-k+1,n/2+k],[n/2-k+1,n/2+k]);
    G=A([1:n/2-k,n/2+k+1:n],[n/2-k+1,n/2+k]);
    F=A([n/2-k+1,n/2+k],[1:n/2-k,n/2+k+1:n]);
    H=A([1:n/2-k,n/2+k+1:n],[1:n/2-k,n/2+k+1:n]);
    B([1:n/2-k,n/2+k+1:n],[n/2-k+1,n/2+k])=G*inv(E);
    S=H-G*inv(E)*F;
    A([1:n/2-k,n/2+k+1:n],[n/2-k+1,n/2+k])=0;
    A([1:n/2-k,n/2+k+1:n],[1:n/2-k,n/2+k+1:n])=S;
end
C=A;
end
```

Remark 3. *Let $A \in \mathbb{Z}^{n\times n}$. If the nested submatrices $A([\frac{n}{2}-k+1 : \frac{n}{2}+k], [\frac{n}{2}-k+1 : \frac{n}{2}+k])$ are unimodular, for $k = 1,\dots,\frac{n}{2}$, then A has an integer ZW factorization.*

5 Numerical experiments

Example 1. Consider the following matrix:

$$A = \begin{bmatrix} 0.6256 & 0.3379 & 0.7228 & 0.9845 & 0.9512 & 0.3806 & 0.4522 \\ 0.5751 & 0.2752 & 0.6681 & 0.8859 & 0.2490 & 0.9259 & 0.8492 \\ 0.7510 & 0.0060 & 0.1788 & 0.2138 & 0.3864 & 0.7408 & 0.3904 \\ 0.1535 & 0.8019 & 0.5505 & 0.0346 & 0.4314 & 0.7376 & 0.7384 \\ 0.3568 & 0.4974 & 0.9599 & 0.4511 & 0.8309 & 0.9469 & 0.9764 \\ 0.1440 & 0.5378 & 0.5960 & 0.0138 & 0.8246 & 0.5101 & 0.5233 \\ 0.8506 & 0.8709 & 0.8086 & 0.4737 & 0.4530 & 0.7919 & 0.4299 \end{bmatrix}.$$

Upon an application of Algorithm 1 with $\alpha_i = i$ and $\beta_i = j_i$, we have the matrix factorization $A = BC$, with

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.9193 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1.2004 & -0.3517 & 1 & 0 & 0 & 0 & 0 \\ 0.2454 & 1.4474 & -1.8692 & 1 & 0 & 0 & 0 \\ 0.5703 & 1.6574 & -0.8818 & 0.2836 & 1 & 0 & 0 \\ 0.2302 & 0.9670 & -1.1994 & 0.6237 & 0.8882 & 1 & 0 \\ 1.3597 & -0.4266 & -0.9618 & 1.3727 & 2.1767 & 24.0238 & 1 \end{bmatrix},$$

and

$$C \begin{bmatrix} 0.6256 & 0.3379 & 0.7228 & 0.9845 & 0.9512 & 0.3806 & 0.4522 \\ 0 & -0.0354 & 0.0036 & -0.0191 & -0.6254 & 0.5760 & 0.4335 \\ 0 & -0.4121 & -0.6876 & -0.9748 & -0.9754 & 0.4865 & 0 \\ 0 & 0 & -0.9174 & -2.0013 & -0.72 & 0.7198 & 0 \\ 0 & 0 & 0.1954 & -0.3708 & 0.6690 & 0 & 0 \\ 0 & 0 & 0 & 0.2141 & -0.1046 & 0 & 0 \\ 0 & 0 & 0 & -3.3997 & 0 & 0 & 0 \end{bmatrix}.$$

With the parameter choices $\alpha_i = j_i$ and $\beta_i = n - i + 1$, we have:

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.8779 & 1 & 0 & 0 & 0 & 0 & 0.4910 \\ 0.8633 & 0.0017 & 1 & 0 & 0 & 0.2047 & 0.9585 \\ 1.6329 & 0.7600 & -0.0824 & 1 & -1.5948 & 0.9486 & 0.2700 \\ 2.1592 & 0.8297 & -0.2424 & 0 & 1 & 0.9920 & 0.2909 \\ 1.1572 & 0.1543 & 0 & 0 & 0 & 1 & 0.1620 \\ 0.9507 & 0 & 0 & 0 & 0 & 0 & 1] \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0.6256 & 0.3379 & 0.7228 & 0.9845 & 0.9512 & 0.3806 & 0.4522 \\ -0.7254 & -0.6292 & -0.7489 & -0.7360 & -1.3157 & 0 & 0 \\ 0.0712 & -0.8432 & -0.5308 & 0 & 0 & 0 & 0 \\ 0.1927 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0561 & -0.2280 & 0 & 0 & 0 & 0 & 0 \\ -0.5095 & 0.1549 & -0.1445 & -0.9370 & 0 & 0 & 0 \\ 0.2559 & 0.5497 & 0.1214 & -0.4622 & -0.4513 & 0.4301 & 0 \end{bmatrix}.$$

Example 2. Let

$$A = \begin{bmatrix} 0.0965 & 0.8212 & 0.5470 & 0.7802 & 0.5085 & 0.3507 & 0.4709 & 0.3111 \\ 0.1320 & 0.0154 & 0.2963 & 0.0811 & 0.5108 & 0.9390 & 0.2305 & 0.9234 \\ 0.9421 & 0.0430 & 0.7447 & 0.9294 & 0.8176 & 0.8759 & 0.8443 & 0.4302 \\ 0.9561 & 0.1690 & 0.1890 & 0.7757 & 0.7948 & 0.5502 & 0.1948 & 0.1848 \\ 0.5752 & 0.6491 & 0.6868 & 0.4868 & 0.6443 & 0.6225 & 0.2259 & 0.9049 \\ 0.0598 & 0.7317 & 0.1835 & 0.4359 & 0.3786 & 0.5870 & 0.1707 & 0.9797 \\ 0.2348 & 0.6477 & 0.3685 & 0.4468 & 0.8116 & 0.2077 & 0.2277 & 0.4389 \\ 0.3532 & 0.4509 & 0.6256 & 0.3063 & 0.5328 & 0.3012 & 0.4357 & 0.1111 \end{bmatrix}.$$

Using the function *GenSchurWZ*, we have a *WZ* factorization as follows:

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.1412 & 1 & 0 & 0 & 0 & 0 & 0 & -0.4845 \\ 0.4768 & 0.7895 & 1 & 0 & 0 & 0 & -0.6421 & 2.5371 \\ -0.4130 & -0.1148 & 0.0991 & 1 & 0 & 9.5446 & 1.8450 & 2.8198 \\ 2.5787 & 0.3148 & -0.3475 & 0 & 1 & 1.0307 & 2.0490 & 0.9240 \\ 3.4226 & 0.4403 & 0 & 0 & 0 & 1 & 1.2522 & -0.7658 \\ 1.3003 & 0 & 0 & 0 & 0 & 0 & 1 & 0.3095 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$C=\begin{bmatrix} 0.0965 & 0.8212 & 0.5470 & 0.7802 & 0.5085 & 0.3507 & 0.4709 & 0.3111 \\ 0 & -2.3457 & -1.1188 & -2.2213 & -0.8284 & -0.0167 & -1.0376 & 0 \\ 0 & 0 & -0.5644 & 1.1086 & -0.1320 & -0.2616 & 0 & 0 \\ 0 & 0 & 0 & 2.9264 & 4.8960 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8319 & -0.3262 & 0 & 0 & 0 \\ 0 & 0 & -0.0453 & -0.1922 & -0.5709 & 0.0524 & 0 & 0 \\ 0 & -0.5596 & -0.5364 & -0.6625 & -0.0145 & -0.3415 & -0.5195 & 0 \\ 0.3532 & 0.4509 & 0.6256 & 0.3063 & 0.5328 & 0.3012 & 0.4357 & 0.1111 \end{bmatrix}.$$

We illustrate that B is a W matrix and C is a Z matrix. By applying the function *GenSchurZW* for computing the ZW factorization, we obtain the following result:

$$B=\begin{bmatrix} 1 & -0.6566 & 1.8535 & 2.2604 & -1.9992 & -1.2902 & 0.1916 & 0 \\ 0 & 1 & -2.2447 & -1.7400 & 2.9393 & 2.7711 & 0 & 0 \\ 0 & 0 & 1 & 1.7790 & -0.9256 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.8554 & -0.4675 & 1 & 0 & 0 \\ 0 & 0 & -0.7163 & -0.9498 & 2.4314 & -1.0903 & 1 & 0 \\ 0 & 0.0037 & -0.0365 & -0.5494 & 1.5047 & -0.7754 & 0.6319 & 1 \end{bmatrix},$$

and

$$C=\begin{bmatrix} -0.3187 & 0 & 0 & 0 & 0 & 0 & 0 & -0.3269 \\ 0.9520 & -3.2962 & 0 & 0 & 0 & 0 & 1.1882 & -2.7562 \\ -0.2264 & 0.3431 & 1.0442 & 0 & 0 & 0.4733 & 0.7068 & 0.9390 \\ 0.9561 & 0.1690 & 0.1890 & 0.7757 & 0.7948 & 0.5502 & 0.1948 & 0.1848 \\ 0.5752 & 0.6491 & 0.6868 & 0.4868 & 0.6443 & 0.6225 & 0.2259 & 0.9049 \\ -0.4891 & 0.8906 & 0.3429 & 0 & 0 & 0.4074 & 0.1097 & 1.2447 \\ -0.9510 & 0.4469 & 0 & 0 & 0 & 0 & 0.4894 & 0.4440 \\ 0.2229 & 0 & 0 & 0 & 0 & 0 & 0 & -0.4199 \end{bmatrix}.$$

The results show that B is a Z matrix and C is a W matrix.

References

[1] J. Abaffy, C.G. Broyden, E. Spedicato, *A class of direct methods for linear systems*, Numer. Math. **45** (1984) 361–376.

[2] T. Ando. *Generalized Schur complements*, Numer. Linear Algebra Appl. **27** (1979) 173–186.

[3] R.B. Bapat. *A refinement of Oppenheim′s inequality*, In current Trends in Matrix Theory: Proceedings of the Third Auburn Matrix Theory Conference, (1987) 29–32.

[4] D. Carlson, E. Haynsworth, T. Markham, *A generalization of the Schur complement by means of the Moore−Penrose inverse*, SIAM J. Appl. Math. **26** (1974) 169–175.

[5] H. Esmaeili, N. Mahdavi-Amiri, E. Spedicato, *A class of ABS algorithms for linear Diophantine systems*, Numer. Math. **90** (2001) 101–115.

[6] H. Esmaeili, N. Mahdavi-Amiri, E. Spedicato, *Generating the integer null space and conditions for determination of an integer basis using the ABS algorithms*, Bull. Iranian Math. Soc. **27** (2001) 1–18.

[7] D.J. Evans, A. Hadjidimos and , D. Noutsos, *The parallel solution of banded linear equations by the new quadrant interlocking factorization (Q.I.F.) method*, Int. J. Comput. Math. **9** (1981) 151–161.

[8] D.J. Evans, M. Hatzopoulos, *A parallel linear system solver*, Int. J. Comput. Math. **7** (1979) 227–238.

[9] E. Golpar-Raboky, N. Mahdavi-Amiri, *A generalization of the ABS algorithms and its application to some special real and integer matrix factorizations*, submitted.

[10] E. Golpar-Raboky, N. Mahdavi-Amiri, *A new interpretation of the integer and real WZ factorization using block scaled ABS algorithms*, Stat. Optim. Inform. Comput. **2** (2014) 243–256.

[11] E. Golpar-Raboky, N. Mahdavi-Amiri, *Diophantine quadratic equation and Smith normal form using scaled extended integer Abaffy-Broyden-Spedicato algorithms*, J. Optim. Theory Appl. **152** (2012) 75–96.

[12] E. Golpar-Raboky, N. Mahdavi-Amiri, *WZ factorization via Abaffy-Broyden-Spedicato algorithms*, Bull. Iranian Math. Soc. **40** (2014) 399–411.

[13] M. Khorramizadeh, N. Mahdavi-Amiri, *Integer extended ABS algorithms and possible control of intermediate results for linear Diophantine systems*, 4OR **7** (2009) 145–167.

[14] S.C.S. Rao, *Existence and uniqueness of WZ factorization*, Parallel Comput. **23** (1997) 1129–1139.

[15] R. L. Smith, *Some interlacing properties of the Schur complement of a Hermitian matrix*, Linear Algebra Appl. **177** (1992) 137–144.

[16] E. Spedicato, E. Bodon, X. Zunquan, N. Mahdavi-Amiri, *ABS methods for continous and integer linear equations and optimization*, CEJOR **18** (2010) 73–95.

[17] E. Spedicato, E. Bodon, A. Del Popolo, N. Mahdavi-Amiri, *ABS methods and ABSPACK for linear systems and optimization: A review*, 4OR **1** (2003) 51–66.

[18] B.Y. Wang, X. Zhang, F. Zhang, *Some inequalities on generalized Schur complements*, Linear Algebra Appl. **302** (1999) 163–172.

[19] F. Zhang, *The Schur Complement and its Applications*, Springer Science and Business Media, 2006.